vector, covariance matrix and relation matrix, see [5]. Any property defined in terms of the distribution of the complex Gaussian vector, could hence be parameterised in terms of these three quantities that would directly and fully determine the *joint density* of the *complex* vector components. We could hence consider properties of vector realizations of a stationary stochastic process, and for a Gaussian process consider defining strict and wide sense stationarity (cf. [17, p. 77]) in terms of the moments of the complex vector, and then relate the moments to assigning a density for the complex quantity that only depended on the absolute distance in time between observations. For non-Gaussian joint distributions, we can then either consider discussing the properties of the vectors in terms of their joint density, or consider moment based definitions, that then have an identical correspondence to strict stationarity for the complex Gaussian case.

V. Conclusion

We have considered the definition and interpretation of a density function for complex variables. We show that two functions could be considered as possible density functions, but that only one of these two could formally be interpreted as a density. This density function can be used to characterize the properties of the complex variables directly in a complex formulation. The second function illuminates the scalar nature of the complex variate, and the role played by the complex conjugate in formulating a distribution for a complex variable, settling confusion prevalent in the literature.

Acknowledgment

The author would like to thank the anonymous referees for their many helpful comments and suggestions that improved the correspondence substantially. She would like to thank Dr. J.-L. Thiffeault of Imperial College London for his suggested stylistic improvements to the manuscript, and many illuminating discussions. The author would like to express her gratitude to Prof. A. Fokas of the University of Cambridge, U.K., for teaching her complex variable methods, and she would also like to thank Prof. A. T. Walden of Imperial College London for introducing her to the area of complex random variables.

References

[1] A. van den Bos, "The multivariate complex normal distribution—A generalization," *IEEE Trans. Inf. Theory*, vol. 41, no. 2, pp. 537–539, Mar. 1995.

[2] A. van den Bos, "The real-complex normal distribution," *IEEE Trans. Inf. Theory*, vol. 44, no. 4, pp. 1670–1672, Jul. 1998.

[3] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inf. Theory*, vol. 39, no. 4, pp. 1293–1302, Jul. 1993.

[4] S. Olhede and A. T. Walden, "Wavelet denoising for signals in quadrature," *Integr. Comput. Aided Eng.*, pp. 1–9, 2005.

[5] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Trans. Signal Process.*, vol. 44, no. 10, pp. 2637–2640, Oct. 1996.

[6] ——, "On circularity," *IEEE Trans. Signal Process.*, vol. 42, no. 12, pp. 3473–3482, Dec. 1994.

[7] R. Arens, "Complex processes for envelopes of normal noise," *IRE Trans. Inf. Theory*, vol. IT-3, pp. 104–107, Sep. 1957.

[8] P. O. Amblard, P. O. Gaeta, and P. O. Lacoume, "Statistics for complex variables and signals—Part I: Variables," *Signal Process.*, vol. 53, pp. 1–13, 1996.

[9] P. J. Schreier and L. L. Scharf, "Second order analysis of improper complex random vectors and processes," *IEEE Trans. Signal Process.*, vol. 51, pp. 714–725, 2003.

[10] P. J. Schreier and L. L. Scharf, "Stochastic time-frequency analysis using the analytic signal: Why the complementary distribution matters," *IEEE Trans. Signal Process.*, vol. 51, no. 3, pp. 3071–3079, Mar. 2003.

[11] K. S. Miller, "Complex Gaussian processes," in *SIAM Rev.*, vol. 11, 1969, pp. 544–567.

[12] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*. Oxford, U.K.: North-Holland, 1990.

[13] A. van den Bos, "A cramér-rao lower bound for complex parameters," *IEEE Trans. Signal Process.*, vol. 42, no. 10, p. 2859, Oct. 1994.

[14] A. van den Bos, "Price's theorem for complex variates," *IEEE Trans. Inf. Theory*, vol. 42, no. 1, pp. 286–287, Jan. 1996.

[15] M. J. Ablowitz and A. S. Fokas, *Introduction and Applications of Complex Variables*, 2nd ed. Cambridge, MA: Cambridge Univ. Press, 2003.

[16] T. Needham, *Visual Complex Analysis*. Oxford, U.K.: Clarendon, 1997.

[17] J. L. Doob, *Stochastic Processes*. New York: Wiley, 1990.

On Quasi-Orthogonal Signatures for CDMA Systems

Robert W. Heath, Jr., *Member, IEEE*, Thomas Strohmer, and Arogyaswami J. Paulraj, *Fellow, IEEE*

***Abstract*—Sum capacity optimal signatures in synchronous code-division multiple-access (CDMA) systems are functions of the codebook length as well as the number of active users. A new signature set must be assigned every time the number of active users changes. This correspondence considers signature sets that are less sensitive to changes in the number of active users. Equiangular signature sequences are proven to solve a certain max-min signal-to-intererference-plus-noise problem, which results from their interference invariance. Unions of orthonormal bases have subsets that come close to satisfying the Welch bound. Bounds on the maximum number of bases with minimum maximum correlation are derived and a new construction algorithm is provided. Connections are made between these signature design problems, Grassmannian line packing, frame theory, and algebraic geometry.**

***Index Terms*—Code-division multiple-access (CDMA) systems, Walsh sequences.**

I. Introduction

Direct spread code-division multiple-access (CDMA) systems assign signature sequences to distinguish between the signals of different users. Information theoretic results have shown that for the uplink of a single cell synchronous CDMA systems, optimal signature sequences are nonorthogonal meaning that they can support more users than chip periods. Sum capacity optimal signature sequences have been characterized for Gaussian channels (so-called Welch bound equality sequences) [1]–[3], fading channels with white noise [4], fading channels with colored noise [5], [6], and with different receivers [7], [8]. Constructions have been proposed using iterative algorithms [9], [6],

Manuscript received September 15, 2002; revised July 8, 2005. The work of T. Strohmer was supported by the National Science Foundation under DMS Grant 0208568. The material in this correspondence was presented in part at the 36th Asilomar Conference on Circuits, Signals, and Systems, Pacific Grove, CA, November 2002 and at the 2003 Global Telecommunications Conference.

R. W. Heath, Jr. is with the Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, TX 78712 USA (e-mail: rheath@ece.utexas.edu).

T. Strohmer is with the Department of Mathematics, University of California, Davis, CA 95616 USA (e-mail: strohmer@math.ucdavis.edu).

A. J. Paulraj is with the Information Systems Laboratory, Stanford University, Stanford, CA 94305 USA (e-mail: apaulraj@stanford.edu).

Communicated by D. N. C. Tse, Associate Editor for Communications.

Digital Object Identifier 10.1109/TIT.2005.864469

0018-9448/$20.00 © 2006 IEEE

[10], [11], finite-step algorithms [4], [5], [12], [13], and algebraic approaches [14] (see also the references in [3]). Unfortunately, optimal signature sequences are fundamentally a function of the number of active signatures K and the number of chips N. Practical application of such signatures requires that new signatures be assigned to all users whenever any user arrives or departs the cell. Alternatively, the subset of remaining sequences will no longer be optimal. It is of interest to find signature sequences that perform well, without reassigning all the signatures, when users enter or leave the system [3], [15]–[18].

In this correspondence, we consider the problem of designing Welch bound equality sequence (WBE) sets that perform robustly with varying numbers of users. We suggest two designs that are up to the task: equiangular signature sets and unions of orthonormal basis (which we call mutually unbiased bases [19]). We prove that equiangular signature solve a certain worst-case SINR when considering all possible subsets of active users. The resulting signatures (when they exist) inherit the interference invariance of WBEs but also exhibit this property for subsets of active users. This means that the interference experienced by each user is only a function of K, N, and the number of active users but not the specific signature sequences. This means that the interference experienced by each user is a function of the number of active users , the spreading actor, and maximum number of users but not on the specific active signature sequences. We relate the problem of equiangular signature design to finding maximum Welch bound equality sequences (MWBEs), packing lines in the Grassmann manifold, and Grassmannian frames. We discuss constructions for exact and nearly exact equiangular signature sequences. For the case of mutually unbiased bases, we compare how they perform relative to WBE signature sequences in terms of correlation, study how many sequences we can add to the Walsh sequences with minimal interference, and present an explicit construction algorithm. Simulations compare both designs in terms of total squared correlation, proximity to the Welch bound, and variability of the interference.

Sarwate summarizes the findings of a number of authors on the related topic of MWBEs (a special type of equiangular signature set) and near MWBE sets in an excellent overview article [3]. Therein he describes some algebraic MWBE and near-MWBE constructions as well as some properties of MWBE signature sets. In this correspondence, we consider nonalgebraic signatures and signature sequences that are equiangular but not maximally spaced. We prove a new result about a certain max min optimality of equiangular signatures. We discuss relations to Grassmannian line packing and Grassmannian frames that do not appear to have been recognized before. We discuss the interference invariance property of MWBEs in the context of CDMA systems with varying numbers of active users and present numerical simulations to illustrate these benefits.

