21D 2. Midterm Exam May 25, 2018

Name: SOLUTIONS **Section:**

Instructions: Read each question carefully and answer it in the space provided. **YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT!!!** Calculators, cell phones, iPads, books, or notes are not allowed.

Make sure that your exam contains 4 problems. Read through the entire exam before beginning work.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

1. Problem:

Find the unit tangent vector **T**, the principal unit normal vector **N**, ~~the binormal vector **B**~~, and the curvature κ for the space curve given by

$$\boldsymbol{r}(t) = (6\sin 2t)\boldsymbol{i} + (6\cos \overset{2t}{\cancel{t}})\boldsymbol{j} + 5t\boldsymbol{k}.$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (12\cos 2t)\vec{i} - (12\sin 2t)\vec{j} + 5\vec{k}$$

$$\Rightarrow |\vec{v}| = \sqrt{12^2\underbrace{(\cos^2 2t + \sin^2 2t)}_{=1} + 5^2} = \sqrt{144+25} = \sqrt{169} = \underline{13}$$

$$\Rightarrow \vec{T} = \frac{\vec{v}}{|\vec{v}|} = \underline{\frac{12}{13}(\cos 2t)\vec{i} - \frac{12}{13}(\sin 2t)\vec{j} + \frac{5}{13}\vec{k}}$$

$$\frac{d\vec{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\vec{i} - \left(\frac{24}{13}\cos 2t\right)\vec{j}$$

$$\Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13}\right)^2(\sin^2 2t + \cos^2 2t)} = \underline{\frac{24}{13}}$$

$$\Rightarrow \vec{N} = \frac{d\vec{T}}{dt} \Big/ \left|\frac{d\vec{T}}{dt}\right| = \underline{(-\sin 2t)\vec{i} - (\cos 2t)\vec{j}}$$

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{13}\cdot\frac{24}{13} = \underline{\underline{\frac{24}{169}}}$$

2. Problem:

Find the work done by $\mathbf{F}$ over the curve C (parametrized by $\boldsymbol{r}(t)$) in the direction of increasing t for

$$\mathbf{F} = 4z\boldsymbol{i} - y^2\boldsymbol{j} + 8x\boldsymbol{k}, \qquad \boldsymbol{r}(t) = \underbrace{(\sin t)}_{=x}\boldsymbol{i} + \underbrace{(\cos t)}_{=y}\boldsymbol{j} + \underbrace{\frac{t}{2}}_{=z}\boldsymbol{k},\ 0 \le t \le 2\pi.$$

$$\vec{F} = 4z\vec{i} - y^2\vec{j} + 8x\vec{k} = 4 \cdot \frac{t}{\cancel{2}}\vec{i} - (\cos^2 t)\vec{j} + 8(\sin t)\vec{k}$$

$$= 2t\vec{i} - (\cos^2 t)\vec{j} + (8\sin t)\vec{k}$$

$$\frac{d\vec{r}}{dt} = \cos t\,\vec{i} - \sin t\,\vec{j} + \frac{1}{2}\vec{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = 2t \cdot \cos t + \sin t \cdot \cos^2 t + 4\sin t$$

$$\Rightarrow \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt = \int_0^{2\pi} 2t\cos t + (\sin t)\cos^2 t + 4\sin t \; dt =$$

$$\int t\cos t\,dt = \cos t + t\sin t$$

$$= \left[2\cos t + 2t\sin t + \frac{1}{3}\cos^3 t - 4\cos t\right]_0^{2\pi} =$$

$$= \left(2 + 2\cdot 2\pi \cdot 0 + \frac{1}{3}\cdot 1 - 4\right) - \left(2 + 0 + \frac{1}{3}\cdot 1 - 4\right) = \underline{\underline{0}}$$

3. Problem:

Evaluate the line integral $\int_C x + yz + z\, ds$ along the curve C parametrized by $\boldsymbol{r}(t) = \underbrace{(2-2t)}_{x}\boldsymbol{i} + \underbrace{2t}_{y}\boldsymbol{j} + \underbrace{t}_{z}\boldsymbol{k}$, $0 \leq t \leq 1$.

$$\frac{d\vec{r}}{dt} = -2\vec{i} + 2\vec{j} + \vec{k} \Rightarrow \left|\frac{d\vec{r}}{dt}\right| = \sqrt{4+4+1} = \underline{3}$$

$$x + yz + z = (2-2t) + 2t \cdot t + t = 2t^2 - t + 2$$

$$\Rightarrow \int_C f(x,y,z)\, ds = \int_0^1 (2t^2 - t + 2) \cdot 3\, dt = 3\left[\frac{2}{3}t^3 - \frac{t^2}{2} + 2t\right]_0^1 =$$

$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = 2 - \frac{3}{2} + 6 = \underline{\underline{\frac{13}{2}}}$$

4. Problem:

a) Verify whether the vector field **F** is conservative, for

$$\mathbf{F} = (2z + 4y)\boldsymbol{i} + 2z\boldsymbol{j} + (2y + x)\boldsymbol{k}.$$

b) Find a potential function for $\mathbf{F} = (2xy)\boldsymbol{i} + (x^2 + 2yz)\boldsymbol{j} + (y^2 + 2z)\boldsymbol{k}$.

a) Component test : $M = 2x + 4y$, $N = 2z$, $P = 2y + z$

$$\frac{\partial P}{\partial y} = 2 \;,\; \frac{\partial N}{\partial z} = 2 \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \;\checkmark$$

$$\frac{\partial M}{\partial z} = 0 \;,\; \frac{\partial P}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial z} \neq \frac{\partial P}{\partial x} \Rightarrow \underline{\text{not conservative}}$$

b) $$\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y,z) = x^2 y + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz \Rightarrow \frac{\partial g}{\partial y} = 2yz$$

$$\Rightarrow g(y,z) = y^2 z + h(z)$$

$$\Rightarrow f(x,y,z) = x^2 y + y^2 z + h(z)$$

$$\frac{\partial f}{\partial z} = y^2 + \frac{\partial h}{\partial z} = y^2 + 2z \Rightarrow \frac{\partial h}{\partial z} = 2z = h(z) = z^2 + C$$

$$\Rightarrow f(x,y,z) = \underline{\underline{x^2 y + y^2 z + z^2 + C}}$$