Pseudodifferential operators and Banach algebras in mobile communications

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Abstract

We study linear time-varying operators arising in mobile communication using methods from time-frequency analysis. We show that a wireless transmission channel can be modeled as pseudodifferential operator H_σ with symbol σ in $\mathcal{F}L^1_w$ or in the modulation space $M^{\infty,1}_w$ (also known as weighted Sjöstrand class). It is then demonstrated that Gabor Riesz bases $\{\varphi_{m,n}\}$ for subspaces of $L^2(\mathbb{R})$ approximately diagonalize such pseudodifferential operators in the sense that the associated matrix $[\langle H_\sigma \varphi_{m',n'}, \varphi_{m,n}\rangle]_{m,n,m',n'}$ belongs to a Wiener-type Banach algebra with exponentially fast off-diagonal decay. We indicate how our results can be utilized to construct numerically efficient equalizers for multicarrier communication systems in a mobile environment.

1 Introduction

Mobile wireless communication is an extremely rapidly growing sector of the telecommunication industry, which is underscored by more than a billion cell phone users worldwide predicted for the near future. However the design of reliable mobile wireless communication systems that can provide high data rates to many users poses a number of new technical challenges. In this paper we show that pseudodifferential operators and Banach algebras, two concepts

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that at first sight seem to have little in common with cellular phones, can in fact provide valuable insight into some aspects of the design of mobile communication systems.

Transmission over mobile wireless channels is impaired by *multipath propagation* and Doppler effect [34]. Signal multipath occurs when the transmitted signal arrives at the receiver via multiple propagation paths with different *delays* and different attenuation due to reflections from the ground and surrounding structures. This *delay spread* leads to *time dispersion*. That means the transmission pulses are spread out in time, which can cause *intersymbol interference*, i.e., interference between successive (blocks of) data symbols that were originally separated in time. Furthermore, relative motion between transmitter and receiver as well as moving objects in the channel paths result in Doppler effect on each of the multipath components. Thus a "pure frequency" located at ω spreads over a finite bandwidth $[\omega-\omega_{\max},\omega+\omega_{\max}]$ where $\omega_{\max}$ is the maximal Doppler shift to be defined later. These Doppler shifts can cause *interchannel interference*, i.e., interference between (blocks of) data symbols that were originally separated in frequency.

If these channel distortions are left uncompensated at the receiver they will cause high error rates. *Equalization* is the process of compensating or reducing intersymbol interference and interchannel interference in the received signal. Efficient equalization methods play a central role in the design of modern wireless communications systems.

We first analyze wireless transmission channels from the viewpoint of pseudodifferential operator theory. We will show that the mobile radio channel can be modeled as pseudodifferential operator H_σ whose symbol belongs to $\mathcal{F}L^1_w(\mathbb{R}^2)$ or alternatively to a specific modulation space, also known as weighted Sjöstrand's class. For such pseudodifferential operators H_σ we derive an approximate diagonalization via Gabor systems $\{\varphi_{k,l}\}_{k,l\in\mathbb{Z}}$ for properly chosen window φ. More specifically, we show that the matrix $[\langle H_\sigma\varphi_{k,l},\varphi_{k',l'}\rangle]_{k,l,k',l'\in\mathbb{Z}}$ belongs to a Wiener-type Banach algebra of matrices with very fast off-diagonal decay. This approximate diagonalization provides useful insight into the design of transmission pulses and equalizers for mobile wireless communication systems. Part of the research presented in this paper has recently already found practical application in the modem design for short radio wave communications.

Ignited by the influential work of Coifman and Meyer on Calderon-Zygmund operators and wavelets [33] approximate diagonalization of pseudodifferential operators has become an important theme in harmonic analysis [36].

This research, while intriguing and exciting, has been mainly of theoretical nature. This paper bridges the gap between theory and practice by showing that recent theoretical work pseudodifferential operators—in particular Sjöstrand's seminal work [38, 39]—combined with Banach algebra theory and time-frequency analysis has direct application in the area of telecommunications.

1.1 Basic concepts from time-frequency analysis

We introduce a few tools from time-frequency analysis used throughout this paper. We follow mostly the notation of the highly recommended book [15] and also refer to this source for more details about the tools presented in this section.

The Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ is formally defined as

$$(\mathcal{F}f)(\omega) = \hat{f}(\omega) = \int\limits_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t}.$$

$\mathcal{F}_k f$ denotes the Fourier transform of f with respect to its k-th variable. For instance, for $f(x,y) \in L^2(\mathbb{R}^2)$ we formally define

$$(\mathcal{F}_1 f)(\omega, y) = \int\limits_{\mathbb{R}} f(x,y) e^{-2\pi i x \omega}\, dx.$$

Convolution of two functions f, g is denoted by $f * g$. For $x, \omega \in \mathbb{R}$ we define the *translation operator* T_x and the *modulation operator* M_ω by

$$T_x f(t) = f(t-x), \qquad M_\omega f(t) = e^{2\pi i \omega t} f(t). \tag{1}$$

The cross Wigner distribution of two functions $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$\mathcal{W}(f,g)(t,\omega) = \int\limits_{\mathbb{R}^d} f(t + \frac{s}{2}) \overline{g(t - \frac{s}{2})} e^{-2\pi i \omega \cdot s}\, ds. \tag{2}$$

The cross ambiguity function of $f, g \in L^2(\mathbb{R}^d)$ is

$$\mathcal{A}(f,g)(t,\omega) = \int\limits_{\mathbb{R}^d} f(x + \frac{t}{2}) \overline{g(x - \frac{t}{2})} e^{-2\pi i \omega \cdot x}\, dx. \tag{3}$$

For $f = g$ definitions (2) and (3) reduce to the usual Wigner distribution and ambiguity function of f, respectively.

Let $f \in L^2(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}$ denotes the Schwartz space. The *short-time Fourier transform* (STFT) of f with respect to the *window* g is defined by

$$\mathcal{V}_g f(s,\omega) = \int\limits_{\mathbb{R}^d} f(t) g(s-t) e^{-2\pi i \omega \cdot t} dt, \qquad (s,\omega) \in \mathbb{R}^{2d}. \tag{4}$$

A simple calculation shows that

$$\mathcal{V}_g f(s,\omega) = (\hat{f} * M_{-s}\hat{g}^*)(\omega). \tag{5}$$

We assume for convenience that $\|g\|_2 = 1$ in which case $\mathcal{V}_g$ becomes a unitary transform.