Several sequence designs have been proposed that are unions of multiple orthonormal basis with an emphasis on unions of two orthonormal basis [20]–[24]. Variations have been incorporated into WCDMA [21] and IS-2000 [22]. Extensions to multiple basis are described in [25]. Compared with prior work, we consider unions of more than two basis. We consider the specific problem of adding sequences to the Walsh set and present a new construction algorithm. Our work is also related to the theory of Kerdock codes. Elsewhere Kerdock codes have appeared with application to CDMA but in different contexts, e.g., binary signatures in [14] and biorthogonal coding in [26].

This correspondence is organized as follows. In Section II, we motivate the problem of finding good Welch bound equality sequences for changing numbers of users. In Section III, we examine equiangular signature sequences while in Section IV we address mutually unbiased bases. We present some numerical results in Section V and summarize our findings in Section VI.

II. Welch Bound Equality Sequences

Consider the uplink of a single cell, short code, synchronous CDMA system with K unit norm signatures, and a processing gain $N \leq K$. Let $\boldsymbol{s}_n$ denote a (potentially complex) $N \times 1$ signature and let $\boldsymbol{S} := [\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K]$ be the signature matrix. Let P denote the received power (assumed the same for all users) and σ^2 the noise power. With a matched-filter receiver, the signal-to-noise-plus-interference ratio (SINR) for the kth user is

$$\mathrm{SINR}_k = \frac{1}{\frac{\sigma^2}{P} + \sum_{n \neq k} |\langle \boldsymbol{s}_n, \boldsymbol{s}_k \rangle|^2}. \tag{1}$$

The second term corresponds to the sum cross correlation between signatures and is nonzero when $K > N$.

In this correspondence, we are interested in the class of quasi-orthogonal signatures that satisfy Welch's bound on the total squared correlation

$$\mathrm{TSC}(\boldsymbol{S}) = \sum_{k=1}^{K} \sum_{m=1}^{K} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 \geq \frac{K^2}{N} \tag{2}$$

with equality. A necessary and sufficient condition for a set of sequences to achieve equality is [2]

$$\boldsymbol{S}\boldsymbol{S}^* = \frac{K}{N}\boldsymbol{I}_N \text{ and } \|\boldsymbol{s}_k\| = 1, \qquad \text{for } k = 1, 2, \ldots, K \tag{3}$$

where * stands for conjugate transpose. Perhaps the most interesting property of such sequences is that the interference term in (1)

$$\begin{aligned} I(k) &= \sum_{n \neq k} |\langle \boldsymbol{s}_n, \boldsymbol{s}_k \rangle|^2 \\ &= \sum_{l=1}^{K} |\langle \boldsymbol{s}_k, \boldsymbol{s}_l \rangle|^2 - 1 \\ &= \frac{K - N}{N}, \qquad k = 1, 2, \ldots, K \end{aligned} \tag{4}$$

is *the same for every user*. Thus the SINR is also constant and only a function of the SNR, N, and K. This means that the cross correlation between signatures does not need to be considered in system functions such as power control.

While WBE signatures are capacity optimal and have a nice interference invariance property, their practical implementation may be limited since a WBE signature sequence designed for K users ceases to satisfy Welch's bound with equality if any $n < N$ signatures are removed. The resulting signature set loses both its capacity optimality and interference invariance property. To see this let us recall the following.

Theorem 1: (from [3]) If $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are both Welch bound equality signature matrices then $[\boldsymbol{S}_1, \boldsymbol{S}_2]$ is also a Welch bound equality signature matrix.

Now, (3) shows that the smallest WBE occurs for $K = N$, where $\boldsymbol{S}$ is orthogonal. Thus, if $n < N$ signatures are removed from a WBE then it could not possibly still be a WBE. Combining this result with Theorem 1 the corollary follows.

Corollary 2: Let $\boldsymbol{S}$ be a WBE signature matrix with K signatures and N chip periods. If any $n < N$ signatures are removed then the resulting signature matrix ceases to satisfy Welch's bound with equality.

The implication of Corollary 2 is that as the number of users changes in a cell, a WBE set designed for N active users is no longer optimal. Thus, a system that fully exploits WBE sequences would need (i) a set of sequences for every possible N and (ii) would need to reassign *all* sequences every time a user arrived or departed from the system. In this correspondence, we argue that a viable solution to this problem is to consider WBE signatures with additional structure. We consider two candidate signature sequences in the following two sections.

III. Equiangular Signatures for Interference Invariance

In this section, we consider the problem of designing signature sequences to maximize a tight lower bound on the SINR . The resulting signatures (when they exist) are equiangular, maximally spaced, and satisfy Welch's bound on the maximum correlation with equality. Their main utility is that they satisfy the interference invariance property in (4) even when subsets of users are active. We discuss the benefits of such signature sets, connections with Grassmannian frames, and constructions.

A. Max-Min Signature Design

Consider the problem of maximizing the minimum SINR when a subset of users is active. Let $\mathcal{P}$ denote the power set of integers $\{1, 2, \ldots, K\}$. This set will index all possible subsets of active users. An element $\mathcal{K} \in \mathcal{P}$ will denote a possible subset of active users. Let the SINR for the k-th user in the subset $\mathcal{K} \in \mathcal{P}$ be

$$\mathrm{SINR}(k, \mathcal{K}, \boldsymbol{S}) = \frac{1}{\sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 - 1 + \alpha} \tag{5}$$

where α is the inverse signal-to-noise ratio and $\langle \cdot \rangle$ stands for the usual vector inner product.

Consider the problem of finding a signature sequence that maximizes

$$\min_{\mathcal{K} \in \mathcal{P}} \min_{k \in \mathcal{K}} \mathrm{SINR}(k, \mathcal{K}, \boldsymbol{S}). \tag{6}$$

The maximizer is equivalent to the minimizer of the total squared correlation experienced by signature k. Using norm inequalities note that for a given $\mathcal{K}$

$$\max_k \sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 \geq \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2$$

where equality holds when $\sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2$ is the same for every $\boldsymbol{s}_k$. Further note that

$$\begin{aligned} &\max_{\mathcal{K} \in \mathcal{P}} \max_{k \in \mathcal{K}} \sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 \geq \\ &\max_{\mathcal{K} \in \mathcal{P}} \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 \end{aligned} \tag{7}$$

where equality holds when $1/|\mathcal{K}| \sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2$ is the same for every $\boldsymbol{s}_k$ and $\mathcal{K} \in \mathcal{P}$.

Hence maximizing the expression in (6) requires that the total squared correlation matrix (or Gram matrix) of $\boldsymbol{S}$ defined as $\mathrm{TSC}(\boldsymbol{S}) := \boldsymbol{S}^* \boldsymbol{S}$ has the form

$$\boldsymbol{S}^* \boldsymbol{S} = \begin{bmatrix} 1 & e^{j\vartheta_{1,2}}\rho & \cdots & e^{j\vartheta_{1,K}}\rho \\ e^{j\vartheta_{2,1}}\rho & 1 & e^{j\vartheta_{2,3}}\rho & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ e^{j\vartheta_{K,1}}\rho & e^{j\vartheta_{K,2}}\rho & \cdots & 1 \end{bmatrix} \tag{8}$$

where $\vartheta_{k,m} = -\vartheta_{m,k}$ due to Hermittian symmetry. To see this note that for $|\mathcal{K}| = 1$ the property is trivial. Next consider $|\mathcal{K}| = 2$. In this case the 2×2 submatrix formed from signatures $\boldsymbol{s}_k$ and $\boldsymbol{s}_m$ has the form

$$\begin{bmatrix} 1 & \rho_{k,m} e^{j\vartheta_{k,m}} \\ \rho_{k,m} e^{-j\vartheta_{k,m}} & \end{bmatrix}$$

where $\rho_{k,m}$ is real and nonnegative. For equality to hold in (7), $(1 + \rho_{k,m}^2)/2$ must be a constant and the same for every user, thus $\rho_{k,m}$ is the same for every user. Signature sequences with total squared correlation matrix with the structure in (8) are *equiangular* since they satisy $|\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle| = \rho$ for all k and $m \neq k \in \{1, 2, \ldots, K\}$. With these results we state the following proposition.

Proposition 3: Equiangular signature sequences (when they exist) maximize the worst case SINR for all possible subsets of active users

$$\min_{\mathcal{K} \in \mathcal{P}} \min_{k \in \mathcal{K}} \frac{1}{\sum_{m \in \mathcal{K}} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2 - 1 + \alpha}.$$

A side benefit is that we can write the SINR exactly.

