FINAL EXAM
MAT17C, Spring 2013, Walcott

Note: There are EIGHT questions. Points for each question are shown in a table below and also in parentheses just after each question (and part). Be sure to budget your time accordingly.

Student ID _____________________________________________

Name __________________________________________________

Section TA/time/number ______________________________________

• Please write your name at the top of each page (in case the staple fails).
• Please ensure you have all 12 pages (including this page, the two equation pages and the scrap paper at the end).
• Please box your answers.
• Scrap paper is provided at the end of the exam; if you need more, just ask.
• Calculators, books, etc. are not allowed; you may only have a pen or pencil/eraser.
• Partial credit is available for all questions, but only if you show your work and it is legible.
• Read every question carefully and completely

June 11, 2013; 3:30 pm – 5:30 pm

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For these equations, unless otherwise stated, assume there is a function of two variables $f(x, y)$. The vector $\mathbf{x}$ is

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

the notation $f(\mathbf{x}) = f(x, y)$, the vector $\mathbf{f(\mathbf{x})}$ is

$$\mathbf{f(\mathbf{x})} = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f(x, y) \approx f(x^*, y^*) + \frac{\partial f}{\partial x} \bigg|_{x^*, y^*} (x - x^*) + \frac{\partial f}{\partial y} \bigg|_{x^*, y^*} (y - y^*) = f(x^*) + \nabla f|_{x^*} \cdot (x - x^*)$$

$$f(\mathbf{x}) \approx f(\mathbf{x}^*) + \mathbf{J}|_{\mathbf{x}^*} \cdot (\mathbf{x} - \mathbf{x}^*)$$

where the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\nabla f \equiv \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

The slope along the unit vector $\hat{e}_\theta$ is

$$\nabla f \cdot \hat{e}_\theta$$

At all critical points $\mathbf{x}_c$,

$$\nabla f|_{\mathbf{x}_c} = 0$$

If $x(t)$ and $y(t)$, then

$$\frac{df(x, y)}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$
\[
\int_c^d \int_a^b f(x,y) \, dxdy \equiv \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} f(x_i, y_j) \Delta x \Delta y
\]

where \(N_x = (b-a)/\Delta x, N_y = (d-c)/\Delta y\). \(x_i\) and \(y_j\) are the points at which the function is evaluated. They will typically be something like \(x_i = a + i \Delta x\) and \(y_j = c + j \Delta y\).

\[
\int_c^d \int_a^b f(x,y) \, dxdy = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} g(r,\theta) r \, d\theta \, dr
\]

The bounds on the integrals and the conversion between \(f(x,y)\) and \(g(r,\theta)\) depend on the problem.

The solution to the system of ODEs
\[
d\mathbf{x}/dt = A\mathbf{x}
\]
where the matrix \(A\) has eigenvectors \(s_1\) and \(s_2\) with associated eigenvalues \(\lambda_1\) and \(\lambda_2\) is
\[
\mathbf{x} = a s_1 e^{\lambda_1 t} + b s_2 e^{\lambda_2 t}
\]

The constants \(a\) and \(b\) are determined by the initial condition: \(\mathbf{x}(0) = a s_1 + b s_2\).

If the real parts of every eigenvalue of the matrix \(M\) are less than zero, then all trajectories head toward \(\mathbf{x} = 0\). This point is then said to be stable.

- Eigenvalues real, positive – Unstable Node.
- Eigenvalues real, negative – Stable Node.
- Eigenvalues real, one positive, one negative – Saddle (unstable)
- Eigenvalues imaginary, with negative real part – Stable Spiral
- Eigenvalues imaginary, with positive real part – Unstable Spiral
- Eigenvalues imaginary with no real part – Center.

Linearizing about a fixed point \(\mathbf{x}^*\), you have
\[
d\mathbf{x}/dt \approx J_{|\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*)
\]
or, if \(\varepsilon = \mathbf{x} - \mathbf{x}^*\)
\[
\frac{d\varepsilon}{dt} \approx J_{|\mathbf{x}^*} \varepsilon
\]

There are \(P(n,k) = \frac{n!}{(n-k)!}\) different ways of arranging \(k\) out of \(n\) different objects, if order matters.

There are \(C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}\) different ways of arranging \(k\) out of \(n\) different objects, if order does not matter.

\[
P(A \text{ given } B) = P(A|B) = \frac{P(A \text{ and } B)}{P(B)}
\]

If \(A\) and \(B\) are independent
\[
P(A \text{ and } B) = P(A)P(B)
\]

\[
P(A) = \sum_{i=1}^{N} P(A|B_i)P(B_i)
\]
1. (8 pts.) Suppose that a fruit fly has red eyes if it has the genotype $RR$, brown eyes if it has the genotype $Rr$ and white eyes with the genotype $rr$. Further, suppose that a fruit fly has normal wings if it has the genotype $WW$ or $Ww$. It has crumpled wings if it has the genotype $ww$. Assuming the genes are independent, what’s the probability that a fruit fly whose parents are both $RrWw$ will have white eyes and normal wings?

2. (8 pts.) Suppose a short protein contains the following amino acids: ABCCCDDE. How many distinct ways could you arrange these amino acids?

3. (8 pts.) You’re standing on a mountain whose altitude is described by the function $a(x, y)$. Where you’re standing, the gradient is

$$\nabla a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

How steep is it? (NOTE: this is equivalent to asking for the slope in the direction of steepest ascent).
4. (8 pts.) Look at the contour plot below left, which shows altitude as a function of \( x \) and \( y \), \( z(x, y) \). Notice the dashed line drawn just below \( y = 200 \). On the axes below right, draw the altitude as a function of \( x \) along this dashed line.