A Gabor system consists of functions of the form

$$\varphi_{k,l}(t) = e^{2\pi i l b t}\varphi(t - ka), \;\; (k,l) \in \mathbb{Z} \times \mathbb{Z} \tag{6}$$

where $\varphi \in L^2(\mathbb{R})$ is a given *window*, and $a, b > 0$ are the time- and frequency-shift parameters [15]. We denote this system by (φ, a, b). The associated analysis operator or coefficient operator $C : L^2(\mathbb{R}) \mapsto \ell^2(\mathbb{Z} \times \mathbb{Z})$ is defined as

$$Cf = \{\langle f, \varphi_{k,l} \rangle\}_{(k,l)\in\mathbb{Z}\times\mathbb{Z}}. \tag{7}$$

C is also known as *Gabor transform* and is obviously just an STFT that has been sampled at the time-frequency lattice $a\mathbb{Z} \times b\mathbb{Z}$. The adjoint C^*, which is also known as synthesis operator, can be expressed as

$$C^*\{c_{k,l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} = \sum_{k,l} c_{k,l}\varphi_{k,l} \qquad \text{for } \{c_{k,l}\}_{k,l\in\mathbb{Z}\times\mathbb{Z}} \in \ell^2(\mathbb{Z}\times\mathbb{Z}). \tag{8}$$

We define a *weight function* w on $\mathbb{R}^d$ as a continuous, non-negative, even function which is submultiplicative, i.e., $w(s+t) \le w(s)w(t)$ for all $s, t \in \mathbb{R}^d$. We say that w satisfies the Gelfand-Raikov-Shilov (GRS) condition [14], if

$$\lim_{n\to\infty} w(nt)^{\frac{1}{n}} = 1 \qquad \text{for all } t \in \mathbb{R}^d. \tag{9}$$

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the *modulation space* $M^{p,q}_w(\mathbb{R}^d)$ if

$$\|f\|_{M^{p,q}_w} := \left(\int\limits_{\mathbb{R}^d} \Big(\int\limits_{\mathbb{R}^d} |\mathcal{V}_g f(t,\omega)|^p w(t,\omega)^p dt \Big)^{q/p} d\omega \right)^{1/q} \tag{10}$$

is finite, cf. [9]. Here w is some weight function on $\mathbb{R}^{2d}$. It can be shown that the norm is independent of the choice of $g \in \mathcal{S}(\mathbb{R}^d)$, see [9, 15]. For weight functions of at most polynomial growth $M^{p,q}_w$ is a subspace of $\mathcal{S}'(\mathbb{R}^d)$, otherwise it is a subspace of ultra-distributions. We refer the reader to Chapter 11.4 in [15] for the necessary adjustments for the latter case.

Furthermore we will make use of *Wiener amalgam spaces* [15]. A measurable function F on $\mathbb{R}^{2d}$ belongs to the amalgam space $W(L^{p,q}_w)$ if the sequence of local suprema

$$a_{k,n} = \operatorname*{ess\,sup}_{x,\omega \in [0,1]^d} |F(x+k, \omega+n)| = \|F \cdot T_{(k,n)} \chi_{[0,1]^{2d}}\|_\infty$$

belongs to $\ell^{p,q}_w$, where w is some weight function. The norm on $W(L^{p,q}_w)$ is $\|F\|_{W(L^{p,q}_w)} = \|\{a_{k,n}\}_{k,n\in\mathbb{Z}}\|_{\ell^{p,q}_w}$. Amalgam spaces are convenient when dealing with sampling of functions [1] and it is exactly this property which we will utilize.

We also introduce a matrix algebra called the Baskakov-Sjöstrand-algebra.

Definition 1.1 *Let $A = [A_{i,j}]_{i,j \in \mathcal{I}\times\mathcal{I}}$ be a matrix, where $\mathcal{I}$ is some index set. Let w be a weight function, which satisfies the GRS condition. The Baskakov-Sjöstrand matrix algebra $\mathcal{M}_w$ consists of all matrices A for which*

$$\|A\|_{\mathcal{M}_w} := \sum_{k\in\mathcal{I}} \sup_{i-j=k} |A_{i,j}| w(k) < \infty. \tag{11}$$

It is easy to verify that

$$\sum_{k\in\mathcal{I}} \sup_{i-j=k} |A_{i,j}| w(k) = \inf_{a \in \ell^1_w(\mathcal{I})} \{|A_{i,j}| \le a(i-j), i, j \in \mathcal{I}\}. \tag{12}$$

We will make use of the following result which was derived by Baskakov in [3] and for the case $w = 1$ independently by Sjöstrand [39].

Theorem 1.2 *Let $A = [A_{i,j}]_{i,j\in\mathcal{I}}$ be a matrix that is invertible on $\ell^2(\mathcal{I})$ and let $A \in \mathcal{M}_w$. If w satisfies the GRS-condition* (9) *then $A^{-1} \in \mathcal{M}_w$.*

2 Mobile wireless channels: multipath propagation and Doppler effect

The presence of reflecting objects and scatterers in the wireless channel results in multiple versions of the transmitted signal that arrive at the receiving antenna with different delays, different spatial orientations and different amplitudes [34]. Multipath propagation leads to *time dispersion*, since the transmission pulses are spread out in time. This *delay spread* can cause *intersymbol interference* (ISI), i.e., interference between successive (blocks of) data symbols. Furthermore, relative motion between transmitter and receiver results in *Doppler spread* due to different Doppler shifts on each of the multipath components. This *Doppler spread* causes *frequency dispersion*, which can lead to interchannel interference (ICI), i.e., interference between transmission pulses that were originally confined to disjoint frequency intervals. ISI is usually much more pronounced than ICI. However with the current development of mobile communication systems with larger and larger bandwidth, Doppler effect will become more severe, as it increases with frequency. Doppler spread is also of considerable concern in underwater communications and satellite communications.

Let us briefly consider the simple case of a linear *time-invariant* communication channel H, as it is arising e.g., in wired or fixed wireless (i.e., the location of transmitter and receiver is fixed) communication systems. In this case H can be expressed as

$$y(t) = (Hx)(t) = (h * x)(t) = \int_{-\infty}^{+\infty} h(t-s)x(s)ds, \tag{13}$$

where h is the *impulse response* between transmitter and receiver, i.e., $h(t)$ is the response at time t to a unit pulse (a δ-distribution) transmitted at time 0. Here x denotes the transmitted signal and the received signal is denoted by y. In (13) above we have ignored additive noise and we will do so throughout the paper, since additive noise has no influence on the subsequent analysis.