Corollary 4: Let $\boldsymbol{S}$ be a signature matrix with equiangular signatures, i.e., $|\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle| = \rho$ for all k and $m \neq k$. Then

$$\mathrm{SINR}(k, \mathcal{K}, \boldsymbol{S}) = \frac{1}{|\mathcal{K}|\rho^2 - 1 + \alpha} \tag{9}$$

and is independent of k the number of active users.

The main utility of equiangular signatures is that the interference experienced by any user is exactly the same and depends only on $|\mathcal{K}|$, the current number of active users. Thus equiangular signatures preserve the *interference invariance* of WBE signature sequences over the range of possible active users.

B. Existence of Equiangular Signature Sequences

The utility of equiangular signatures is apparent. The main question at hand is when do such signature sequences exist? To review, the design problem is to find a signature matrix such that $\boldsymbol{S}\boldsymbol{S}^* = \frac{K}{N}\boldsymbol{I}$, $|\langle \boldsymbol{s}_k, \boldsymbol{s}_k \rangle|^2 = 1$ (this guarantees that the signature sequence is a Welch bound equality signature set), and $|\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle| = \rho$ for $k \neq m$ (this is the equiangular property). Notice that this poses a spectral constraint on the singular values of $\boldsymbol{S}$ as well as a structural constraint on $\mathrm{TSC}(\boldsymbol{S})$.

Existence of equiangular signature sequences is equivalent to the problem of designing an equiangular unit norm tight frame and is a topic of investigation in the frame theory community. Equiangular signature designs for different combinations of K and N are known (typically for small values of N). Existence (or lack there of) has been established for some choice of parameters [27], [28]. A table that shows the existence of equiangular unit norm tight frames is available in [29]. We comment more on constructions in Section III-D.

C. Maximally Spaced Signature Sequences

The amount of interference contributed by the other active signatures is a function of ρ. A smaller value of ρ results in less interference overall. The question at hand is: What is the smallest possible value of ρ as a function of K and N?

Let $\rho(K, N) := \max_{k,m,k \neq m} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|$. The problem of finding the smallest possible $\rho(K, N)$ is equivalent to find a lower bound on the maximum angle between the lines generated by $\boldsymbol{s}_k$ and $\boldsymbol{s}_l$. In particular, this problem relates to the notion of line packing. This problem has been addressed by a number of researchers in algebra, coding theory, and graph theory. The pertinent result is summarized in the following theorem.

Theorem 5: Let $\mathcal{S} = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N\}$ be a set of unit vectors in E^m with $N \geq m$. Then

$$\rho(K, N) \geq \sqrt{\frac{K - N}{N(K - 1)}}. \tag{10}$$

If $E = \mathbb{R}$ then equality in (10) can only hold if $N \leq m(m+1)/2$ and if $E = \mathbb{C}$ then equality in (10) can only hold if $N \leq m^2$. Furthermore, if equality holds in (10) then

$$\text{the vectors in } \mathcal{S} \text{ are equiangular.} \tag{11}$$

Proof: See [30]–[32] for example. □

The bound in (10) is often known as Welch's bound on the maximum correlation [32]. Signature sets that satisfy (10) with equality are known as maximum Welch bound equality signature sequences [3]. They are equiangular, maximally spaced, and most importantly are a special case of WBE sequences. Such vectors can also be viewed as an optimal packing of lines in the Grassmann manifold $\mathcal{G}(N,1)$. In this case, the resulting signatures form a Grassmannian frame [33]. The connection to line packings and Grassmannian frames is especially important because we can leverage the results of prior researchers to find signatures, as will be discussed in Section III-D.

The bound in (10) provides us with a bound on the SINR for any equiangular signature sequences, achieved when the sequence is maximally spaced. Substituting the right-hand side (RHS) of (10) into [9]

$$\text{SINR}(k, \mathcal{K}, \boldsymbol{S}) \leq \frac{1}{\frac{(K-N)(|\mathcal{K}|-1)}{N(K-1)} + \alpha}. \tag{12}$$

This value should be compared with the SINR achieved by redesigning a WBE sequence for $|\mathcal{K}|$ active users

$$\text{SINR} \leq \begin{cases} \frac{1}{\frac{|\mathcal{K}|-N}{K}+\alpha} |\mathcal{K}| > N \\ \alpha \; |\mathcal{K}| \leq N. \end{cases} \tag{13}$$

An important point is that when there are $|\mathcal{K}| \leq N$ users, the optimal WBE signature sequence consists of orthonormal vectors and thus there is no interference. Equiangular signatures, however, always have a residual interference term.

It is important to mention that satisfying (10) with equality is a sufficient condition for the signature sequence to be equiangular but is not necessary. Specifically, there may exist other signature sequences that are equiangular but do not satisfy this bound with equality. An example there is an equiangular signature sequence composed of five signatures in $\mathbb{R}^3$ but it's maximal inner product is $1/\sqrt{6}$ not $1/\sqrt{5}$. Thus it still satisfies the nice interference invariance property but incurs a slight SINR loss compared with a theoretical maximally spaced signature sequence.

It is also worth mentioning that the best line packings, tabulated in [34], do not typically form a WBE sequence (that is their $\boldsymbol{S}$ does not satisfy the spectral constraint). Consequently arbitrary line packings do not necessarily lead to the nearest equiangular WBE (see [29] for details). This impacts the construction of such sequences.

D. Construction of Equiangular Signature Sets

Given the utility of equiangular signatures, some comments on their construction are in order. From a matrix theoretic point-of-view, the problem of finding a WBE is equivalent to finding a matrix $\boldsymbol{S}$ such that $\boldsymbol{S}\boldsymbol{S}^* = K/N\boldsymbol{I}_N$ with unit-norm columns. This is in fact equivalent to solving for a uniform normal tight frame [35] where the columns of $\boldsymbol{S}$ are the elements of the frame. The equiangular property adds an additional constraint, specifically that the off-diagonal entries of the Gram matrix TSC($\boldsymbol{S}$) have entries of constant modulus less than or equal to $\rho(K,N)$ in (10). Such frames are called equiangular unit norm tight frames [36] or Grassmannian frames [33]. Construction of equiangular signature sequences is thus equivalent to finding an equiangular unit norm tight frame.

A natural question related to construction is when do equiangular signature sets exist? Theorem 5 provides necessary conditions for a maximally spaced signature set to exist, but it yields no insight for nonmaximally spaced equiangular signatures. Research has established the existence (or lack there of) for various dimensions [27], [28] and real or complex space using related results in graph theory and algebra. This is still an ongoing area of investigation.

In some cases it is possible to find maximally spaced equiangular signature sets. The simplex is the standard example, though not very useful for CDMA. Sarwate reviews some algebraic constructions for maximum Welch bound equality sets [3] (notably m-sequence codes) as well as near maximum Welch bound equality sets. The authors discuss some nonalgebraic constructions including one based on difference sets and another based on conference matrices [33] in the context of Grassmannian frames. As discussed before, the real line packings tabulated in [34] are equiangular signatures when they are maximally spaced.

Perhaps the most relevant construction with respect to CDMA is the conference matrix construction. This construction provides a convenient way to find $K = 2N$ vectors in $\mathbb{C}^N$ or $\mathbb{R}^N$, which is useful because for quasi-orthogonal CDMA K is typically less than $2N$. Since we use this construction in the numerical results, we summarize this construction from [33] here without proof.

Recall that an $n \times n$ conference matrix $\boldsymbol{C}$ has zeros along its main diagonal and ± 1 as its other entries, see [37]. In addition a conference matrix satisfies $\boldsymbol{C}\boldsymbol{C}^T = (n-1)\boldsymbol{I}_n$. If $\boldsymbol{C}_{2N}$ is a symmetric conference matrix, then there exist $2N$ vectors in $\mathbb{R}^N$ such that the bound (10) holds with $\rho(2N,N) = 1/\sqrt{2N-1}$. If $\boldsymbol{C}_{2N}$ is a skew-symmetric conference matrix (i.e., $\boldsymbol{C} = -\boldsymbol{C}^T$), then there exist $2N$ vectors in $\mathbb{C}^N$ such that the bound (10) holds with $\rho(2N,N) = 1/\sqrt{2N-1}$, see [31, Example 5.8]. The following Lemma describes a simple algorithm that uses conference matrices to construct a Gram matrix then to extract the signatures $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_N$ from this matrix.

