SOLUTIONS

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td>****</td>
</tr>
</tbody>
</table>
1. **Problem:** Compute the following limits. Give each answer as a finite number, $+\infty$ or $-\infty$.

a) 
\[
\lim_{x \to 3} \frac{9 - x^2}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(3 + x)}{x - 3} = \lim_{x \to 3} (3 + x) = -6
\]

b) 
\[
\lim_{x \to 0} \frac{\sin(6x)}{\sin(2x) \cos(4x)} = \lim_{x \to 0} \frac{\sin(6x)}{\sin(2x) \cos(4x)} = \lim_{x \to 0} \frac{2x}{\sin(2x)} \cdot \frac{1}{2x} \cdot 6x = 3.
\]

c) 
\[
\lim_{x \to \infty} \frac{3x^2 - 2x + 10}{-4x^3 + 20} = 0
\]

since the degree of the polynomial in the numerator is smaller than the degree of the polynomial in denominator.
2. **Problem:** (a) Can $a$ be chosen so that

$$f(x) = \begin{cases} \frac{x^3 - 2x^2}{x - 2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases}$$

is a continuous function at every $x$?

Yes, choose $a = 4$.

(b) Show that the function $f(x) = x + |x + 1|$ is continuous at every $x$.

Clearly the domain of $f$ is the entire real line and the limit exists for every $x$. Hence we need to show that for every $x$ there holds

$$\lim_{x \to c} x + |x + 1| = c + |c + 1|.$$ 

Case 1: Assume that $|x + 1| \geq 0$. Then $x + |x + 1| = 2x + 1$. This is a polynomial (or a line), thus continuous for every $x$.

Case 2: Assume that $|x + 1| \leq 0$. Then $-(x + 1) \geq 0$ and $x + |x + 1| = x - (x + 1) = -1$. This is a constant function, hence continuous at every $x$. 

20 points
3. **Problem:** Given is the function \( f(x) \) defined by

\[
f(x) = \begin{cases} 
  x, & \text{if } x \geq 0, \\
  1 - x, & \text{if } x < 0.
\end{cases}
\]

Graph the function and determine the following limits. Give each answer as a finite number, \(+\infty\), \(-\infty\) or as “limit does not exist”.

a) \( \lim_{x \to 0^-} f(x) \).

b) \( \lim_{x \to 0^+} f(x) \).

c) \( \lim_{x \to 0^-} f(x) \).

(a) limit does not exist, since \( \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) \).

b) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \)

c) \( \lim_{x \to 0^-} f(x) \lim_{x \to 0^-} 1 - x = 1 \)
4. Problem: 

Let \( f(x) = \frac{1}{x} \), \( L = \frac{1}{4} \), and \( x_0 = 4 \). For given \( \varepsilon = 0.05 \) find a \( \delta > 0 \) such that 

\[
|x - x_0| < \delta \implies |f(x) - L| < \varepsilon.
\]

\[
\left| \frac{1}{x} \right| < 0.05 \iff -0.05 < \frac{1}{x} - \frac{1}{4} < 0.05
\]

\[
\iff 0.2 < \frac{1}{x} < 0.3 \iff \frac{10}{2} > x > \frac{10}{3}
\]

or

\[
\frac{10}{3} < x < 5
\]

Now consider

\[
|x - 4| < \delta \iff 4 - \delta < x < \delta + 4
\]

Hence \( 4 - \delta = \frac{10}{3} \) which means \( \delta = \frac{2}{3} \)

or \( \delta + 4 = 5 \) which means \( \delta = 1 \).

Since \( \frac{2}{3} < 1 \) we need to choose \( \delta = \frac{2}{3} \).
5. **Problem:** Let \( f(x) = \frac{2x}{x+1} \). Compute vertical and horizontal asymptotes, and sketch the graph of this function. On your graph clearly indicate all intercepts (those points, where the graph intersects the \( x \)-axis and the \( y \)-axis) and asymptotes.

\[
\lim_{x \to \infty} \frac{2x}{x+1} = 2 \text{ by comparing leading coefficients of the polynomials in numerator and denominator, hence the vertical asymptote given by } y = 2.
\]

\( x + 1 = 0 \) implies \( x = -1 \), and \( \lim_{x \to -1^+} \frac{2x}{x+1} = -\infty, \lim_{x \to -1^-} \frac{2x}{x+1} = \infty \), hence we have a horizontal asymptote at \( x = -1 \).

Intercept at \((0,0)\).