Considering, as usual, a baseband communication system [34], i.e.,

$$\hat{x}(\omega) \in [-\Omega, \Omega], \qquad \text{for some } \Omega > 0,$$

we can express (13) equivalently as

$$y(t) = \int_{-\infty}^{+\infty} \hat{h}(\omega)\hat{x}(\omega)e^{2\pi i\omega t}\, d\omega \tag{14}$$

where $\hat{h}(\omega)$ is also known as the *transfer function* of the channel. Thus multipath propagation has the effect of attenuating different frequency components of the transmitted signal differently, which is also referred to as *frequency-selective fading*. Equation (14) makes it easy — at least in theory — to remove the frequency-selective fading effect of the channel via deconvolution. If $\hat{h}(\omega) \neq 0$ we can write

$$\frac{\hat{y}(\omega)}{\hat{h}(\omega)} = \hat{x}(\omega). \tag{15}$$

This also explains the term *equalization*, since the goal is to make the product of the "equalizer" $g(\omega) := 1/\hat{h}(\omega)$ with $\hat{h}(\omega)$ equal to 1 for $\omega \in [-\Omega, \Omega]$.

If either transmitter or receiver is moving the relative location of reflectors in the transmission path is varying with time and so is the impulse response h. The input-output relation of a general mobile radio channel can be represented by the linear time-varying system [22]

$$y(t) = (Hx)(t) = \int_{-\infty}^{+\infty} h_t(s)x(t-s)ds, \tag{16}$$

where h_t is the impulse response at time t. In contrast to a time-invariant system where $h_0 = h_t$ for all t, the impulse response is now modeled by a time-dependent family of functions (tempered distributions) h_t. By interpreting h_t as function of two variables, i.e., $h_t(s) = h(t, s)$ and renaming variables, we can rewrite (16) as

$$(Hx)(t) = \int_{-\infty}^{+\infty} h(t, t-s)x(s)ds. \tag{17}$$

Formula (17) is the analog of (13) with $h(t)$ replaced by the time-varying impulse response $h(t, s)$. We now consider the analog of (14). Introducing the notation $\sigma = \mathcal{F}_2 h$, i.e.,

$$\sigma(t, \omega) = \int_{-\infty}^{+\infty} h(t, s)e^{-2\pi i\omega s}\, ds, \tag{18}$$

we obtain

$$(Hx)(t) = \int_{-\infty}^{+\infty} \sigma(t, \omega)\hat{x}(\omega)e^{2\pi i\omega t}d\omega. \tag{19}$$

Here σ can be interpreted as time-varying transfer function. In this formulation the operator $H = H_\sigma$ becomes a pseudodifferential operator with *Kohn-Nirenberg symbol* σ. The mapping $\sigma \mapsto H_\sigma$ is usually called the *Kohn-Nirenberg correspondence*. The representation (19) was first introduced and analyzed by Zadeh [47]. We write $H_\sigma \in \mathrm{Op}(S)$ if $\sigma \in S$ for some function space S.

Alternatively we can write [13, 15]

$$H_\sigma x = \iint \hat{\sigma}(\eta, u) M_\eta T_{-u} x du d\eta. \tag{20}$$

The function $\hat{\sigma}$ is called the *spreading function* in communications engineering [22], since it describes how much the transmitted signal x gets "spread out" in time and frequency due to delay spread and Doppler spread and we will adopt this terminology (even though $\hat{\sigma}$ may often be a distribution).

Clearly, for the identity operator $H_\sigma = Id$ we have $\hat{\sigma} = \delta_{0,0}$ and $\sigma \equiv 1$. If H_σ is a time-invariant channel then $\hat{\sigma}(\eta, u) = \delta_\eta h_0(u)$ and $\sigma = 1 \otimes \hat{h}_0$. These special cases already show that—unlike common engineering practice—we have to allow (ultra-)distributional symbols to model radio channels rigorously and with sufficient generality.

3 Mobile wireless channels and pseudodifferential operators

In the study of pseudodifferential operators one usually proceeds by analyzing properties of the pseudodifferential operator based on certain growth or smoothness conditions of its symbol σ. In the following we want to identify an appropriate symbol class for those pseudodifferential operators that arise in wireless communications. Therefore we will study the effects of multipath and Doppler spread in more detail.

While in ideal free space propagation the signal energy drops quadratically with traveled distance, in realistic mobile environments the signal power loss is more severe. A classical model that quantifies the phenomenon of multipath propagation yields an exponentially decaying power delay profile for the impulse response [22]. That is, for fixed t there exist constants $a, c > 0$ such that

$$|h(t, s)| \leq c e^{-a|s|}. \tag{21}$$

This exponential decay profile can be easily understood by observing that every time the signal is reflected from an object only a certain percentage, say $100r\%$ (with $0 \leq r \leq 1$) of the signal energy is in fact reflected and the remaining $100(1-r)\%$ of the enery are absorbed. After n reflections (assuming for simplicity the same energy loss in each bounce) the signal energy is reduced to $100r^n\%$ of its original energy[1].

Now let us consider the Doppler effect. Recall that the Doppler shift ω_d is given by

$$\omega_d = \frac{v}{\lambda}\cos\theta, \tag{22}$$

where v is the velocity v of the object, θ is the angle between the direction of the moving object and the direction of arrival of the radio wave, and λ is the wave length. Doppler shift will be positive or negative depending on whether the mobile receiver is moving toward or away from the base station. Due to multipath propagation we observe different Doppler shifts on each of the multipath components. A widely used model due to Jakes [22] predicts a U-shaped "Doppler spectrum", however in reality significant deviations from this model have been observed. We refrain from a definition of the term "Doppler spectrum" and an analysis of its shape, since for our purposes we only need the following observation. In a baseband communication model equation (22) implies that the spreading function $\hat{\sigma}(\eta, u)$ is always compactly supported in $[-\omega_{\max}, \omega_{\max}]$ with respect to η, where $\omega_{\max} = \frac{v}{\lambda}$ denotes the maximal Doppler shift. Let $\mathcal{I}$ denote the coordinate reflection operator $\mathcal{I}f(x,y) = f(x,-y)$. By combining the support condition on $\hat{\sigma}$ with (21) and using the relation

$$\hat{\sigma} = \mathcal{F}_1\mathcal{I}h \tag{23}$$

we conclude that the symbol σ of a mobile wireless channel H_σ satisfies

$$\hat{\sigma}(\eta, u) = 0 \text{ if } |\eta| > \omega_{\max} \text{ and } |\hat{\sigma}(\eta, u)| \leq ce^{-a|u|}, \text{ for some } c, a > 0. \tag{24}$$

While this property is not new to engineers (usually stated under the restriction to Hilbert-Schmidt operators), it may be less familiar among mathemati-

[1]For narrowband signals it can be assumed that the energy loss is constant across the bandwidth. However this may no longer be true for signals with large bandwidth as e.g. in currently developed ultrawideband systems, where the fluctuation in absorbed energy changes slowly with frequency. This frequency-selective absorption can also be modeled as convolution with a rapidly decaying function and therefore does not change the overall exponential decay profile.

cians. Therefore and because it is essential for the following considerations we have included the derivations that have lead us to (24).