Lemma 6: Let $\boldsymbol{C}_{2N}$ be a (skew)symmetric conference matrix and denote $\alpha := 1/\sqrt{2N-1}$. Compute

$$\boldsymbol{R} = \begin{cases} \alpha \boldsymbol{C}_{2N} + \boldsymbol{I}_{2N} & \text{if } E = \mathbb{R}, \\ \boldsymbol{R} = j\alpha \boldsymbol{C}_{2N} + \boldsymbol{I}_{2N} & \text{if } E = \mathbb{C}. \end{cases} \tag{14}$$

Let $w_1, \ldots, w_{2N}$ be the eigenvectors of $\boldsymbol{R}$. Then the $2N$ vectors $v_1, \ldots, v_{2N}$ in E^N given by

$$v_k := \sqrt{2}[w_1(k), \ldots, w_N(k)]^T, \qquad k = 1, \ldots, 2m, \tag{15}$$

constitute a set of vectors that achieve the bound (10).

Proof: See [33] for the proof. □

A necessary condition for the existence of a symmetric conference matrix $\boldsymbol{C}_n$ is that $n = 2 \mathrm{mod} 4$. A sufficient condition due to Paley [38] is $n = p^\alpha + 1$ where p is a prime number and $\alpha \in \mathbb{N}$. In this case it is not difficult to explicitly construct the conference matrix. Thus for instance there exist 50 equiangular lines in $\mathbb{R}^{25}$ with angle $\operatorname{acos}(1/\sqrt{49})$.

A sufficient condition for the existence of a skew-symmetric conference matrix $\boldsymbol{C}_n$ is that $n = 2^k$ for $k \in \mathbb{N}$. Here is a simple recursive procedure to construct a skew-symmetric conference matrix whose size is a power of two. Initialize

$$\boldsymbol{C}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{16}$$

and compute recursively

$$\boldsymbol{C}_{2N} = \begin{bmatrix} \boldsymbol{C}_N & \boldsymbol{C}_N - \boldsymbol{I}_N \\ \boldsymbol{C}_N + \boldsymbol{I}_N & -\boldsymbol{C}_N \end{bmatrix} \tag{17}$$

then $\boldsymbol{C}_{2N}$ is a skew-symmetric conference matrix. This construction is reminiscent of the construction of Hadamard matrices. Indeed, a simple calculation shows that $\boldsymbol{C}_N + \boldsymbol{I}_N$ is a skew-symmetric Hadamard matrix.

In many cases it is not possible to find WBEs that are exactly equiangular. While [34] presents the best packings of lines, these packings do not necessarily satisfy the WBE constraint. We would like to find elements of the space of WBEs that is most nearly equiangular. An example of a near-equiangular construction is given in [33, Sec. 2.1.3]. The general case, however, is a difficult problem. A numerical procedure is available [36]. In brief, this algorithm performs an alternating projection between the space of WBEs and the space of Gram matrices with constant modulus off-diagonal elements. The details of this approach are beyond the scope of this correspondence.

IV. Sequence Sets From Mutually Unbiased Orthonormal Bases

While equiangular signature sequences preserve the interference invariance of WBE sequence sets, they do not satisfy an important property: orthogonality when there are $K < N$ users in the system. In this section, we discuss an alternative construction based on unions of orthonormal basis. We first motivate the problem then present some new results on their construction.

A. Motivation and Preliminaries

Early CDMA communication systems used only a single orthonormal basis, typically the Walsh basis. Orthonormal basis are very desirable because in the absence of multipath interference all the users are orthogonal and thus experience a simple Gaussian channel. Quasi-orthogonal signatures are capacity optimal only when there are $K > N$ users. When the system is lightly loaded, an orthogonal signature sequence is sufficient and in fact desirable for the reasons mentioned. Thus it is of interest to develop signature sequences that contain a subsequence that corresponds to an orthonormal basis.

One approach taken in third generation systems was to add an additional basis obtained by modifying the Walsh codes with a cover code. The cover code is chosen so that the new set of orthogonal codes has optimal correlation with Walsh codes of the same length and further has low correlation to Walsh codes of smaller lengths. The latter property is desirable when multiple spreading factors are used. A thorough discussion of cover code construction and related results are available in [14], [22], [21].

Using a cover code is a nice algebraic method of converting an orthonormal basis into another basis that is less correlated with the original basis. This is a special case of constructing a signature sequence from unions of specially chosen orthonormal basis (ONB). Unions of two orthonormal bases have been extensively investigated [20]–[24]; extensions to more than two basis has received less attention [25]. As previous work shows, merely replicating the original ONB is not practical since this would lead to rather poor signal-to-interference ratio performance. Instead one design these ONBs such that the maximal correlation between members of different ONBs is minimal, i.e., equals $\frac{1}{\sqrt{N}}$. This property is well-known in quantum physics, where two (or more) orthonormal bases $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_N\}, \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_N\}$ which satisfy

$$|\langle \boldsymbol{s}_k, \boldsymbol{u}_l \rangle| = \frac{1}{\sqrt{N}}, \qquad k = 1, \dots, N; l = 1, \dots, N \tag{18}$$

are known as *mutually unbiased bases* (MUB), cf. [19], and we adopt this term here.

These MUB sequences have the nice property that for $k \le N$ users taken from a single basis, there is no interference. For any k that is a multiple of N they are optimal from the point-of-view of sum capacity as follows from Theorem 1. They are also optimal for $K = k$ and thus retain all the nice properties of WBE sets in this case. It is clear, however, that they do not have the constant interference property of MWBE sequences.

In the next section we pursue two questions that have not been completely addressed in the CDMA literature: i) How well do MUB sequences behave compared to WBE sequences? ii) How many sequences can we add to the Walsh sequences so that the minimal interference $\frac{1}{\sqrt{N}}$ is maintained and how can we construct them?

B. Analysis and Construction of MUB Signature Sequences

Construction of a MUB sequence set begins with a set of N orthonormal signatures. First we consider the special case of adding $k \le N$ signatures to this set. This is the approach taken in WCDMA [21] and IS-2000 [22]. In particular, we would like to know how to chose these n signatures and how far the resulting set is from a WBE set. This will determine the loss in optimality (if any). The following lemma addresses this point.

Lemma 7: Let $\boldsymbol{S} = [\boldsymbol{s}_1, \dots, \boldsymbol{s}_N]$ with $\boldsymbol{S}\boldsymbol{S}^* = \boldsymbol{S}^*\boldsymbol{S} = \boldsymbol{I}_N$ and let $\boldsymbol{V}_k = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_k]$ with $\|\boldsymbol{v}_n\| = 1$ for $n = 1, \dots, k, k \le N$. Denote $\boldsymbol{W}_k = [\boldsymbol{S}, \boldsymbol{V}_k]$. Then TSC$(\boldsymbol{W}_k)$ is minimized if $\boldsymbol{V}_k^*\boldsymbol{V}_k = \boldsymbol{I}_k$. In this case TSC$(\boldsymbol{W}_k) = N + 3k$ and $\mathrm{cond}(\boldsymbol{W}_k) = \sqrt{2}$.

Proof: We have $\boldsymbol{W}_k\boldsymbol{W}_k^* = \boldsymbol{I}_N + \boldsymbol{V}_k\boldsymbol{V}_k^*$, and since the nonzero eigenvalues of $\boldsymbol{V}_k\boldsymbol{V}_k^*$ coincide with those of $\boldsymbol{V}_k^*\boldsymbol{V}_k$ it follows that the eigenvalues of $\boldsymbol{W}_k\boldsymbol{W}_k^*$ are

$$\lambda_1 = \cdots = \lambda_{N-k} = 1 \quad \text{and} \quad \lambda_{N-k+1} = \cdots = \lambda_N = 2.$$

Since TSC$(\boldsymbol{W}_k) = \sum_{n=1}^{N} \lambda_n^2$ and $\mathrm{cond}(\boldsymbol{W}_k) = \sqrt{\lambda_N/\lambda_1}$, both results follow. □

Thus, if we want to add k sequences to an ONB, then the k sequences should form an orthonormal system. Note that the Welch bound for $N + k$ vectors is $N + 2k + \frac{k^2}{N}$. Hence for an almost fully loaded system, there is little difference between WBE sequences and MUB sequences, in particular if $k = N$ then the MUB set becomes a WBE set. Furthermore, adding $1 < k < N$ vectors only moderately increases the condition number from 1 to $\sqrt{2}$, which is important when employing MMSE receivers. Lemma 7 can easily be extended to the case when we add $k \le N$ vectors to a sequence set that is composed of n sets of basis vectors so that $K = Nn$.

Lemma 7 shows how close MUB sequence sets are to WBE sequence sets. However, it does not indicate how to construct the additional ONBs. Furthermore one often wants to construct sequences whose entries belong to a finite alphabet, such as binary or quaternary sequences. While some interesting constructions are given in prior work, we will not only provide new constructions but simultaneously answer the question of how many ONBs can be concatenated and yet retain the minimum inner product.