5. (8 pts.) Set up, but do not solve, a double integral that, if evaluated, would give the volume of a cylinder of radius \( R = 0.5 \) and height \( h = 2 \).
6. (8 pts.) Suppose you have two coupled non-linear ODEs

\[
\frac{dx}{dt} = f_1(x, y) \\
\frac{dy}{dt} = f_2(x, y)
\]  

There are eight separate plots below. Each one in the top row shows null clines. Each one in the bottom row shows a trajectory, labeled a), b), c) and d), from left to right. Each trajectory plot at the bottom corresponds to ONE of the null cline plots at the top. In the box at the top left corner of each null cline plot, write the letter of the corresponding trajectory plot.

NOTES: For your reference, the flow is indicated at the same point in all plots. The null clines are labeled \(dx/dt = 0\) or \(dy/dt = 0\) as appropriate. The scale on all plots is the same.
7. (25 pts.) ENERGY. For many physical systems, it is useful to think about energy. Here, you’ll see why.

This is a drawing of a pendulum.

The angle $\theta$ and angular velocity $\omega$ of the particle are described by the following equations

\[ \frac{d\omega}{dt} = -\omega - \sin(\theta) \]
\[ \frac{dx}{dt} = \omega \]  

(2)

The energy of the particle $E(\omega, \theta)$ is given by

\[ E(\omega, \theta) = \frac{1}{2}\omega^2 + (1 - \cos(\theta)) \]

(3)

where we’ve assumed $m = 1, L = 1, g = 1$ and $b = 1$.

a) (8 pts.) Show that every critical point of $E(\omega, \theta)$, is a fixed point of Eqs. 2.
For this system, it turns out that when $\frac{\partial^2 E}{\partial \theta^2} > 0$, the critical point is a minimum; when $\frac{\partial^2 E}{\partial \theta^2} < 0$, the critical point is a saddle.

b) (6 pts.) From your answer to c), find a critical point that’s a saddle. Show that it is an unstable fixed point.

c) (6 pts.) From your answer to c), find a critical point that’s a minimum. Show that it is a stable fixed point.

d) (5 pts.) Why do you think energy minima are stable fixed points while saddles are unstable fixed points? (HINT: Recall that the damped pendulum does not conserve energy, but rather loses it at a rate $\frac{dE}{dt} = -\omega^2$)
8. (27 pts.) ZOMBIE APOCALYPSE! It’s the zombie apocalypse. Can humanity defeat the zombie hordes?

Consider a community of \( S(t) \) people. People wander in at a constant rate \( \gamma \). They are born at a rate \( \mu S \) and, due to limited resources (food, beds, etc.), they die from non-zombie related effects at a rate \( \mu S^2 \).

When someone encounters a single zombie, she or he kills the zombie. However, if someone encounters two zombies, he or she is turned into a zombie. Thus, people become zombies at a rate \( \alpha SZ^2 \), and zombies die at a rate \( \beta SZ \). The situation is shown schematically below.

Mathematically, we can represent the situation with the following two coupled non-linear ODEs

\[
\frac{dS}{dt} = \mu S (1 - S) - \alpha SZ^2 + \gamma \\
\frac{dZ}{dt} = \alpha SZ^2 - \beta SZ
\]  

(4)

where \( S(t) \) is the number of people (in thousands) and \( Z(t) \) is the number of zombies (in thousands).

a) (5 pts) In terms of \( \alpha, \beta, \mu \) and/or \( \gamma \), find all null-clines along which \( dZ/dt = 0 \).
b) (5 pts) Using the same parameters ($\alpha = 1$, $\beta = 2$, $\gamma = 0.11$ and $\mu = 1$), there are two fixed points, one of which is $(Z = 0, S = 1.1)$. Show that this fixed point is stable.

c) (6 pts) The other fixed point is $(Z = 2, S \approx 0.0025)$. If you linearized the equations about this point, you would get

$$\frac{dx}{dt} = \begin{bmatrix} -3 & -0.01 \\ 0 & 0.005 \end{bmatrix} \begin{bmatrix} x - \begin{bmatrix} 0.0025 \\ 2 \end{bmatrix} \end{bmatrix}$$

given that $x = \begin{bmatrix} S \\ Z \end{bmatrix}$. Classify this fixed point (e.g. center, saddle, stable node, ... etc.) AND sketch the phase portrait near this fixed point.
d) (8 pts) Below is a computer-generated plot of the null-cline along which $dS/dt = 0$ (still using $\alpha = 1$, $\beta = 2$, $\gamma = 0.11$ and $\mu = 1$). On the plot do ALL of the following:

1. Sketch the null-cline(s) along which $dZ/dt = 0$ (from part a).
2. Indicate all fixed points.
3. At each numbered, filled point, sketch your best guess of the flow direction (to get full credit, your flow vector must only point in the correct quadrant; note that I have already drawn in one at the upper left).
4. At the two hollow points, labeled “a” and “b” sketch your best guess of the trajectory (i.e. how $(S(t), Z(t))$ evolves with time from this initial condition).

![Plot of null-clines](image)

Figure 1: The box in the upper left shows a zoomed-in version of the box in the plot. Notice that I have drawn the approximate flow direction in this region.

e) (3 pts) Based on your work, describe what it takes for a community to survive the zombies.
Scrap Paper