These quantitative characterizations of multipath propagation and Doppler effect in connection with the representation (20) have inspired engineers to introduce the class of *underspread operators.* Since $|h(t,s)|$ decays exponentially in s, one can argue that $\hat{\sigma}(\eta, u)$ is approximately compactly supported in u. In particular, we may define the *maximal delay spread* as the time delay at which the energy of the impulse response stays below a predefined level (such as the noise level). Together with the compact support of the Doppler spectrum this leads to a spreading function $\hat{\sigma}(\eta, u)$ that is approximately compactly supported in the interval $[-\tau_{\max}, \tau_{\max}] \times [-\omega_{\max}, \omega_{\max}]$. In case $\hat{\sigma}$ is truly compactly supported with $\tau_{\max} \cdot \omega_{\max} < 1$ the operator H_σ is called *underspread.* In wireless communications one often assumes that $\tau_{\max} \cdot \omega_{\max} \ll 1$. Underspread operators have been analyzed by time-frequency methods in the pioneering work of Kozek [25, 26, 27, 28] and also in [31], however almost exclusively with the restriction to Hilbert-Schmidt operators. This restriction is quite severe, both from a practical and a theoretical perspective since it excludes the case where H_σ is invertible and thus one of the most important cases, in particular since the condition $\tau_{\max} \cdot \omega_{\max} \ll 1$ includes the practically relevant case when H_σ is just a small perturbation of the identity. Furthermore, it is easy to see that neither underspread operators nor pseudodifferential operators whose symbol satisfies (24) form a Banach algebra. While the Banach algebra property is not a *must*, it is a quite convenient and powerful property, that often leads to important structural insight.

Thus, instead of projecting H_σ into a smaller space (such as underspread operators) we will embed H_σ (or rather its symbol) into somewhat larger spaces and by doing so we will obtain much richer algebraic properties and deeper analytical insight.

We have the following simple, but useful, observation.

Lemma 3.1 *Any operator H_σ whose symbol satisfies* (24) *belongs to the Banach algebra* $\mathrm{Op}(\mathcal{F}L^1_w(\mathbb{R}^2))$ *where* $w(\eta, u) = e^{|\eta|^b + |u|^b}$ *with* $0 < b < 1$.

Proof: It is trivial to see that $H_\sigma \in \mathrm{Op}(\mathcal{F}L^1_w(\mathbb{R}^2))$. It is well-known that composition of two operators H_σ, H_τ corresponds to twisted convolution of their spreading functions [13, 15], i.e., $H_\sigma \circ H_\tau = H_{\mathcal{F}^{-1}(\hat{\sigma} \natural \hat{\tau})}$, where

$$(\hat{\sigma} \natural \hat{\tau})(\eta, u) = \iint\limits_{\mathbb{R}^2} \hat{\sigma}(\eta', u') \hat{\tau}(\eta - \eta', u - u') e^{\pi i (\eta u' - \eta' u)} \, d\eta' du'. \qquad (25)$$

We use (25) and note that twisted convolution is dominated by ordinary convolution $*$. Since $(L^1_w, *)$ forms a Banach algebra, cf. [35], so does $(L^1_w, \natural)$. $\square$

Above we have chosen $b < 1$ instead of $b = 1$, so that $\sigma \in \mathcal{F}L^1_w(\mathbb{R}^2)$ simultaneously for *all* σ.

We can also embed H_σ into a somewhat larger space, known as *Sjöstrand's class*, a very interesting symbol space which has recently attracted the attention of harmonic analysts [39, 6, 45, 16]. Sjöstrand's class can be seen as a member of the family of *modulation spaces*. Modulation spaces have been used as symbol classes for pseudodifferential operators in e.g. [44, 18, 15, 29, 45, 21]. By using the modulation space $M^{p,q}_w$ defined in (10) as symbol class we characterize the time-frequency contents of the symbol σ via the decay properties of its STFT, i.e., by considering $\|\mathcal{V}_\Psi \sigma\|_{L^{p,q}_w}$. Since the STFT of a symbol $\sigma(t, \omega)$ with $(t, \omega) \in \mathbb{R}^2$ is a function on $\mathbb{R}^4$ some care has to be taken in the notation in order to avoid confusion of the various univariate and bivariate time- and frequency variables. To keep our notation manageable, we will write $X := (t, \omega)$ for the "time variable"and $\Omega := (\eta, u)$ for the "frequency variable" in the definition of the STFT in (4).

The weighted *Sjöstrand class* is $M^{\infty,1}_w$. Explicitly, $\mathrm{Op}(M^{\infty,1}_w(\mathbb{R}^2))$ is the class of pseudodifferential operators whose symbol σ satisfies

$$\|\sigma\|_{M^{\infty,1}_w} = \int\limits_{\mathbb{R}^2} \sup_{X \in \mathbb{R}^2} |\mathcal{V}_\Psi \sigma(X, \Omega)| \, w(\Omega) d\Omega < \infty \tag{26}$$

with $\Psi \in \mathcal{S}(\mathbb{R}^2)$.

The following theorem provides a link between mobile radio channels and $M^{\infty,1}_w$.

Theorem 3.2 *Let H_σ satisfy* (24). *Then $H_\sigma \in \mathrm{Op}(M^{\infty,1}_w(\mathbb{R}^2))$ where the weight function is given by*

$$w(\Omega) = e^{|\Omega|^\alpha}, \qquad \text{with } \alpha < 1. \tag{27}$$

Proof: We have to show that

$$\int \sup_X |\mathcal{V}_\Psi \sigma(X, \Omega)| w(\Omega) d\Omega < \infty, \tag{28}$$

where Ψ is a function in $\mathcal{S}(\mathbb{R}^2)$ such that

$$|\Psi(X)| \le c_1 e^{-c_2|X|}, \qquad |\hat{\Psi}(\Omega)| \le c_3 e^{-c_4|\Omega|}, \tag{29}$$

with constants $c_1, c_2, c_3, c_4 > 0$ (e.g., Ψ can be a two-dimensional Gaussian).