We answer these questions by utilizing a result that was derived in the context of finite group theory and algebraic geometry. In the ingenious paper [39] Cameron *et al.* showed the existence of $K + 1$ MUB for $\mathbb{C}^K, K = 2^N$ with $0 < N \in \mathbb{N}$ and the existence of $K/2 + 1$ MUB for $\mathbb{R}^K, K = 2^N$ for even N. Some of these results can already by found in [40] and in [19]. However the approach in [39] is more general, provides greater flexibility, and demonstrates various appealing cross connections between different fields. The authors of [39] use advanced concepts from discrete geometry (orthogonal and symplectic spreads) and finite group theory (properties of extraspecial groups) to derive their results and construct those bases. Since symplectic spreads and extraspecial groups are not part of the standard toolbox of communications engineers we translate the approach by Cameron *et al.* into a simple and efficient algorithm that does not require any finite group theory and uses only very little Galois field theory. We restrict ourselves to the case of $K = 2^N$, the case of p^N for odd prime numbers is quite similar.[1]

Algorithm for constructing MUB sequences We start with the vector space $\mathbb{Z}_2^N$, where N is a positive integer and some ordering is chosen for $\mathbb{Z}_2^N$, we denote $K = 2^N$. The first ONB is given by the identity matrix in $\mathbb{C}^K$, i.e., $\mathrm{S}_0 = I_K$. The second ONB is the $K \times K$ Sylvester-Hadamard matrix, $\mathrm{S}_1 = H_K$, where H_K is normalized such that $H_K H_K^* = I_K$. The other ONBs $\mathrm{S}_n, n = 2, \dots, N$ can be computed from H_K via $\mathrm{S}_n = D_n H_K$, where the D_n are certain diagonal matrices, which we have to construct now.

Let $1, \alpha, \alpha^2, \dots, \alpha^{N-1}$ be a basis for the Galois field $\mathrm{GF}(2^m)$ (choose α to be a root of a primitive polynomial in $\mathrm{GF}(2^m)$). Let $B_0, \dots, B_{N-1}$ be the m matrices of dimension $N \times N$ obtained from the multiplication table

$$\begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{N-1} \end{bmatrix} [1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{N-1}] =: B_0 + \alpha B_1 + \alpha^2 B_2 + \cdots + \alpha^{N-1} B_{N-1}. \quad (19)$$

Next we construct a set of binary symmetric matrices $P_n, n = 0, \dots, K-1$ by computing

$$P_n = \left(\sum_{l=0}^{N-1} c_{n,l} B_l \right)_{\mathrm{mod}\ 2} \quad (20)$$

where $c_n = \{c_{n,l}\}_{l=0}^{N-1}$ is the binary vector representing the number n, for $n = 0, \dots, K-1$. The subscript mod(2) means that the result of the summation is taken modulo 2.

We define a $\mathbb{Z}_4$*-quadratic form*, i.e., a map $T_P : \mathbb{Z}_2^N \to \mathbb{Z}_4$ by

$$T_P(v) := \left(\sum_{n=0}^{N-1} P_{n,n} v_n^2 + 2 \sum_{l<n} P_{l,n} v_l v_n \right)_{\mathrm{mod}\ 4} \quad (21)$$

where $v = [v_1, \dots, v_N]^T \in \mathbb{Z}_2^N$. The diagonal matrices D_n are

$$D_n = \mathrm{diag}\left([i^{T_{P_n}(v)}]_{v \in \mathbb{Z}_2^N} \right)$$

and $B_n = D_n H$. This gives 2^N ONBs in $\mathbb{C}^K$ with $K = 2^N$.

When $N+1$ is even one can easily construct MUB with binary entries once the matrices P_n are computed. For denote $K' = 2^{N+1}$ (thus, $K = K'/2$) and define the $2^{N+1} \times 2^{N+1}$ skew-symmetric binary matrices M_n by

$$M_n = \begin{bmatrix} P_n + d_{P_n} d_{P_n}^* & d_{P_n} \\ d_{P_n}^* & 0 \end{bmatrix}_{\mathrm{mod}\ 2} \quad (22)$$

for $n = 0, \dots, K-1$, where P_n are the matrices defined in (20) and d_{P_n} denotes the diagonal of P_n. Let the quadratic form $Q_M(v) : \mathbb{Z}_2^{N+1} \to \mathbb{Z}_2$ be given by

$$Q_M(v) = \sum_{n \le l} M_{n,l} v_n v_l, \qquad v \in \mathbb{Z}_2^{N+1}. \quad (23)$$

Similar to before we introduce diagonal matrices D_n by setting

$$D_n = \mathrm{diag}\left([(-1)^{Q_{M_n}(v)}]_{v \in V} \right), \qquad \text{for } n = 0, \dots, K-1.$$

The K binary orthonormal bases in $\mathbb{R}^{K'}$ are now given by $B_n = D_n H_{K'}$. We note that compared to the quaternary case above, we have only half as many binary ONBs.

[1]Furthermore, we only consider the case where the symplectic spread underlying the algorithm produces a desarguesian affine plane, but we note that other choices for the symplectic spread are possible.

Example 8: We illustrate the above algorithm by an example. Let $N = 3$, i.e., $K = 2^3 = 8$ and we construct 9 ONBs for $\mathbb{C}^8$. As is common practice, we represent polynomials $c_0 + c_1\alpha + c_2\alpha^2$ (with $c_n \in \{0,1\}$) for $\mathrm{GF}(2^3)$ either in binary notation or by its corresponding decimal number. For example, $1 + \alpha \leftrightarrow (0,1,1) \leftrightarrow 3$. Then the matrix on the left-hand side of (19) can be written as

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 3 \\ 4 & 3 & 6 \end{bmatrix}.$$

(In MATLAB A can be computed via the commands a = gf(2.^[0:2]',3),A = a*a'.) An easy calculation shows that the matrices B_0, B_1, B_2 on the right-hand-side of (19) are

$$B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The coefficients vectors c_n in (20) are $c_0 = [0,0,0], c_1 = [0,0,1], \dots, c_7 = [1,1,1]$, and the matrices $P_0, \dots, P_7$ of (20) are given by

$$P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$P_6 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

To save space we list the diagonals of the diagonal matrices $D_n, n = 0, \dots, 7$ computed via the $\mathbb{Z}_4$-quadratic form (21) as column vectors of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & j & j & 1 & 1 & j & j & 1 \\ 1 & j & 1 & j & 1 & j & 1 & j \\ 1 & -1 & j & j & -1 & 1 & j & j \\ 1 & 1 & 1 & 1 & j & j & j & j \\ 1 & j & j & -1 & j & 1 & -1 & j \\ 1 & j & -1 & j & j & -1 & j & 1 \\ 1 & 1 & j & j & j & j & 1 & -1 \end{bmatrix}. \quad (24)$$

The nine ONBs are now given by $B_0 = I_8, B_n = D_n H_K, n = 1, \dots, K-1$. We also compute one of the matrices M_n, say for $n = 1$. By (22)

$$M_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad (25)$$

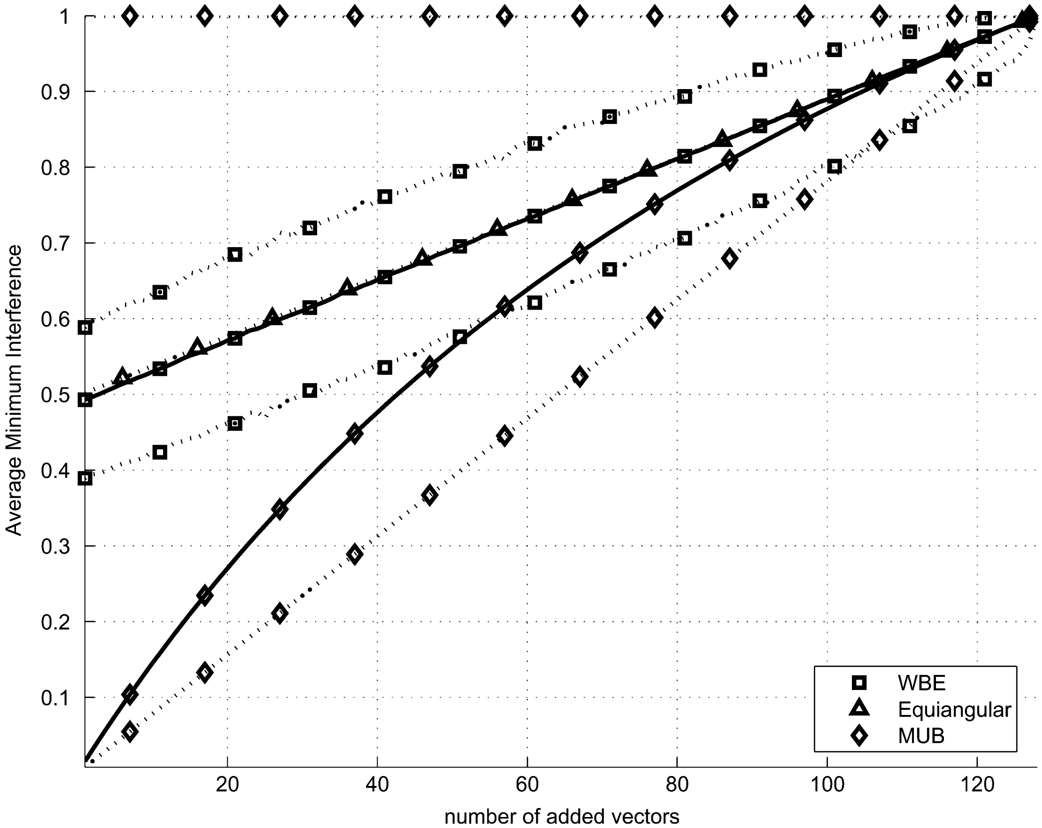

Fig. 1. Comparison of the average mean, maximum, and minimum interference levels as the number of active codes increases. For the equiangular case the mean, minimum, and maximum are the same for all signatures. For the other cases we take the average over a series of random trials. The mean for each set of vectors is plotted with a solid line. The min/max are plotted with dashed lines.

