1.1 15. j
16. c
17. g
18. b
19. h
20. e

1.2 13 The half-life of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation \( Q' = -rQ \), the half-life \( \tau \) and the decay rate \( r \) satisfy the equation \( r \tau = \ln 2 \).

Ans: By using separation of variables, we get the equation

\[
\frac{dQ}{Q} = -r dt
\]

Integrating both sides gives \( \ln(Q(t)) = -rt + C \), or \( Q(t) = e^{C}e^{-rt} \), where \( C \) is some constant. Let \( Q_0 \) be the initial amount of the material. Then plugging in \( t = 0 \) gives \( Q_0 = e^{C}e^{0} = e^{C} \), so \( Q(t) = Q_0e^{-rt} \). When the amount of material has reduced to half its original value, we have \( Q(\tau) = Q_0/2 \), so

\[
\frac{Q_0}{2} = Q_0e^{-r\tau}
\]

\[
\ln(1/2) = -r\tau
\]

\[
\ln(2^{-1}) = -r\tau
\]

\[
-\ln(2) = -r\tau
\]

\[
\ln(2) = r\tau
\]

14 Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.

Ans: From problem 13, we have that \( r \tau = \ln(2) \), so \( r = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{1620 \text{ yrs}} \). Let \( t_{3/4} \) be the time at which the amount of material has reduced by one-quarter. Then, \( Q(t_{3/4}) = \frac{3Q_0}{4} \), and using the solution to the differential equation in problem 13 gives

\[
\frac{3Q_0}{4} = Q_0e^{-\frac{\ln(2)}{1620 \text{ yrs}} t_{3/4}}
\]

\[
\ln(3/4) = -\frac{\ln(2)}{1620 \text{ yrs}} t_{3/4}
\]

\[
-\frac{1620 \text{ yrs} \ln(3/4)}{\ln(2)} = t_{3/4}
\]

Therefore, \( t_{3/4} = \frac{1620 \text{ yrs} \ln(3/4)}{\ln(2)} \approx 6.72 \times 10^2 \).

1.3 29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.

(a) Assume that the mass is in an arbitrary displaced position, indicated by the angle \( \theta \). Draw a free-body diagram showing the forces acting on the mass.

See the attached diagram at the end of the solutions.
(b) Apply Newton’s law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $L \frac{d^2 \theta}{dt^2}$, where $L$ is the length of the rod.

Newton’s law of motion is $F = ma$. We have that $a = L \frac{d^2 \theta}{dt^2}$ and from the figure, $F = -mg \sin(\theta)$. Notice, $F$ is negative since it is in the opposite direction of increasing $\theta$. Therefore, we have

$$-mg \sin(\theta) = mL \frac{d^2 \theta}{dt^2}$$

(c) Simplify the result from part (b) to obtain Eq. (12) in the text.

$$L \frac{d^2 \theta}{dt^2} = -g \sin(\theta)$$

$$\frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin(\theta) = 0$$

30. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

(a) Show that the kinetic energy $T$ of the pendulum in motion is

$$T = \frac{1}{2} mL^2 \left(\frac{d\theta}{dt}\right)^2$$

The general equation for kinetic energy is $T = \frac{1}{2}mv^2$, where $v$ is velocity. At any point in time, the velocity of the mass at the end of the pendulum is $L \frac{d\theta}{dt}$. Therefore, the kinetic energy of the pendulum is

$$T = \frac{1}{2} mL^2 \left(\frac{d\theta}{dt}\right)^2$$

(b) Show that the potential energy $V$ of the pendulum, relative to its rest position, is

$$V = mgL(1 - \cos(\theta))$$

From the attached figure, we have that at angle $\theta$, the height difference from the bottom of the pendulum is $h = L(1 - \cos(\theta))$. Therefore, the potential energy is $V = mgh = mgL(1 - \cos(\theta))$.

(c) By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate $dE/dt$, set it equal to zero, and show that the resulting equation reduces to Eq. (12).

From parts (a) and (b), the total energy is $E = \frac{1}{2} mL^2 \left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos(\theta))$. So, taking the derivative gives

$$\frac{dE}{dt} = mL^2 \frac{d\theta}{dt} \frac{d^2 \theta}{dt^2} + mgL \sin(\theta) \frac{d\theta}{dt}$$

And therefore,

$$0 = mL^2 \frac{d\theta}{dt} \frac{d^2 \theta}{dt^2} + mgL \sin(\theta) \frac{d\theta}{dt}$$

$$= L \frac{d^2 \theta}{dt^2} + g \sin(\theta)$$
29.

30.