By (24) $\hat{\sigma}(\eta, u)$ is compactly supported in η and has exponential decay in u. Thus there exist constants $C, \alpha > 0$ such that

$$|\hat{\sigma}(\eta, u)| \le C e^{-\alpha(|\eta|+|u|)}, \qquad (\eta, u) \in \mathbb{R}^2. \tag{30}$$

Using (5) we estimate

$$\sup_X |\mathcal{V}_\Phi \sigma(X, \Omega)| = \sup_X |(\hat{\sigma} * M_{-X}\hat{\Psi}^*)(\Omega)| \le (|\hat{\sigma}| * |\hat{\Psi}^*|)(\Omega), \tag{31}$$

Furthermore

$$\begin{aligned}\int_{\mathbb{R}^2} \big(|\hat{\sigma}| * |\hat{\Psi}^*|\big)(\Omega) w(\Omega) d\Omega &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\hat{\sigma}(\Omega - \Gamma)||\hat{\Psi}^*(\Gamma)| w(\Omega - \Gamma) w(\Gamma) d\Gamma d\Omega \\ &\le \int_{\mathbb{R}^2} |\hat{\Psi}^*(\Gamma)| w(\Gamma) \int_{\mathbb{R}^2} |\hat{\sigma}(\Omega - \Gamma)| w(\Omega - \Gamma) d\Omega d\Gamma < \infty\end{aligned} \tag{32}$$

where we have used the submultiplicativity of the weight function, (24) and (29). Finally, (31) and (32) imply (28) and the proof is complete. ❑

Sjöstrand has shown that pseudodifferential operators with symbol $\sigma \in M^{\infty,1}$ form a Wiener-type Banach algebra. Using an entirely different approach consisting of an ingenious combination of time-frequency analysis and Banach algebra theory Gröchenig [17] was able to extend Sjöstrand's result to the case $M^{\infty,1}_w$ for weight functions w that satisfy the GRS condition (9). From an abstract point of view it is this Wiener algebra property that paves the way for numerically efficient equalizers in mobile communications, as described in the next section.

4 Equalization, approximate diagonalization, and Banach algebras

Up to now we have analyzed the mobile radio channel and how it affects the transmission of signals. Now it is time to have a closer look at the formation of the transmission signal x itself and how the receiver may extract the transmitted information from the received signal $H_\sigma x$.

We consider a *multicarrier communication system*. This means, simply spoken, that the available transmission bandwidth is not occupied by a single transmission pulse φ, but by a set of transmission pulses $\{\varphi_l\}_{l=0}^{N-1}$, where the index l is usually related to the carrier frequency of φ_l. A typical choice for φ_l is $\varphi_l(t) = \varphi(t)e^{2\pi i l b t}$, where φ is some prototype transmission pulse with well-localized Fourier transform, and $0 < b \in \mathbb{R}$ is referred to as *carrier separation*.

Let us assume that $\{c_n\}_{n\in\mathbb{Z}}$ are the data symbols to be transmitted. Typically the c_n are chosen from some "finite alphabet" $\mathbb{A}$, such as the p-th roots of unity, in which case $\{c_n\}_{n\in\mathbb{Z}} \notin \ell^2(\mathbb{Z})$. However since in practice any transmitted data stream has finite length, we will assume $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. We rearrange the data into data blocks of length N, $\{c_{k,l}\}_{l=0}^{N-1}$ for $k \in \mathbb{Z}$. A data block $\{c_{k,l}\}_{l=0}^{N-1}$ is also called *data symbol* in engineering jargon (but the term "symbol" in this context has obviously nothing to do with the "symbol" σ of H_σ).

Considering a baseband model and using $\varphi_l(t) = \varphi(t)e^{2\pi i l b t}$, the signal emitted at the transmitter can be written as

$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{N-1} c_{k,l}\varphi(t-ka)e^{2\pi i l b t}, \tag{33}$$

where $0 < a \in \mathbb{R}$ is called *symbol period*[2], i.e., the time between the transmission of two data symbols $\{c_{k,l}\}_{l=0}^{N-1}$ and $\{c_{k+1,l}\}_{l=0}^{N-1}$. It is easy to see that the possibility of recovery the discrete data $c_{k,l}$ requires the set of functions $\{\varphi_{k,l}\}$ to be linearly independent. We will furthermore assume for convenience that the $\varphi_{k,l}$ are mutually orthonormal.

This setup corresponds to a *pulse-shaping orthogonal frequency division multiplex system* (OFDM) with pulse shape φ, cf. [12, 20, 27, 43]. If we let $N \to \infty$ and consider an infinite number of subcarriers then the set of functions $\{\varphi_{k,l}\}_{k,l\in\mathbb{Z}}$ in (33) corresponds also to a Gabor system (φ, a, b) as defined in (6). We know from Gabor theory that a necessary condition for a Gabor system $\{\varphi_{k,l}\}_{k,l\in\mathbb{Z}}$ to be linear independent is that $ab \geq 1$ and we will make the assumption $ab \geq 1$ henceforth.

[2]In the communications literature symbol period and carrier separation are often denoted by the letters T and F respectively. However to be consistent with the standard notation in Gabor analysis we use a, b instead of T, F.

The emitted signal passes through the mobile radio channel and arrives at the receiver as $y = H_\sigma x$. While the data $c_{k,l}$ "live" in the discrete world, the signals x and y "live" in the continuous (analog) world. The received signal y is transformed back into the discrete world by computing the inner product of y with the functions $\varphi_{k,l}$, i.e., $d_{k,l} := \langle y, \varphi_{k,l}\rangle$. Using (33) and (16) we formally compute

$$d_{k,l} = \langle H_\sigma x, \varphi_{k,l}\rangle = \sum_{k',l'} c_{k,l} \langle H_\sigma \varphi_{k',l'}, \varphi_{k,l}\rangle, \tag{34}$$

where the interchange of summation and integration is justified under mild smoothness and decay conditions on φ and H_σ. Denoting $c := \{c_{k,l}\}_{k,l\in\mathbb{Z}}$, $d := \{d_{k,l}\}_{k,l\in\mathbb{Z}}$ and $R = [R_{k,l,k',l'}]_{k,l,k',l'\in\mathbb{Z}}$ with $R_{k,l,k',l'} = \langle H_\sigma \varphi_{k',l'}, \varphi_{k,l}\rangle$, we can write (34) as the linear system of equations

$$Rc = d. \tag{35}$$

Thus the attempt to recover the coefficients $c_{k,l}$ boils down to the solution of (35). A *maximum likelihood* equalizer solves $\min \|Rc - d\|_2$ (in presence of additive white Gaussian noise) by calculating $Rc - d$ for all possible choices of $c \in \mathbb{A}$, since $\mathbb{A}$ has finite, albeit large, cardinality [34]. A *minimum mean square error* equalizer computes a (regularized) least squares solution to $Rc = d$ and then picks that element in $\mathbb{A}$ which is closest to the least squares solution. Regardless of which equalization method we use, since R is a potentially very large matrix, our goal is to design the transmission pulses $\varphi_{k,l}$ such that R has a simple (e.g., diagonal, sparse, ...) structure.