and the diagonal of D_{M_1} is $d_{M_1} = [0\,0\,0\,1\,0\,1\,1\,1\,0\,0\,1\,0\,0\,1\,0\,0]$. Hence $D_{M_1}H_{16}$ gives one of the binary MUB for $\mathbb{R}^{16}$.

A different construction of MUB sequences for the case of prime $p \geq 5$ was proposed by Alltop [41]. The *Alltop-sequences* can be constructed as follows. Let $p \geq 5$ be a prime number and set

$$\boldsymbol{s}_{k,l}(n) = \frac{1}{\sqrt{p}} e^{j2\pi(n^3-k)/p} e^{j2\pi nl/p}, \qquad k,l,n = 1,\dots,p. \quad (26)$$

We obtain $p+1$ orthonormal bases for $\mathbb{C}^p$ given by the sets $\boldsymbol{U}_k = \{\boldsymbol{s}_{k,l}\}_{l=1}^p$, $k = 1,\dots,p$ and the standard basis in $\mathbb{C}^p$. Due to the specific construction rule the correlation of any Alltop sequence with any of its cyclically shifted versions is $\frac{1}{\sqrt{p}}$. Furthermore we note that the Alltop sequences cannot be derived from the construction proposed in [39].

A hitherto overlooked but quite useful property of Alltop-sequences is that their Fourier transform is nearly unimodular, as the following lemma shows.

Lemma 9: Let the vector $\boldsymbol{s}$ be an Alltop-sequence. Then

$$|\hat{\boldsymbol{s}}(k)| \leq \frac{2}{\sqrt{p}}, \quad \text{for } k = 1,\dots,p. \quad (27)$$

Proof: An easy calculation shows that

$$\hat{\boldsymbol{s}}_{k,l}(m) = \frac{1}{p} \sum_{n=1}^{p} e^{-j2\pi[(n^3-k)+n(m-k)]/p} e^{-j2\pi ml/p}, \quad (28)$$

for $m = 1,\dots,p$ hence

$$|\hat{\boldsymbol{s}}_{k,l}(m)| = \frac{1}{p} \left| \sum_{n=1}^{p} e^{-j2\pi[(n^3-k)+n(m-k)]/p} \right|. \quad (29)$$

Inequality (27) follows now by applying to (29) an estimate of A. Weil [42] who showed that for prime numbers p which satisfy $p > r, r \in \mathbb{N}$ and $\gcd(a_r, p) = 1$ there holds

$$\left| \sum_{n=1}^{p} e^{j2\pi \frac{a_1 n + \cdots + a_r n^r}{p}} \right| \leq (r-1)\sqrt{p}. \quad (30)$$

□

Except for the fact that the length of Alltop sequences is not a power of 2, they have several nice properties including nice shift-correlations as well as good spreading properties.

V. Numerical Results

Here, we compare the interference generated by different signatures for the case of $N = 64$ and $K = 128$. We compare a randomly generated complex WBE (generated using the procedure in [11], initialized with a randomly chosen $N \times K$ complex Gaussian matrix), a maximally spaced equiangular signature sequence found through the conference matrix construction, and a MUB constructed from the algorithm described in Section IV-B.

Since we are interested in the overloaded case, we focus on the interference variability as we increase the number of active sequences from N to K. We begin with N signatures. For the WBE sequence set, we choose the N signatures that have $\boldsymbol{S}$ with the smallest condition number. For the equiangular signature set we pick an arbitrary N signatures. For the MUB set, we pick any N signatures that form an orthonormal basis. Then we add $0 < n \leq N$ vectors, randomly chosen, to that set to form the active signature set $\mathcal{K}$ and then compute $I(k) = \sum_{m \in \mathcal{K}, m \neq k} |\langle \boldsymbol{s}_k, \boldsymbol{s}_m \rangle|^2$ for $k = 1,2,\dots,N+n$. For each set we compute $I_{\min} := \min_k I(k), I_{max} := \max_k I(k)$, and $I_{\text{avg}} := \frac{1}{N+n} \sum_{l=1}^{N+nz} I(l)$. We averaged each of these metrics over 20 different randomly chosen sets of vectors and plot all three in Fig. 1.

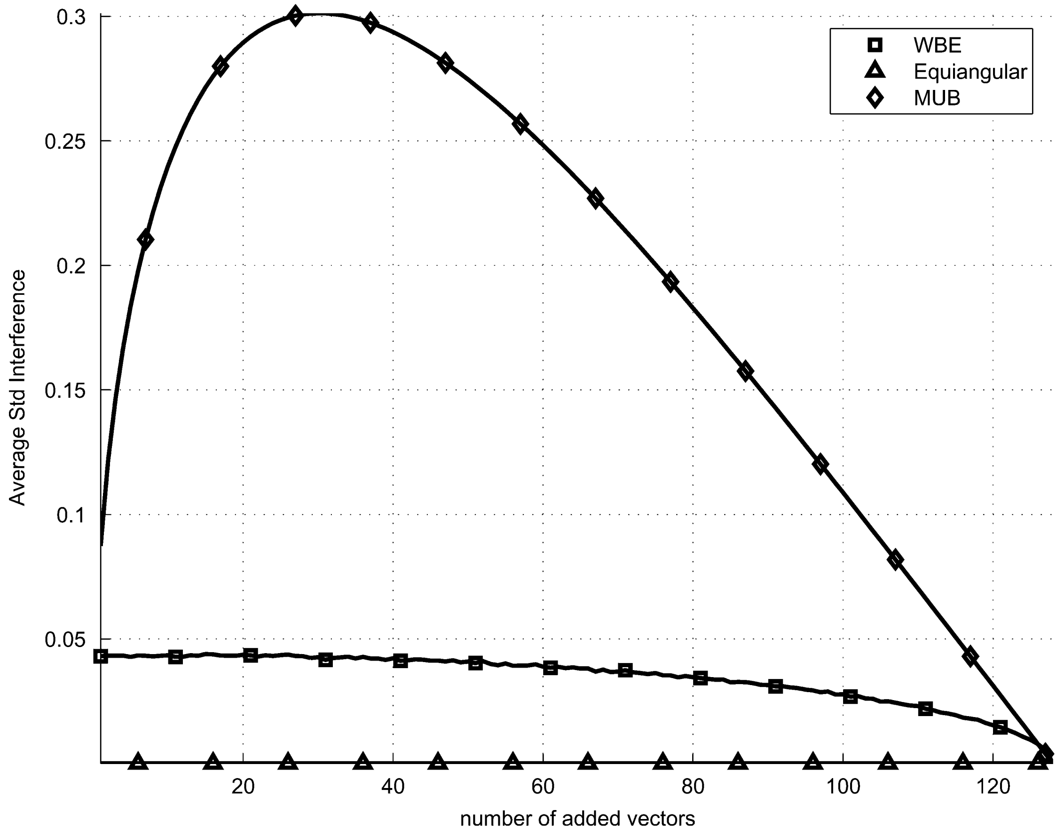

Fig. 2. Comparison of the average standard deviation as the number of active codes increases. For the equiangular case the standard deviation is zero. For the other cases we take the average over a series of random trials.

