21D 1. Midterm Exam May 2, 2018

Name: SOLUTIONS **Section:**

Instructions: Read each question carefully and answer it in the space provided. **YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT!!!** Calculators, cell phones, iPads, books, or notes are not allowed.

Make sure that your exam contains 4 problems. Read through the entire exam before beginning work.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

1. Problem:

(a) In the integral below, change the Cartesian integral into an equivalent polar integral, and evaluate the polar integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dydx.$$

(b) Find the average height of the (single) cone $z=\sqrt{x^2+y^2}$ above the disk $x^2+y^2\le 9$ in the xy-plane.

a) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{2\pi}\int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta =$$

$$= \int_0^{2\pi}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \underline{\underline{\pi}}$$

b) $$\text{average} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^{a} r^2\,dr\,d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^{a} d\theta =$$

$$= \frac{1}{\pi a^2}\int_0^{2\pi}\frac{a^3}{3}\,d\theta = \frac{2\pi a^3}{3\pi a^2} = \underline{\underline{\frac{2a}{3}}}$$

2. Problem:
Find the center of mass and the moments of inertia about the y-axis of a thin plate bounded by the x-axis, the lines $x = 1$, $x = -1$ and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.

$$M = \int_{-1}^{1}\int_{0}^{x^2} (7y+1)\,dy\,dx = \int_{-1}^{1} \left[\frac{7y^2}{2} + y\right]_0^{x^2} dx =$$

$$= \int_{-1}^{1}\left(\frac{7x^4}{2} + x^2\right)dx = \left[\frac{7x^5}{10} + \frac{x^3}{3}\right]_{-1}^{1} = 2\left(\frac{7}{10} + \frac{1}{3}\right) = \underline{\underline{\frac{31}{15}}}$$

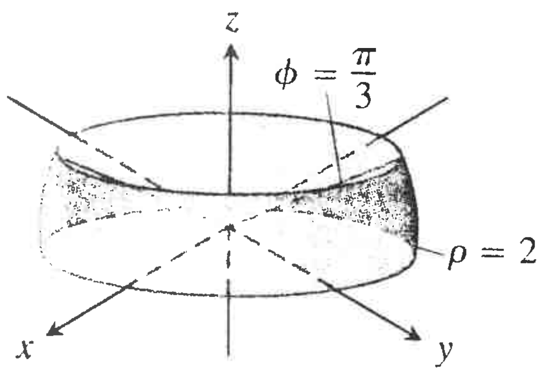
$$M_x = \int_{-1}^{1}\int_{0}^{x^2} y(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^6}{3} + \frac{x^4}{2}\right)dx = \left[\frac{7x^7}{21} + \frac{x^5}{10}\right]_{-1}^{1} = \underline{\underline{\frac{13}{15}}}$$

$$M_y = \int_{-1}^{1}\int_{0}^{x^2} x(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^5}{2} + x^3\right)dx = \left[\frac{7x^6}{12} + \frac{x^4}{2}\right]_{-1}^{1} = \underline{\underline{0}}$$

$$\Rightarrow \bar{x} = \frac{M_y}{M} = \underline{\underline{0}}, \quad \bar{y} = \frac{M_x}{M} = \frac{\frac{13}{15}}{\frac{31}{15}} = \underline{\underline{\frac{13}{31}}} \Rightarrow \underline{\underline{(\bar{x}, \bar{y}) = \left(0, \frac{13}{31}\right)}}$$

$$I_y = \int_{-1}^{1}\int_{0}^{x^2} x^2(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^6}{2} + x^4\right)dx = \left[\frac{7x^7}{14} + \frac{x^5}{5}\right]_{-1}^{1} = \underline{\underline{\frac{7}{5}}}$$

3. Problem:
Given is the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$ (see the picture below).



$\Rightarrow 0 \le \rho \le 2$

$\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$

from drawing   from xy-plane

(a) Find the spherical coordinate limits for the integral that calculates the volume of the given solid. (b) Evaluate the integral.

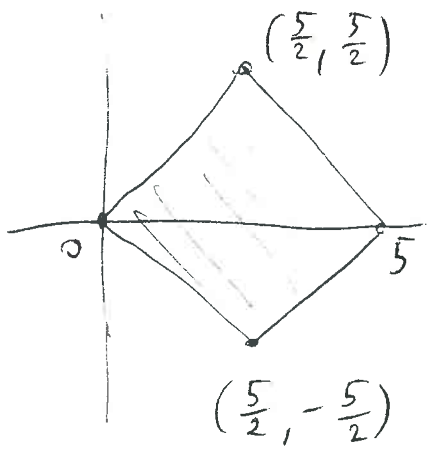
a) $V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

b) $V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \frac{8}{3} \sin\phi \, d\phi \, d\theta =$

$= \frac{8}{3} \int_0^{2\pi} [-\cos\phi]_{\pi/3}^{\pi/2} \, d\theta = \frac{8}{3} \int_0^{2\pi} (0 + \frac{1}{2}) \, d\theta = \frac{4}{3} [\theta]_0^{2\pi} = \underline{\frac{8\pi}{3}}$

[ok to leave $\cos\frac{\pi}{3}$ as $\cos\frac{\pi}{3}$ and get as answer $\underline{\frac{16\pi}{3} \cdot \cos\frac{\pi}{3}}$]

4. Problem:
Evaluate $\int_R (x+y)dxdy$, where R is the trapezoidal region bounded by the lines $y=x$, $y=-x$, $y=-x+5$, and $y=x+5$, using the transformation $x=2u+3v$ and $y=2u-3v$.



$y=x \Rightarrow 2u-3v=2u+3v \Rightarrow v=0$

$y=-x \Rightarrow 2u-3v=-2u-3v \Rightarrow u=0$

$y=-x+5 \Rightarrow 2u-3v=-2u-3v+5 \Rightarrow u=\frac{5}{4}$

$y=x+5 \Rightarrow 2u-3v=2u+3v+5 \Rightarrow v=-\frac{5}{6}$

Jacobian: $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$

$$\Rightarrow \iint_R (x+y)dxdy = \int_{-\frac{5}{6}}^{0}\int_{0}^{5/4} ((2u+3v)+(2u-3v))\cdot|-12|\,du\,dv =$$

$$= \int_{-\frac{5}{6}}^{0}\int_{0}^{\frac{5}{4}} 48u\,du\,dv = \int_{-\frac{5}{6}}^{0} \left[24u^2\right]_0^{5/4} dv = \int_{-\frac{5}{6}}^{0} 24\,\frac{25}{16}\,dv = \int_{-5/6}^{0} \frac{75}{2}\,dv =$$

$$= \frac{\overset{25}{\cancel{75}}}{2}\cdot\frac{5}{\underset{2}{\cancel{6}}} = \underline{\underline{\frac{125}{4}}}$$