Homework groups: You will complete each of seven homework assignment as part of a three- or four-person group. Group members are assigned randomly and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (the writer), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines (described below).

Homework groups don’t work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus in not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert the TA. That person will not receive credit.

Homework guidelines for writers: (Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, you must hand homework in by the end of class Monday, the day it is due. Homework is available on my website Wednesday evenings, and is due the following week in class (unless stated otherwise). Late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting:
   On the top left corner, please put the course information, homework number and date, e.g.:
   MAT207C
   HW 1
   Due April 10, 2019.
   On the top right corner, please put the names of your group members, with the writer at the top and clearly indicated. Non-participating group members should also be indicated, e.g.:
   Jaromir Jagr (writer)
   Sarah Jessica Parker
   Michelle Wie
   James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. CITE YOUR HELP. At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it,” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. Box your answers.

6. Grading and regrading. We have a reasonable grading and regrading policy (see syllabus).
The topics of this homework are 1. Non-dimensionalization/scaling; 2. ODE review; and 3. An introduction to Matlab.

Non-dimensionalization is only briefly covered in §1.1 in Holmes; you may find a handout that will familiarize you with Matlab, along with two programs, on my website.

1. **The characteristic length scale for a mass on a spring.**

In class, we discussed a harmonic oscillator with damping. We found that there were two characteristic time scales, \( \tau_1 \) and \( \tau_2 \), and that the characteristic length scale, \( \xi \), would have to be determined from the initial conditions.

I argued that the characteristic length scale should not simply be the initial displacement, \( x_0 \), because the initial displacement could be zero and, given a non-zero initial velocity, the system would have non-zero displacement. This would result in problems.

Perhaps an intuitive characteristic length scale would be the maximum displacement of the system. This is a good idea, but hard to figure out without explicitly solving the differential equation. In this problem, you’ll find a characteristic length scale.

Let’s consider the non-dimensional form of the equation, where we’ve used the characteristic time scale \( \tau_1 = \sqrt{m/k} \):

\[
\frac{d^2 X}{dT^2} = -X - \alpha \frac{dX}{dT} \tag{1}
\]

a) Find a conserved quantity, \( E(X, dX/dT) \), for the case that \( \alpha = 0 \). Show that when \( \alpha > 0 \), this quantity decreases along trajectories (i.e. \( dE/dt \leq 0 \)) and is therefore a Lyapunov function.

b) Based on your answer to a) show that \( X(T) \leq \Xi \), where \( \Xi \) satisfies the equation \( E(\Xi, 0) = E\left(X(0), \frac{dX}{dT} \bigg|_{T=0}\right) \).

c) The value for \( \Xi \) you found in part b) is the characteristic length scale, but it’s in non-dimensional form (since the velocity we used was \( dX/dT \)). Turn it into dimensional form to find \( \xi \). (Write your answer in terms of the initial displacement, \( x_0 \), and initial velocity, \( v_0 \)).

d) Physically, the characteristic length scale, \( \xi \), is the initial displacement if the system’s initial energy were only in potential energy. For a mass on a spring, potential energy is \( V(x) = kx^2/2 \) and kinetic energy is \( T(\dot{x}) = m\dot{x}^2/2 \). Show that, if all of the system’s initial energy \( E(0) = V(x_0) + T(v_0) \) were in potential energy, you’d get the value of \( \xi \) you found in part c). (That is, show \( V(\xi) = E(0) \)).

2. The following equation is a simple model of resource-limited population growth:

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)
\]

a) Non-dimensionalize the equation.

b) How many parameters does your non-dimensionalized equation have?
c) Solve your non-dimensionalized equation, for $N(0) = N_0 > 0$.

d) Consider the following two cases. Case 1: $r = 0.1$, $K = 100$, $N_0 = 1$; Case 2: $r = 0.5$, $K = 300$, $N_0 = 1$. Is the solution to your non-dimensionalized equation the same for both cases? Why or why not? If not, what could you change to make the solutions identical?

3. Muscle contraction

Muscle contracts when the molecular motor myosin interacts with the filamentous protein actin. During this process, a myosin molecule binds to an actin filament. Myosin binds in such a way that, upon attachment, it is biased to have positive extension, $x$. This extension, in turn, causes myosin to apply a force on actin, sliding the actin filament until myosin’s extension returns to 0. The action of trillions of myosin molecules underlies all voluntary (and most involuntary) human movement.

In 1957, Andrew Huxley developed a very simple model for this process, expressed in the following PDE

$$\frac{\partial n}{\partial t} + v \frac{\partial n}{\partial x} = k_a(x)(1 - n) - k_d(x)n$$

where $n(x, t)$ is the (dimensionless) probability a myosin molecule will bind to actin with extension $x$ at time $t$, $k_a(x)$ and $k_d(x)$ are attachment and detachment rate constants, respectively, with units of inverse time. The parameter $v$ is actin’s velocity (assumed, for the purposes of this problem, to be a negative constant, $v < 0$).

a) Solve the equation in steady-state, $\partial n/\partial t = 0$ (your answer will include ugly integrals of the functions $k_d(x)$ and $k_a(x)$).

b) In his paper, Huxley chose the following functions for $k_a(x)$ and $k_d(x)$:

$$k_a(x) = \begin{cases} 0 & : x < 0, x > h \\ k'_a x & : 0 \leq x \leq h \end{cases}$$

$$k_d(x) = \begin{cases} k^0_d & : x < 0 \\ k'_d x & : x \geq 0 \end{cases}$$

where the constants $k'_a$ and $k'_d$ have units of inverse time times inverse distance, $h$ has units of distance and $k^0_d$ has units of inverse time. Using these parameters, non-dimensionalize Huxley’s PDE (note, you can do this in several ways – just pick one).

c) How many free parameters are in your non-dimensionalized equation?

d) Solve the steady-state equation, using Huxley’s functions. The boundary condition is $n(h) = 0$. (Recall that $v < 0$).
4. Consider the following ODE

\[ 0.01 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0 \]

with boundary conditions \( y(0) = 0 \) and \( y(1) = 1 \).

a) Using a computer, find \( y(x) \) numerically. Your algorithm should be something like this

1. Transform the second order ODE to two first order ODEs;
2. Write a function that takes an input \( a \), that goes in the initial condition \( y(0) = 0 \), \( \frac{dy}{dx}(0) = a \) and returns \( y(1) - 1 \), where \( y(1) \) is found by numerically approximating the solution to the two first order ODEs from step 1;
3. Perform a root find on your function in step 2.

b) As you can see, the coefficient in front of \( \frac{d^2 y}{dx^2} \) is small. Solve the equation analytically, neglecting this term.

c) Your answer from part b) should have a constant. You’ll need to determine the constant from a boundary condition, but which one should you pick? Make a plot with the numerical solution from part a) and two approximations from part b), one using \( y(0) = 0 \) and the other using \( y(1) = 1 \). Which boundary condition would you pick? Explain.