2.4 5. \((4 - t^2)y' + 2ty = 3t^2, y(1) = -3\)

**Ans:** This ODE also can be written in the form \(y' + p(t)y = g(t)\) as

\[
y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}
\]

So \(p(t) = \frac{2t}{4-t^2}\) and \(g(t) = \frac{3t^2}{4-t^2}\). By Theorem 2.4.1, the solution to the ODE exists and is unique wherever \(p\) and \(g\) are continuous. The functions \(p\) and \(g\) are discontinuous at \(t = \pm 2\), and since the initial condition is at \(t = 1\), the solution \(y(t)\) exists on the interval \(-2 < t < 2\).

13. \(y' = -4t/y, y(0) = y_0\)

**Ans:** Using separation of variables gives

\[
ydy = -4tdt
\]

Integrating both sides then gives \(\frac{1}{2}y^2 = -2t^2 + C\) or multiplying through by 2 and taking the square root yields \(y(t) = \pm \sqrt{-4t^2 + C}\) where \(C\) is now a new constant. For \(t = 0, y_0 = y(0) = \pm \sqrt{C}\), so \(C = y_0^2\) and if \(y_0 > 0\),

\[
y(t) = \sqrt{-4t^2 + y_0^2}
\]

otherwise

\[
y(t) = -\sqrt{-4t^2 + y_0^2}
\]

Now \(y(t)\) becomes undefined when the radicand becomes negative, and the radicand is equal to zero at \(t = \pm y_0/2\). Therefore the solution is exists for \(-y_0/2 \leq t \leq y_0/2\).

25. \(y = y_1(t)\) is a solution of the equation \(y' + p(t)y = 0\) and \(y = y_2(t)\) is a solution of the equation \(y' + p(t)y = g(t)\). Show that \(y = y_1 + y_2\) is a solution to \(y' + p(t)y = g(t)\).

**Ans:** To check that \(y_1 + y_2\) is a solution, we plug in \(y_1 + y_2\) for \(y\) in the given equation and check that we do in fact get \(g(t)\). Plugging in gives

\[
(y_1 + y_2)' + p(t)(y_1 + y_2) = y_1' + y_2' + p(t)y_1 + p(t)y_2 = (y_1' + p(t)y_1') + (y_2' + p(t)y_2) = 0 + g(t) = g(t)
\]

The second to last equality is given by the fact that \(y_1\) is a solution to \(y' + p(t)y = 0\), which means that \(y_1' + p(t)y_1 = 0\) and the fact that \(y_2\) is a solution to \(y' + p(t)y = g(t)\), which means \(y_2' + p(t)y_2 = g(t)\).

2.5 7. (a) Consider the equation

\[
dy/dt = k(1 - y)^2
\]

where \(k\) is a positive constant. Show that \(y = 1\) is the only critical point, with the corresponding equilibrium solution \(\phi(t) = 1\).

**And:** Setting the right hand side equal to zero and solving for \(y\) gives

\[
0 = k(1 - y)^2
\]

\[
0 = (1 - y)^2
\]

\[
0 = 1 - y
\]

\[
y = 1
\]

So, the only critical point is at \(y = 1\), and if \(\phi\) is a solution to the ODE with \(\phi(0) = 1\), then \(\phi(t) = 1\).
(b) Sketch $f(y)$ versus $y$. Show that $y$ is increasing as a function of $t$ for $y < 1$ and also for $y > 1$. The phase line has upward-pointing arrows both below and above $y = 1$. Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore, $\phi(t) = 1$ is semistable.

And: For $k = 1$, the phase line looks like the following figure.

(c) Solve the ODE subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part (b).

And: Separation of variables yields $dy/(1 - y)^2 = kdt$ and integrating both sides gives

\[
\int \frac{dy}{(1 - y)^2} = \int kdt
\]

\[
- \int \frac{du}{u^2} = kt + C \quad \text{using } u = 1 - y
\]

\[
\frac{1}{u} = kt + C
\]

\[
\frac{1}{kt + C} = 1 - y
\]

\[
y(t) = 1 - \frac{1}{kt + C}
\]

Assuming that $y_0 \neq 1$, then letting $t = 0$ gives $y_0 = 1 - 1/C$. So $1/C = 1 - y_0$, or $C = 1/(1 - y_0)$. Therefore,

\[
y(t) = 1 - \frac{1}{kt + 1/(1 - y_0)}
\]

\[
= 1 - \frac{1 - y_0}{(1 - y_0)(kt - 1) + 1}
\]

\[
= \frac{(1 - y_0)(kt - 1) + 1}{(1 - y_0)kt + 1}
\]

So, for $y_0 < 1$, the denominator is never equal to zero for $t \geq 0$, which means the solution exists for all $t \geq 0$, and as $t \to \infty$, $y \to 1$. For $y_0 > 1$, the denominator is equal to zero at $t = \frac{1}{k(y_0 - 1)}$. At $t = \frac{1}{k(y_0 - 1)}$, the numerator is not equal to zero, so the solution blows up, we just need to figure out in which direction. Taking the derivative gives

\[
y'(t) = \frac{(1 - y_0)k[(1 + 1 - y_0)kt + 1] - (1 - y_0)k[(1 - y_0)(kt - 1) + 1]}{[(1 - y_0)kt + 1]^2}
\]
\[
(1 - y_0)k[(1 - y_0)kt + 1 - (1 - y_0)(kt - 1) - 1] \\
= \frac{(1 - y_0)k(1 - y_0)}{(1 - y_0)kt + 1} \\
\]

So, \( y'(t) > 0 \) for \( 0 \leq t \leq \frac{1}{k(y_0 - 1)} \), which means that as predicted, the solution blows up to positive infinity, but it does so in finite time.