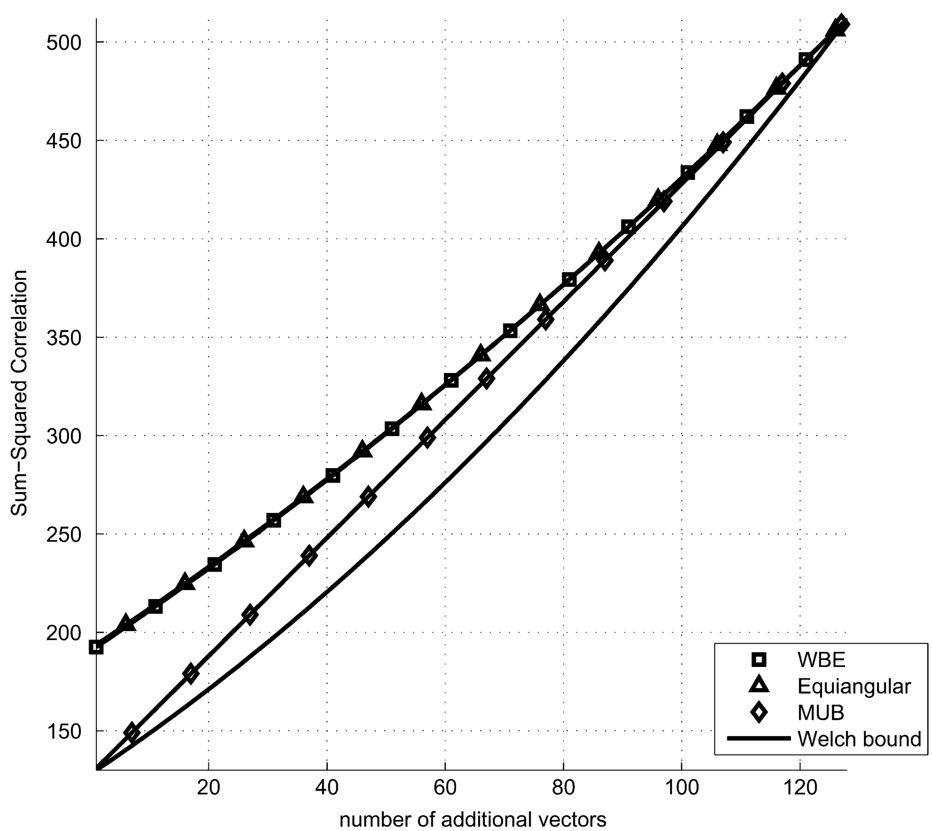

Fig. 3. Comparison of the total squared correlation as the number of active codes increases. For the MWBE case the standard deviation is zero due to the equiangular property. For the other cases we compute the standard deviation as we add signatures using the default equiangular signature ordering.

As expected, the equiangular signatures have the same interference in part due to the equiangular property. Interestingly, note that the WBE set has a mean interference level that is almost the same as the equiangular set (this was proven in [15]). The interference experienced by each code, however, varies between the maximum and minimum average interference curves. This extra variability creates additional fluctuations in signal-to-interference ratio that must be reduced in the power control algorithms. The MUB set performs quite differentlythan the equiangular set. The average interference for the MUB set is actually lower than the average interference for either equiangular or the WBE sets. The maximum and minimum, however, are significantly more extreme. The maximum is 1 for the MUB because the absolute value of the inner product between any signature $\boldsymbol{a}$ in one basis and $\boldsymbol{s}$ in the other basis is $|\langle \boldsymbol{a}, \boldsymbol{s} \rangle| = 1/\sqrt{N}$ while signatures within the basis are mutually orthogonal. Thus signatures in the MUB can see higher levels of interference but on average the interference is lower than with a randomly chosen WBE or an equiangular signature.

To illustrate the variability of $I(k)$, in Fig. 2. we plot the standard deviation of $\{I(k)\}_{k=1}^{N+n}$ averaged over 20 different randomly chosen sets of vectors. The standard deviation gives a measure of how much power control would be required to compensate for nonideal-quasi-orthogonality in the signature sequence.

Finally, for comparison with other work on signature design based on the total squared correlation we plot the total squared correlation as a function of the number of added signatures and Welch's bound on total squared correlation in Fig. 3. As we expect, the MUB sequence agrees with Welch's bound at two points corresponding to the inclusion of one or two complete basis. Interestingly the total squared correlation of the WBE and the equiangular signature sequence is quite close. Both approach the bound for larger numbers of signatures but incur a loss at smaller numbers of vectors.

VI. Conclusions and Future Work

In this correspondence we investigated two signature sequences that are robust in different ways to the number of active users. Equiangular sequences are interference invariant and thus do not depend on the specific signatures of the active users. We showed that they satisfy a certain SINR maximizing over all Welch bound equality sequences. Mutually unbiased basis contain multiple orthonormal basis and thus can behave like orthonormal basis with few users, can augment existing sequences, and are Welch bound equality sequences for multiple subsets. We studied the specific case of augmented the Walsh sequences and derived the maximum number of mutually orthonormal bases that can be concatenated with minimal correlation and presented a specific construction algorithm.

Our correspondence was focused on signature design for the AWGN channel. Future work should consider signature sequences with additional structure suitable for fading channels such as generalized Welch bound equality sequences (e.g., [4]). In the asynchronous case or systems with multipath interference, signatures should satisfy additional properties like have low auto- and cross correlation for example. Signature design criteria for these more complicated channels is an active area of research.

References

[1] M. Rupf and J. L. Massey, "Optimum sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. Inf. Theory*, vol. 40, no. 4, pp. 1261–1266, Jul. 1994.

[2] J. L. Massey and T. Mittelholzer, "Welch's bound and sequence sets for code division multiple-access systems," in *Sequences II: Methods In Communcation, Security and Computer Science*. Berlin, Germany: Springer-Verlag, 1994, pp. 63–78.

[3] D. V. Sarwate, "Meeting the Welch Bound with equality," in *Sequences and Their Applications*. London, U.K.: Springer, 1998, pp. 79–102.

[4] P. Viswanath and V. Anantharam, "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 1984–1991, Sep. 1999.

[5] P. Viswanath and V. Anantharam, "Optimal sequences for CDMA under colored noise: A Schur-saddle function property," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1295–1318, Jun. 2002.

[6] P. Anigstein and V. Anantharam, "Ensuring convergence of the MMSE iteration for interference avoidance to the global optimum," *IEEE Trans. Inf. Theory*, vol. 49, no. 4, pp. 873–885, Apr. 2003.

[7] P. Viswanath, A. Anantharam, and D. N. C. Tse, "Optimal sequences, power control, and user capacity of synchronous CDMA systems with linear MMSE multiuser receivers," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 1968–1983, Sep. 1999.

[8] T. Guess, "Optimal sequences for CDMA with decision-feedback receivers," *IEEE Trans. Inf. Theory*, pp. 886–900, Apr. 2003.

[9] S. Ulukus and R. D. Yates, "Iterative construction of optimum signature sequence sets in synchronous CDMA systems," *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 1989–1998, Jul. 2001.

[10] C. Rose, "CDMA codeword optimization: Interference avoidance and convergence via class warfare," *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2368–2382, Sep. 2001.

[11] J. A. Tropp, R. W. Heath Jr., and T. Strohmer, "An alternating projection algorithm for constructing quasi-orthogonal CDMA signature sequences," in *Proc. Int. Symp. Information Theory*, Yokohama, Japan, Jun./Jul. 2003, p. 407.

[12] P. Cotae, "Optimal sequences and total weighted square correlation of synchronous CDMA systems," in *Proc. IEEE Wireless Networks and Communicatio Conf.*, vol. 2, New Orleans, LA, Mar. 2003, pp. 947–952.

[13] J. A. Tropp, I. Dhillon, and R. W. Heath Jr., "Finite-step algorithms for constructing optimal CDMA signature sequences," *IEEE Trans. Inf. Theory*, vol. 50, no. 11, pp. 2916–2921, Nov. 2004.

[14] K. Yang, Y.-K. Kim, and P. Vijay Kumar, "Quasi-orthogonal sequences for code-division multiple-access systems," *IEEE Trans. Inf. Theory*, vol. 46, no. 3, pp. 982–993, Mar. 2000.

[15] V. V. Kravcenko, H. Boche, F. Fitzek, and A. Wolisz, "No need for signaling: Investigation of capacity and quality of service for multi-code CDMA systems usign the WBE++ approach," in *Proc. 4th Int. Workshop on Mobile and Wireless Commun. Networks*, Stockholm, Sweden, 2002, pp. 110–114.

[16] T. Strohmer, R. W. Heath Jr., and A. J. Paulraj, "On the design of optimal sequences for CDMA systems," in *Proc. Asilomar Conf. Signals Systems and Computing*, vol. 2, Pacific Grove, CA, Nov. 2002, pp. 1434–1438.

[17] R. W. Heath Jr., T. Strohmer, and A. J. Paulraj, "Grassmannian signatures for CDMA systems," in *Proc. Global Telecommunications Conf.*, vol. 3, San Francisco, CA, Dec. 2003, pp. 1553–1557.

