7.1 3. Transform \( t^2u'' + tu' + (t^2 - 0.25)u = 0 \) into a system of first order equations.

**Ans:** Let \( u_1 = u \) and \( u_2 = u' = u'_1 \). Then \( u'_2 = u'' \), so plugging into the original ODE gives \( t^2u'_2 + tu_2 + (t^2 - 0.25)u_1 = 0 \). Therefore, the system of first order equations is

\[
\begin{align*}
    u'_1 &= u_2 \\
    u'_2 &= -\frac{1}{t}u_2 - \frac{t^2 - 0.25}{t^2}u_1 
\end{align*}
\]

15. Consider the linear homogeneous system

\[
\begin{align*}
x' &= p_{11}(t)x + p_{12}(t)y, \\
y' &= p_{21}(t)x + p_{22}(t)y.
\end{align*}
\]

Show that if \( x = x_1(t), y = y_1(t) \) and \( x = x_2(t), y = y_2(t) \) are two solutions of the given system, then \( x = c_1x_1(t) + c_2x_2(t), y = c_1y_1(t) + c_2y_2(t) \) is also a solution for any constants \( c_1 \) and \( c_2 \). This is the principle of superposition.

**Ans:** Plugging in gives

\[
\begin{align*}
c_1x'_1(t) + c_2x'_2(t) &= p_{11}(t)[c_1x_1(t) + c_2x_2(t)] + p_{12}(t)[c_1y_1(t) + c_2y_2(t)], \\
c_1y'_1(t) + c_2y'_2(t) &= p_{21}(t)[c_1x_1(t) + c_2x_2(t)] + p_{22}(t)[c_1y_1(t) + c_2y_2(t)].
\end{align*}
\]

Looking at the first equation, we have

\[
c_1x'_1(t) + c_2x'_2(t) = p_{11}(t)[c_1x_1(t) + c_2x_2(t)] + p_{12}(t)[c_1y_1(t) + c_2y_2(t)]
\]

\[
= [p_{11}(t)c_1x_1(t) + p_{12}(t)c_1y_1(t)] + [p_{11}(t)c_2x_2(t) + p_{12}(t)c_2y_2(t)]
\]

\[
= c_1[p_{11}(t)x_1(t) + p_{12}(t)y_1(t)] + c_2[p_{11}(t)x_2(t) + p_{12}(t)y_2(t)]
\]

\[
= c_1x'_1(t) + c_2x'_2(t)
\]

And looking at the second equation gives

\[
c_1y'_1(t) + c_2y'_2(t) = p_{21}(t)[c_1x_1(t) + c_2x_2(t)] + p_{22}(t)[c_1y_1(t) + c_2y_2(t)]
\]

\[
= [p_{21}(t)c_1x_1(t) + p_{22}(t)c_1y_1(t)] + [p_{21}(t)c_2x_2(t) + p_{22}(t)c_2y_2(t)]
\]

\[
= c_1[p_{21}(t)x_1(t) + p_{22}(t)y_1(t)] + c_2[p_{21}(t)x_2(t) + p_{22}(t)y_2(t)]
\]

\[
= c_1y'_1(t) + c_2y'_2(t)
\]

So \( x = c_1x_1(t) + c_2x_2(t), y = c_1y_1(t) + c_2y_2(t) \) is also a solution for any constants \( c_1 \) and \( c_2 \).

7.3 8. Are the given vectors linearly independent? If not, find a linear relation among them.

\( x^{(1)} = (2, 1, 0), \quad x^{(2)} = (0, 1, 0), \quad x^{(3)} = (-1, 2, 0) \)

**Ans:** To see if these vectors are linearly independent, we take their transposes and from a matrix and find the solution to the equation

\[
\begin{pmatrix}
    2 & 0 & -1 \\
    1 & 1 & 2 \\
    0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}
\]
So, we need to do row operations on the matrix

$$
\begin{pmatrix}
2 & 0 & -1 & 0 \\
1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

One row operation gives

$$
\begin{pmatrix}
2 & 0 & -1 & 0 \\
0 & 1 & 5/2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

This means that \( c_2 = -5/2 c_3 \) and \( 2c_1 = c_3 \). Since there are an infinite number of possible solutions, the vectors are linearly dependent. Now, let \( c_3 = 2 \), so \( c_2 = -5 \) and \( c_1 = 1 \). This gives

\[
x^{(1)} - 5x^{(2)} + 2x^{(3)} = 0
\]

or

\[
x^{(1)} = 5x^{(2)} - 2x^{(3)}
\]

as a linear relation.

