12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

Ans: This ODE can be rewritten

$$y'' + \frac{1}{x-2}y' + (\tan x)y = 0$$

which is now in the form $y'' + p(x)y' + q(x)y = g(x)$. The function $q(x) = \tan x$ is discontinuous at $x = (n + 1/2)\pi$ for all $n \in \mathbb{Z}$, and since the derivation of $p, q,$ and $g$ all involved dividing by $x-2$, they are all discontinuous at $x = 2$. Therefore, since the initial conditions are at $x = 3$, the interval we are interested in is $I = (2, \frac{3}{2}\pi]$, because $p, q$, any $g$ are all continuous on $I$ and $x = 3 \in I$. So, by Theorem 3.2.1, there exists a unique solution to the ODE over the interval $I$.

14. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $y y'' + (y')^2 = 0$ for $y > 0$. Then show that $y = c_1 + c_2 t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

Ans: First, plugging in $y_1(t)$ gives $1(0) + (0)^2 = 0$, so $y_1$ is a solution. Next, for $y_2(t)$ we get $t^{1/2}(-\frac{1}{4}t^{-3/2}) + (\frac{1}{2}t^{-1/2})^2 = -\frac{1}{4}t^{-1} + \frac{1}{2}t^{-1} = 0$, so $y_2(t)$ is also a solution. Now, plugging in $y = c_1 + c_2 t^{1/2}$ gives

$$(c_1 + c_2 t^{1/2})(-\frac{c_2}{4}t^{-3/2}) + (\frac{c_2}{2}t^{-1/2})^2 = -\frac{c_1c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} = -\frac{c_1c_2}{4}t^{-3/2} \neq 0$$

so $y = c_1 + c_2 t^{1/2}$ is not a solution to the ODE. This does not contradict Theorem 3.2.2, because the ODE in this problem is not a linear ODE.

21. Let $y_1$ and $y_2$ form a fundamental set of solutions to a differential equation. Given $y_3 = a_1 y_1 + a_2 y_2$ and $y_4 = b_1 y_1 + b_2 y_2$, show that $W(y_3, y_4) = (a_1 b_2 - a_2 b_1) W(y_1, y_2)$ and say whether $y_3$ and $y_4$ form a fundamental set of solutions.

Ans: First we want to compute the Wronskian of $y_3$ and $y_4$

$$W(y_3, y_4) = \begin{vmatrix} y_3 & y_4 \\ y'_3 & y'_4 \end{vmatrix} = \begin{vmatrix} (a_1 y_1' + a_2 y_2') & (b_1 y_1' + b_2 y_2') \\ (a_1 y_1 + a_2 y_2) & (b_1 y_1 + b_2 y_2) \end{vmatrix}$$

$$= (a_1 y_1 + a_2 y_2)(b_1 y_1' + b_2 y_2') - (a_1 y_1' + a_2 y_2')(b_1 y_1 + b_2 y_2)$$
$$= (a_1 b_2 - a_2 b_1)(y_1 y_2' - y_1' y_2)$$
$$= (a_1 b_2 - a_2 b_1)W(y_1, y_2)$$

This is a fundamental set of solutions if and only if $W(y_3, y_4) \neq 0$. Since $y_1, y_2$ is a fundamental set of solutions, we know $W(y_1, y_2) \neq 0$. Thus, $W(y_3, y_4) = 0$ if and only if $a_1 b_2 - a_2 b_1 = 0$. That is, if $(a_1, a_2)$ is a multiple of $(b_1, b_2)$.

31. $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel’s equation

Ans: Dividing the ODE by $x^2$ gives $y'' + \frac{y'}{x} + \frac{x^2 - \nu^2}{x^2} y = 0$, which is of the form $y'' + p(x)y' + q(x)y = g(x)$, and $p, q,$ and $g$ are discontinuous at $x = 0$. So solutions exist on the intervals $(-\infty, 0)$ and
(0, ∞). On these intervals, by Abel’s Theorem, the Wronskian is

\[
W(y_1, y_2)(x) = c \exp \left[ - \int p(x) dx \right] = c \exp \left[ - \int \frac{1}{x} dx \right] = ce^{-\ln x} = cx^{-1} \text{ since } x \neq 0
\]

So, the Wronskian is \(W(y_1, y_2)(x) = c/x\).

33. Show that if \(p\) is differentiable and \(p(t) > 0\), the Wronskian \(W(t)\) of two solutions of \([p(t)y']' + q(t)y = 0\) is \(W(t) = c/p(t)\), where \(c\) is a constant.

**Ans:** The ODE can be rewritten as \(p(t)y'' + p'(t)y' + q(t)y = 0\), and this can be rewritten as \(y'' + \frac{p'(t)}{p(t)} y' + q(t) = 0\). So assuming \(q(t)\) is continuous, by Abel’s Theorem,

\[
W(y_1, y_2)(t) = c \exp \left[ - \int \frac{p'(t)}{p(t)} dt \right] = ce^{-\ln(p(t))} = ce^{\ln(p(t)^{-1})} = cp(t)^{-1}
\]

Therefore, the Wronskian is \(W(t) = c/p(t)\), where \(c\) is a constant.