[18] F. Vanhaverbeke and M. Moeneclaey, "Optimal signature sets for oversaturated quasi-scalable direct-sequence spread-spectrum systems," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 1136–1139, Mar. 2005.

[19] W. K. Wootters and B. D. Fields, "Optimal state-determination by mutually unbiased measurements," *Ann. Phys.*, vol. 191, no. 2, pp. 363–381, May 1989.

[20] H. Sari, F. Vanhaverbeke, and M. Moeneclaey, "Multiple access using two sets of orthogonal signal waveforms," *IEEE Commun. Lett.*, vol. 4, no. 1, pp. 4–6, Jan. 2000.

[21] H. D. Schotten and H. Hadinejad-Mahram, "Analysis of a CDMA downlink with nonorthogonal spreading sequences for fading channels," in *Proc. Vehicula Technology Conf.*, vol. 3, Boston, MA, May 2000, pp. 1782–1786.

[22] A. G. Shanbhag and J. Holtzman, "Optimal QPSK modulated quasi-orthogonal functions for IS-2000," in *IEEE 6th Int. Symp. Spread Spectrum Techniques and Applications*, Parsippany, NJ, Sep. 2000, pp. 756–760.

[23] D. V. Dnonin and V. K. Bharagava, "Spectral efficiency of the feedback receiver for two sets of orthogonal sequences," *IEEE Commun. Lett.*, vol. 6, no. 11, pp. 4–6, Nov. 2002.

[24] F. Vanhaverbeke and M. Moeneclaey, "Sum capacity of equal-power users in overloaded channels," *IEEE Trans. Commun.*, vol. 53, no. 2, pp. 228–233, Feb. 2005.

[25] F. Vanhaverbeke, M. Moeneclaey, and H. Sari, "Increasing CDMA capacity using multiple orthogonal spreading sequence sets and successive interference cancellation," in *Proc. Int. Conf. Communications*, vol. 3, Vancouver, BC, Canada, Apr. 1999, pp. 1516–1520.

[26] G. E. Bottomley, "Signature sequence selection in a CDMA system with orthogonal coding," *IEEE Trans. Veh. Technol.*, vol. 42, no. 1, pp. 62–68, Feb. 1993.

[27] R. B. Holmes and V. I. Paulsen, "Optimal frames for erasures," *Linear Algebra Its Applic.*, vol. 377, pp. 31–51, 2004.

[28] M. Sustik, J. A. Tropp, I. S. Dhillon, and R. W. Heath< Jr., "On the existence of equiangular uniform tight frames," Preprint, 2004. Submitted to *Linear Algebra and Its Applic.*, Available [Online] at http://www.cs.utexas.edu/users/inderjit/public_papers/equiangular.pdf.

[29] R. W. Heath Jr., J. A. Tropp, I. S. Dhillon, and T. Strohmer, "Construction of equiangular signatures for synchronous CDMA systems," in *Proc. Int. Symp. Spread Spectrum Techniques and Applications*, Sydney, Australia, Aug. 2004, pp. 335–340.

[30] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: Packings in Grassmannian spaces," *Exper. Math.*, vol. 5, pp. 139–159, 1996.

[31] P. Delsarte, J. M. Goethals, and J. J. Seidel, "Bounds for systems of lines and Jacobi poynomials," *Philips Res. Repts*, vol. 30, no. 3, pp. 91*–105*, 1975. Issue in honor of C. J. Bouwkamp.

[32] L. R. Welch, "Lower bounds on the maximum cross-correlation of signals," *IEEE Trans. Inf. Theory*, vol. IT-20, no. 3, pp. 397–399, May 1974.

[33] T. Strohmer and R. W. Heath Jr., "Grassmannian frames with applications to coding and communications," *Appl. Computat. Harmonic Anal.*, vol. 14, no. 3, pp. 257–275, May 2003.

[34] N. J. A. Sloane. (2003) Tables of real Grassmannian packings. [Online]. Available: http://www.research.att.com/njas/

[35] S. Waldron, "Generalized welch bound equality sequences are tight frames," *IEEE Trans. Inf. Theory*, vol. 49, no. 9, pp. 2307–2309, Sep. 2003.

[36] J. A. Tropp, I. Dhillon, R. W. Heath Jr., and T. Strohmer, "Designing structured tight frames via an alternating projection method," *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 188–209, Jan. 2005.

[37] P. W. H. Lemmens and J. J. Seidel, "Equi-isoclinic subspaces of Euclidean spaces," in *Nederl. Akad. Wetensch. Proc. Ser. A* $76 = \mathrm{Indag.}$ *Math.*, vol. 35, 1973, pp. 98–107.

[38] R. E. A. C. Paley, "On orthogonal matrices," *J. Math. Phys.*, vol. 12, pp. 311–320, 1933.

[39] A. R. Calderbank, P. J. Cameron, W. M. Kantor, and J. J. Seidel, "Z_4-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets," in *Proc. London Math. Soc. (3)*, vol. 75, 1997, pp. 436–480.

[40] P. J. Cameron and J. J. Seidel, "Quadratic forms over GF (2)," in *Nederl. Akad. Wetensch. Proc. Ser. A* $76 = \mathrm{Indag.}$ *Math.*, vol. 35, 1973, pp. 1–8.

[41] W. O. Alltop, "Complex sequences with low periodic correlations," *IEEE Trans. Inf. Theory*, vol. IT-26, no. 3, pp. 350–354, May 1980.

[42] A. Weil, "On some exponential sums," in *Proc. Nat. Acad. Sci. USA*, vol. 34, 1948, pp. 204–207.

On Information Transmission Over a Finite Buffer Channel

Suhas Diggavi, *Member, IEEE,* and
Matthias Grossglauser, *Member, IEEE*

***Abstract*—We study information transmission through a finite buffer queue. We model the channel as a finite-state channel whose state is given by the buffer occupancy upon packet arrival; a loss occurs when a packet arrives to a full queue. We study this problem in two contexts: one where the state of the buffer is known at the receiver, and the other where it is unknown. In the former case, we show that the capacity of the channel depends on the long-term loss probability of the buffer. Thus, even though the channel itself has memory, the capacity depends only on the stationary loss probability of the buffer. The main focus of this correspondence is on the latter case. When the receiver does not know the buffer state, this leads to the study of deletion channels, where symbols are randomly dropped and a subsequence of the transmitted symbols is received. In deletion channels, unlike erasure channels, there is no side-information about which symbols are dropped. We study the achievable rate for deletion channels, and focus our attention on simple (mismatched) decoding schemes. We show that even with simple decoding schemes, with independent and identically distributed (i.i.d.) input codebooks, the achievable rate in deletion channels differs from that of erasure channels by at most $H_0(p_d) - p_d \log \frac{K}{K-1}$ bits, for $p_d < 1 - K^{-1}$, where p_d is the deletion probability, K is the alphabet size, and $H_0(\cdot)$ is the binary entropy function. Therefore, the difference in transmission rates between the erasure and deletion channels is not large for reasonable alphabet sizes. We also develop sharper lower bounds with the simple decoding framework for the deletion channel by analyzing it for Markovian codebooks. Here, it is shown that the difference between the deletion and erasure capacities is even smaller than that with i.i.d. input codebooks and for a larger range of deletion probabilities. We also examine the noisy deletion channel where a deletion channel is cascaded with a symmetric discrete memoryless channel (DMC). We derive a single letter expression for an achievable rate for such channels. For the binary case, we show that this result simplifies to $\max(0, 1 - [H_0(\theta) + \theta H_0(p_e)])$ where p_e is the cross-over probability for the binary symmetric channel.**

***Index Terms*—, Common subsequences, channel capacity, deletion channels, erasure channels.**

I. Introduction

In a packet-switched communication network, such as the Internet, the source of a session encodes information in a set of packets, which are transported as independent units through a set of links to reach their destination. A packet reaches its destination if there exists a route to the destination, and if there is buffer space available at every node along the path followed by this packet. The context motivating our problem formulation is that of a packet-switched communication network where packet flows *share* resources, which gives rise to random packet losses due to the randomness of packet arrivals to buffers in the network, and the effects of congestion control protocols such as TCP that regulate the packet generation rate of flows. We assume that a) a packet either reaches its destination or is lost completely, and b) the original order of packets is conserved. We propose an abstraction for this finite buffer channel, and examine reliable transmission rates over this channel. We ignore the possibility of information transmission through

Manuscript received September 9, 2004; revised September 6, 2005.
The authors are with the School of Computer and Communication Sciences (I&C), Ecole Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland (e-mail: suhas.diggavi@epfl.ch; matthias.grossglauser@epfl.ch).
Communicated by M. Médard, Associate Editor for Communications.
Digital Object Identifier 10.1109/TIT.2005.864445