22. In the book there is a long explanation of how to model the spread of disease, the result of which is that we obtain the initial value problem \( y' = \alpha y(1 - y) \) with \( y(0) = y_0 \) where \( \alpha \) is a positive proportionality constant and \( y_0 \) is the number of individuals who are initially infected.

(a) Find the equilibrium points for the differential equation and determine whether each is asymptotically stable, unstable, or semistable.

**Ans:** The equilibrium points are where \( y' = 0 \) so plugging that in we get

\[
0 = \alpha y(1 - y) \implies y = 0 \text{ or } y = 1
\]

So our equilibrium points are \( y = 0, 1 \). To check their stability we draw the direction field.

This shows us that the equilibrium point \( y = 0 \) is unstable, while the point \( y = 1 \) is stable.

(b) Solve the initial value problem and determine whether the conclusions you reached in part (a) are correct. show that \( y \to 1 \) as \( t \to \infty \), which means that ultimately the disease spreads through the entire population.

**Ans:**

\[
\frac{dy}{dt} = \alpha y(1 - y) \\
\frac{1}{y(1 - y)} dy = \alpha dt \\
\int \left( \frac{1}{y(1 - y)} \right) dy = \int \alpha dt \\
\]

using partial fractions we get

\[
\int \frac{1}{y} dy + \int \frac{1}{1 - y} dy = \int \alpha dt \\
\ln y - \ln(1 - y) = \alpha t + C
\]

3
\[ \ln \left( \frac{y}{1-y} \right) = \alpha t + c \]

\[ \frac{y}{1-y} = Ce^{\alpha t} \]

\[ y = Ce^{\alpha t} - yCe^{\alpha t} \]

\[ y + yCe^{\alpha t} = Ce^{\alpha t} \]

\[ y(1 + Ce^{\alpha t}) = Ce^{\alpha t} \]

\[ y = \frac{Ce^{\alpha t}}{1 + Ce^{\alpha t}} \]

Plugging in \( t = 0 \) and \( y(0) = y_0 \) and solving gives \( C = \frac{y_0}{1-y_0} \) so our final answer is

\[ y(t) = \frac{\left( \frac{y_0}{1-y_0} \right) e^{\alpha t}}{1 + \left( \frac{y_0}{1-y_0} \right) e^{\alpha t}} \]

Now to calculate the limit.

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \frac{1}{\left( \frac{1-y_0}{y_0} \right) e^{-\alpha t} + 1} = \frac{1}{\left( \lim_{t \to \infty} e^{-\alpha t} \right) + 1} = \frac{1}{0+1} = 1 \]

2.7 20. It can be shown that under suitable conditions on \( f \), the numerical approximation generated by the Euler method for the initial value problem \( y' = f(t, y) \), \( y(t_0) = y_0 \) converges to the exact solution as the step size \( h \) decreases. This is illustrated by the following example. Consider the initial value problem

\[ y' = 1 - t + y, \quad y(t_0) = y_0 \]

(a) Show that the exact solution is \( y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t \).

**Ans:** The ODE can be rewritten as \( y' - y = 1 - t \). So multiplying through by the integrating factor \( \mu = e^{-t} \) gives

\[ (e^{-t}y)' = (1-t)e^{-t} \]

And integrating both sides, we get

\[ e^{-t}y = \int e^{-t} - te^{-t}dt = -e^{-t} + (1+t)e^{-t} + C = te^{-t} + C \]

Therefore, \( y(t) = t + Ce^t \). For \( t_0 \), \( y_0 = y(t_0) = t_0 + Ce^{t_0} \), so \( C = (y_0 - t_0)e^{-t_0} \). This means \( y(t) = t + (y_0 - t_0)e^{t-t_0} \).

(b) Using the Euler formula, show that

\[ y_k = (1+h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \ldots \]

**Ans:** Euler’s formula is \( y_k = y_{k-1} + hf(t_{k-1}, y_{k-1}) \) and for this problem, \( f(t, y) = 1 - t + y \). So, \( y_k = y_{k-1} + h(1 - t_{k-1} + y_{k-1}) = (1+h)y_{k-1} + h - ht_{k-1} \).

(c) Noting that \( y_1 = (1+h)(y_0 - t_0) + t_1 \), show by induction that

\[ y_n = (1+h)^n(y_0 - t_0) + t_n \]

for each positive integer \( n \).
Ans: Assume that \( y_{n-1} = (1 + h)^{n-1}(y_0 - t_0) + t_{n-1} \), then

\[
y_n = (1 + h) \left[ (1 + h)^{n-1}(y_0 - t_0) + t_{n-1} \right] + h - ht_{n-1} \\
= (1 + h)^n(y_0 - t_0) + (1 + h)t_{n-1} + h - ht_{n-1} \\
= (1 + h)^n(y_0 - t_0) + t_{n-1} + h \\
= (1 + h)^n(y_0 - t_0) + t_n
\]

And, since the formula holds for \( n = 1 \), \( y_n = (1 + h)^n(y_0 - t_0) + t_n \) for all \( n \geq 1 \).

(d) Consider a fixed point \( t > t_0 \) and for a given \( n \) choose \( h = (t - t_0)/n \). Then \( t_n = t \) for every \( n \). Note also that \( h \to 0 \) as \( n \to \infty \). By substituting for \( h \) in the equation from part (c) and letting \( n \to \infty \), show that \( y_n \to \phi(t) \) as \( n \to \infty \).

Ans: Plugging in \( h = (t - t_0)/n \) gives

\[
y_n = (1 + (t - t_0)/n)^n(y_0 - t_0) + t
\]

Then taking the limit as \( n \to \infty \) gives

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} (1 + (t - t_0)/n)^n(y_0 - t_0) + t \\
= e^{t-t_0}(y_0 - t_0) + t \\
= \phi(t)
\]