We note that setting up the matrix R requires knowledge of H_σ or rather of the coefficients $\langle H_\sigma \varphi_{k',l'}, \varphi_{k,l}\rangle$. Unlike time-invariant channels, which can be identified (at least in principle) by sending a δ-impulse and recording the received impulse response $h = h * \delta$, the identification or estimation of time-varying channels is more involved [32]. In this regard the concept of underspread operators can provide valuable insight [28]. Since a detailed discussion of channel estimation is beyond the scope of this paper, we will assume that H_σ is known at the receiver.

It is intuitively clear from (34) that the interference of the data symbol $c_{k,l}$ with neighbor symbols will be smaller with better time-frequency localization of $\varphi_{k,l}$ as well as with larger distance between adjacent data symbols in the time-frequency domain, i.e, the larger a and b. Increasing a and/or b results in reduced spectral efficiency, i.e., in a reduced number of bits that can be transmitted per Hertz per second. The observation that functions $\varphi_{k,l}$ with good time-frequency localization are "somehow" the right choice in presence of Doppler spread and delay spread has been made several times in the engineering literature [12, 46, 20, 27]. The following theorems give a rigorous and constructive confirmation of this observation.

Theorem 4.1 *Let $\varphi \in M^{1,1}_w(\mathbb{R})$, where the weight w satisfies the GRS-condition* (9) *and assume that (φ, a, b) is an orthonormal basis for a subspace of $L^2(\mathbb{R})$. Define the weight v by $v(x,y) = w(y,-x)$. If $L_\sigma \in \mathrm{Op}(M^{\infty,1}_v(\mathbb{R}^2))$ then $R \in \mathcal{M}_w$. Furthermore, if H_σ is invertible on $L^2(\mathbb{R})$ and if* $\mathrm{ran}(H_\sigma C^*) = \mathrm{ran}(C^*)$, *where C^* is as defined in* (8), *then $R^{-1} \in \mathcal{M}_w$.*

Proof: Let the unitary operator $\mathcal{U}$ be defined by

$$(\mathcal{U}\sigma)^\wedge(\xi, y) = e^{\pi i y\xi}\hat{\sigma}(\xi, y) =: \hat{\tau}(\xi, y), \tag{36}$$

then the Kohn-Nirenberg symbol and the Weyl symbol are related by [15]

$$H_{\mathcal{F}^{-1}(\widehat{\mathcal{U}\sigma})} = L_\sigma.$$

Modulation spaces are invariant under the operator in (36) that transforms the Kohn-Nirenberg symbol into the Weyl symbol, see Theorem 29 in [11], Lemma 2.1 in [18] and also Proposition 2.1 in [39]. Hence

$$\sigma \in M^{p,q}(\mathbb{R}^{2d}) \Leftrightarrow \mathcal{F}^{-1}(\widehat{\mathcal{U}\sigma}) \in M^{p,q}(\mathbb{R}^2). \tag{37}$$

Relation (37) can be extended to weighted modulation spaces $M^{p,q}_w$. Therefore we can proceed by considering L_σ instead of H_σ.

We set $\alpha = (\alpha_x, \alpha_\xi), \beta = (\beta_x, \beta_\xi)$ with $\alpha_x, \alpha_\xi, \beta_x, \beta_\xi \in \mathbb{R}$. There holds

$$|\langle L_\sigma M_{\beta_\xi} T_{\beta_x}\varphi, M_{\alpha_\xi} T_{\alpha_x}\varphi\rangle| = |\langle \sigma, \mathcal{W}(M_{\alpha_\xi} T_{\alpha_x}\varphi, M_{\beta_\xi} T_{\beta_x}\varphi)\rangle| \tag{38}$$

$$= \Big| \iint \sigma(x,\xi) e^{-\pi i(\alpha_x+\beta'_x)(\alpha_\xi-\beta'_\xi)} e^{-2\pi i x(\alpha_\xi-\beta'_\xi)} e^{2\pi i \xi(\alpha_x-\beta'_x)} \times \tag{39}$$

$$\times\, \mathcal{W}(\varphi,\varphi)(x - \frac{\alpha_x+\beta_x}{2}, \xi - \frac{\alpha_\xi+\beta_\xi}{2})\, dx d\xi \Big| \tag{40}$$

$$= |(T_{\frac{(\alpha_x,\alpha_\xi)+(\beta_x,\beta_\xi)}{2}}\psi \cdot \sigma)^\wedge(\alpha_\xi - \beta_\xi, -(\alpha_x - \beta_x))|, \tag{41}$$

where we have introduced the notation $\psi := \mathcal{W}(\varphi,\varphi)$ and used Proposition 4.3.2(b),(c) in [15]. Furthermore we use that

$$\varphi \in M^{1,1}_w(\mathbb{R}) \Longleftrightarrow \mathcal{W}(\varphi,\varphi) \in M^{1,1}_{1\otimes v}(\mathbb{R}^2), \tag{42}$$

which follows easily by extending Proposition 2.5 in [7] from polynomial weights to weights satisfying the GRS condition. Hence

$$|(T_{\frac{(\alpha_x,\alpha_\xi)+(\beta_x,\beta_\xi)}{2}}\psi\cdot\sigma)^\wedge(\alpha_\xi-\beta_\xi,-(\alpha_x-\beta_x))| = |(T_{\frac{\alpha+\beta}{2}}\psi\cdot\sigma)^\wedge(\mathcal{J}(\alpha-\beta))| \tag{43}$$

$$= |\mathcal{V}_\psi\sigma\big(\frac{\alpha+\beta}{2},\mathcal{J}(\alpha-\beta)\big)|. \tag{44}$$