16. Find all eigenvalues and eigenvectors of the given matrix.

$$
\begin{pmatrix}
5 & -1 \\
3 & 1
\end{pmatrix}
$$

**Ans:** To find the eigenvalues, we want to look at what values of \( \lambda \) solve

\[
\begin{vmatrix}
5 - \lambda & -1 \\
3 & 1 - \lambda
\end{vmatrix} = 0
\]

This equation is the same as

\[
(5 - \lambda)(1 - \lambda) + 3 = 0
\]

\[
\lambda^2 - 6\lambda + 8 = 0
\]

\[
(\lambda - 2)(\lambda - 4) = 0
\]

Therefore \( \lambda = 2 \) or \( 4 \). To find an eigenvector corresponding to \( \lambda = 2 \), we look at the equation

\[
\begin{pmatrix}
3 & -1 \\
3 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

which gives the equation \( 3x_1 - x_2 = 0 \). So, if \( x_1 = 1, x_2 = 3 \) and an eigenvector is

\[
x^{(1)} = \begin{pmatrix}
1 \\
3
\end{pmatrix}
\]

To find an eigenvector corresponding to \( \lambda = 4 \), we look at the equation

\[
\begin{pmatrix}
1 & -1 \\
3 & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

which gives the equation \( x_1 - x_2 = 0 \). So, if \( x_1 = 1, x_2 = 1 \) and an eigenvector is

\[
x^{(2)} = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]
7.5 1a. Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

$$x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

**Ans:** First we find the eigenvalues of the given matrix.

$$(A - \lambda I) = \begin{pmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3 - \lambda)(-2 - \lambda) + 4 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

So $\lambda = -1, 2$ Now we find the eigenvectors that correspond to each of these eigenvalues. For $\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$$

$(A - \lambda I)v = 0$ gives $2v_1 - v_2 = 0$ if $v = (v_1, v_2)$. This relation gives $v_2 = 2v_1$ Choosing the value $v_1 = 1$ gives the eigenvector $v = (1, 2)$

Similarly, for $\lambda = 2$

$$(A - 2I) = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$

The relation $(A - 2I)v = 0$ gives the eigenvector $v = (2, 1)$ Thus our general solution is

$$y(t) = c_1 (1, 2)e^{-t} + c_2 (2, 1)e^{2t}$$

If $c_2 = 0$, Then $y \to 0$ If $c_2 \neq 0$ then $y \to \infty$ as $t \to \infty$

15 Solve the given initial value problem, then describe the behavior as $t \to \infty$

$$x' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

**Ans:** First we solve for the eigenvalues of the given matrix

$$(A - \lambda I) = \begin{pmatrix} 5 - \lambda & -1 \\ 3 & 1 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (5 - \lambda)(1 - \lambda) + 3 = (\lambda - 4)(\lambda - 2)$$

$\lambda = 2, 4$

$$(A - 2I) = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$$

$(A - 2I)v = 0$ gives the relation $3v_1 - v_2 = 0$ which gives $v_2 = 3v_1$ Choosing the value $v_1 = 1$ gives the vector $v = (1, 3)$

$$(A - 4I) = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$$

$(A - 4I)v = 0$ gives the relation $v_1 - v_2 = 0$ which gives $v_1 = v_2$. Choosing $v_1 = 1$ gives the vector $v = (1, 1)$. Thus our general solution is

$$y(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$
Plugging in our initial condition we get

\[
\begin{pmatrix} 2 \\ -1 \end{pmatrix} = y(0) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2x_0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4x_0} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Looking at each coordinate gives the two relations

\[2 = c_1 + c_2 \quad \text{and} \quad -1 = 3c_1 + c_2\]

When solved, this means \(c_1 = -\frac{3}{2}\) and \(c_2 = \frac{7}{2}\). Now our solution is given by

\[y(t) = -\frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}\]

Since both eigenvalues are positive as \(t \to \infty\), \(y \to \infty\).