Homework 1: due Wednesday, April 21 2010

Problem 1: Let $i = \sqrt{-1}$ and set

$$A = \begin{bmatrix} i & 0 & -i \\ 0 & i & -i \end{bmatrix}.$$

Using the null space property, show that ℓ_1-minimization can recover any 1-sparse vector x, given $Ax = y$.

Problem 2: Prove that a unique minimizer of $\|z\|_1$ subject to $Az = y$ is not necessarily m-sparse, where m is the number of rows of A. (Hint: Consider the matrix A and the vector x below.)

$$A = \begin{bmatrix} i & 0 & -i \\ 0 & i & -i \end{bmatrix}, \qquad x = [1,\, e^{2\pi i/3},\, e^{4\pi i/3}]^T.$$

Problem 3: Consider

$$\min \|x\|_1 \qquad \text{subject to} \;\; Ax = b,\; x > 0.$$

Show that if this optimization problem has at least two solutions, it has already infinitely many solutions. (In fact, this still holds true if we drop the condition $x > 0$.)

Problem 4: Find a 2×3 matrix A and a nonsingular 3×3 diagonal matrix D such that A has the first order null space property, but AD does not.

Problem 4: Prove that any unit-norm equiangular frame $\{a_k\}_{k=1}^m$ for $\mathbb{C}^n$, $n \le m$, whose coherence is

$$\mu = \sqrt{\frac{m-n}{n(m-1)}},$$

must be a tight frame.

Problem 5: Given two orthonormal bases $U = \{u_1, \dots, u_n\}$, and $V = \{v_1, \dots, v_n\}$ of $\mathbb{C}^n$, prove that their mutual coherence $\mu(U,V) = \max_{1 \le i,j \le n} |\langle u_i, v_j \rangle|$ satisfies $\frac{1}{n} \le \mu \le 1$.

Problem 6: Prove that the $m+1$ vertices of a regular simplex in $\mathbb{R}^m$ centered at the origin form an equiangular tight frame for $\mathbb{R}^m$.

Problem 7: Let $\{a_k\}_{k=1}^m$ be a frame for $\mathbb{C}^n$ and let $\alpha, \beta > 0$ be the optimal frame bounds satisfying

$$\alpha\|x\|_2^2 \leq \sum_{k=1}^{m} |\langle x, a_k\rangle|^2 \leq \beta\|x\|_2^2, \tag{1}$$

for all $x \in \mathbb{C}^n$. As usual, we identify the frame with the $n \times m$ matrix $A = [a_1|a_2|\ldots|a_m]$. Setting $\delta = \frac{\beta-\alpha}{\beta+\alpha}$ and $\lambda = \frac{\beta+\alpha}{2}$, show that the inequality (1) can be expressed equivalently as

$$\Big\|\frac{1}{\lambda}AA^* - I_n\Big\|_{\mathrm{op}} \leq \delta.$$

(Here $\|B\|_{\mathrm{op}}$ denotes the operator norm of B, i.e., its largest singular value.)