Let $U\in L^1_w(\mathbb{R}^2)$. There holds

$$|(T_{\frac{\alpha+\beta}{2}}\psi\cdot\sigma)^\wedge(\mathcal{J}(\alpha-\beta))| \le U(\alpha-\beta) \tag{45}$$

$$\Leftrightarrow\qquad |\mathcal{V}_\psi\sigma\big(\frac{\alpha+\beta}{2},\alpha-\beta\big)| \le (U\circ\mathcal{J}^{-1})(\alpha-\beta) \tag{46}$$

$$\Leftrightarrow\qquad |\mathcal{V}_\psi\sigma(\alpha',\beta')| \le (U\circ\mathcal{J}^{-1})(\beta'). \tag{47}$$

By definition (26) and (42), relation (47) is equivalent to $\sigma\in M^{\infty,1}_v$. Thus there exists indeed a $U\in L^1_w(\mathbb{R}^2)$ such that (45) holds. Now we sample $\mathcal{V}_\psi\sigma$ at the time-frequency lattice $\{(na,mb)\}_{n,m\in\mathbb{Z}}$, i.e., we set $(\alpha_x,\alpha_\xi)=(ka,lb)$, $(\beta_x,\beta_\xi)=(k'a,l'b)$, $k,k'l,l'\in\mathbb{Z}$. By combining the continuity of $\mathcal{V}_\psi\sigma$ for $\psi\in M^{1,1}_{1\otimes v}$, $\sigma\in M^{\infty,1}_v$ with Theorem 12.2.1 and Proposition 11.1.4 in [15] we conclude that there exists $u\in\ell^1_w(\mathbb{Z}^2)$ such that

$$|R_{k,l,k',l'}| = |\langle L_\sigma M_{l'b}T_{k'a}\varphi, M_{lb}T_{ka}\varphi\rangle| \le u((ka,lb)-(k'a,l'b)), \tag{48}$$

and thus $R\in\mathcal{M}_w$.

Since $R=CH_\sigma C^*$, L^2-invertibility of H_σ combined with the assumption $\mathrm{ran}(H_\sigma C^*)=\mathrm{ran}(C^*)$ and the linear independence of (ψ,a,b) implies that R is invertible on $\ell^2(\mathbb{Z}^{2d})$ and therefore by Theorem 1.2 $R^{-1}\in\mathcal{M}_w$. ❏

Many variations of Theorem 4.1 are possible. For instance, invertibility of R (which is essential for the ability to reconstruct the transmitted data at the receiver) does not require that H_σ is invertible on $L^2(\mathbb{R})$, the assumption that H_σ is invertible on $\mathrm{ran}(C^*)$ is sufficient. This observation is useful for the following reason. Any practical communication system has to operate on a specified *finite* frequency band, B, say. Thus the transmission functions $\varphi_{k,l}$ and the (finite) index set for the frequency index l have to be chosen such that $\mathrm{supp}(\hat\varphi_{k,l})$ is essentially contained in B for all l. In this case it would be sufficient for L_σ to be invertible on the corresponding subspace spanned by $\{\varphi_{k,l}\}$. On the other hand the assumption $\mathrm{ran}(H_\sigma C^*)=\mathrm{ran}(C^*)$

is in general not easy to verify. Fortunately for the in practice relevant case of finite-dimensional submatrices of R, one can show under fairly general assumptions on H_σ that these finite matrices are invertible with probability 1. We will discuss this in more detail elsewhere.

Furthermore, instead of $M^{\infty,1}_w$ we could consider a somewhat smaller symbol class (however with less algebraic structure). As an example we state the following corollary, which follows immediately from Theorem 4.1 and equations (31) and (32).

Corollary 4.2 *Let $\varphi \in M^{1,1}_w(\mathbb{R})$ and assume that (φ, a, b) is an orthonormal basis for a subspace of $L^2(\mathbb{R})$. Define the weight v by $v(x, y) = w(y, -x)$ and assume $H_\sigma \in \mathrm{Op}(\mathcal{F}L^1_v(\mathbb{R}^2))$. Let $R = [\langle H_\sigma \varphi_{k',l'}, \varphi_{k,l}\rangle]_{k,k' \in \mathbb{Z}, |l|,|l'| \le N}$, where $N \in \mathbb{N}$. If w satisfies the GRS-condition, then $R \in \mathcal{M}_w$. If H_σ is invertible on $\mathrm{ran}(\{\varphi_{k,l}\}_{k,k' \in \mathbb{Z}, |l|, l|' \le N})$ and $\mathrm{ran}(H_\sigma C^{ast}) = \mathrm{ran}(C^*)$ then $R^{-1} \in \mathcal{M}_w$.*

To complete the picture we need the following result.

Corollary 4.3 *Let $w(t) = e^{\alpha|t|^r}$ with $0 < \alpha, r$ and $r < 1$. Then there exists a $\varphi \in M^{1,1}_w(\mathbb{R})$ for any $a, b \in \mathbb{R}$ with $ab > 1$ such that (φ, a, b) is an orthonormal basis for its closed linear span.*

Proof: By Gabor duality theory constructing an orthonormal system (φ, a, b) is equivalent to constructing a tight Gabor frame $(\varphi, 1/b, 1/a)$, see e.g. [15]. Furthermore if $(g, 1/b, 1/a)$ is a Gabor frame then $(S^{-\frac{1}{2}} g, 1/b, 1/a)$ is a tight Gabor frame, where S denotes the Gabor frame operator. It has been shown in [19] that if $g \in M^{1,1}_w$ then $\varphi := S^{-\frac{1}{2}} g \in M^{1,1}_w$ as long as the weight satisfies the GRS condition. Finally Gabor frames with $g \in M^{1,1}_w(\mathbb{R})$ exist for any $1/(ab) < 1$, for instance let g be a Gaussian [30, 37] or a hyperbolic secant [24].

❏

Extending work of Sjöstrand, Gröchenig has shown in [17, 16] that

$$L_\sigma \in \mathrm{Op}(M^{\infty,1}_w(\mathbb{R}^{2d})) \Leftrightarrow R \in \mathcal{M}_w. \tag{49}$$

While Sjöstrand used "hard analysis" in his proof, Gröchenig's approach is based on classical tools from time-frequency analysis, which makes the "mathematical forces" behind this important result more transparent.

Taking (49) together with Theorem 4.2 we arrive at the following

Corollary 4.4 *Define the weight v by $v(x,y) = w(y,-x)$ and assume that w satisfies the GRS condition. Then*

$$\mathcal{F}L^1_w \subset M^{\infty,1}_v.$$

Remarks: (i) One can actually prove that the entries of R decay strictly exponentially for H_σ whose symbol's decay conditions are given by (24) (i.e., we can set $r = 1$ in Corollary 4.3). In this case R^{-1} has also truly exponential decay, however with a different (in general smaller) exponent and we lose the closedness under inversion of $\mathcal{M}_w$.
(ii) The canonical tight window $\varphi := S^{-\frac{1}{2}}g$ is optimally close to g in the sense that φ solves the optimization problem [23]

$$\min_{\varphi} \|g - \varphi\|_2 \qquad \text{subject to } (\varphi, 1/b, 1/a) \text{ is a tight frame for } L^2(\mathbb{R}).$$

It is an open problem whether $S^{-\frac{1}{2}}g$ also minimizes $\|g - \gamma\|_{M^{1,1}_w}$ among all $(\gamma, 1/b, 1/a)$ that generate a tight Gabor frame.
(iii) The statement $g \in M^{1,1}_w(\mathbb{R}) \Rightarrow S^{-\frac{1}{2}}g \in M^{1,1}_w$ in [19] also includes the very difficult "irrational case" $ab \in \mathbb{R}\backslash\mathbb{Q}$. Gröchenig and Leinert have proven their result by using advanced methods from symmetric Banach algebra theory. The Baskakov-Sjöstrand algebra allows a drastic simplification of their proof. This can be seen by observing that the matrix representing a certain discrete twisted convolution arising in [19] is dominated by an "ordinary" convolution matrix, as it appears inside the parentheses on the right-hand side of (12), and the applying Theorem 1.2.
(iv) Further results on decay properties of dual and tight Gabor frames can be found in e.g. [10, 15, 5, 42]. Reference [42] seems to be the first paper in which the GRS-condition has been used in connection with time-frequency analysis and Gabor frames. Efficient numerical methods to compute φ can be easily derived from the algorithms in [41, 23], see also [43].

Since the entries of R decay exponentially fast off the diagonal, we can compute the matrix-vector multiplication Rc as well as the solution to the linear system of equations $Rd = c$ in (35) numerically efficiently via sparse matrix techniques. Moreover, since R^{-1} can also be approximated by a sparse matrix, the computation of a data coefficient $d_{k,l} = \sum_{k',l'} R^{-1}_{k,l,k',l'} c_{k',l'}$ involves only a few non-zero coefficients. In engineering terms this corresponds to the desired case of an equalizer with only a *few taps*.

Let us compare this situation to current OFDM systems, which essentially use $\varphi(t) = \chi_{[0,c]}$ with $\tau_{\max} < c < a$ and $b = 1/a$, where $\tau_{\max}$ is the maximal effective delay spread and χ denotes the indicator function. The choice $\tau_{\max} < c$ corresponds to inserting a *guard interval* between two adjacent data symbols, cf [12]. While this approach works well in fixed wireless communications and is a key ingredient of several industry standards in wireless communications, it also has a few serious drawbacks. This approach is very sensible to Doppler spread, which is not avoidable in mobile wireless communications (even though Doppler spread is significantly smaller than delay spread). We note that $\hat{\varphi}(\omega) = \mathrm{sinc}_c(\omega)$. While properly spaced sinc-functions are mutually orthogonal, already a small perturbation can result in severe interference between adjacent sinc-functions and thus can cause significant interchannel interference. In this case $|R_{k,l,k',l'}| = \mathcal{O}(|(k,l) - (k',l')|)^{-1}$ and R is far from being sparse. Furthermore the slow decay of the sinc-function makes the OFDM receiver vulnerable to frequency offset.

Ideally we would like to construct an OFDM system (φ, a, b) that satisfies the following three requirements simultaneously: (i) $R = \mathrm{Id}$ on $\ell^2(\mathbb{Z}^2)$ if $H_\sigma = \mathrm{Id}$ on $L^2(\mathbb{R})$; (ii) $\varphi \in M_w^{1,1}$; (iii) $ab = 1$, since this choice maximizes the spectral efficiency. As we know from the Balian-Low Theorem [8, 4] these three conditions cannot be satisfied simultaneously. Hence we relax condition (iii) and allow $ab > 1$, which is essentially also done in current OFDM systems that employ a guard interval or cyclic prefix for fixed wireless communications. This loss in spectral efficiency is usually an acceptable price to pay to mitigate ISI/ICI as long as ab is not too large[3]. Time-frequency analysis provides a fairly complete and rigorous framework for pulse-shape design for OFDM for mobile wireless communications.

Another case where the time-frequency localization of pulses is important is to combat "out-of-band leakage" for wired or wireless communications. There the time-localization of the transmission pulses is used as usual to combat intersymbol interference. However the frequency localization is not used to mitigate Doppler spread, but to avoid the "leakage" of the carrier functions $\varphi_{k,l}$ located close to the boundary of the assigned transmission bandwidth into frequency bands reserved for other users or other applica-

[3]Alternatively we can use a slightly different modulation scheme called OQAM-OFDM, see [46], which is known as Wilson bases in harmonic analysis. While a comparison between these two approaches is beyond the scope of this paper, the analysis presented in this paper can be easily modified so that Wilson bases can be used instead of Gabor systems.

tions. Usually this is done by either simply not using those carrier functions that are close to the boundaries of the assigned bandwidth (which results in loss of data rates) or by applying some additional filtering (which destroys the orthogonality of the pulses). The methods presented here and in [43] yield a more efficient way to deal with this problem. Indeed, some of the theoretical and numerical methods discussed here and in [43] have already been used in the design of recent modems for short-radio wave communication systems, in which the author was involved.

5 Final remarks

In the representation (20) the Doppler effect manifests itself as frequency shift via the modulation operator M_η. This model is accurate for narrow-band signals, but for wideband signals a more adequate approach consists of modeling the Doppler effect via a dilation operator D_s, cf. [40]. While the compact support condition of the spreading function holds nevertheless true in this case, it seems more appropriate to replace (20) by the operator

$$(H_a x)(t) = \iint a(u,s) D_s T_u x(t) \, ds du, \tag{50}$$

see [2]. One would intuitively expect that wavelets will approximately diagonalize such operators. We plan to address this and related questions in our future research.

While finishing the write-up of this manuscript the author received a preprint of K. Gröchenig in which results related to Theorem 4.1 were obtained [16], see also (49). I am grateful to J. Sjöstrand and K. Gröchenig for stimulating discussions on this topic and K. Gröchenig for pointing out an error in an earlier version of the proof of Theorem 4.1.

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