**Homework groups:** You will complete each of seven homework assignments as part of a three- or four-person group. Group members are assigned randomly and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (*the writer*), and this writer rotates for every assignment.

**Homework groups work best if:** Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the **Homework guidelines** (described below).

**Homework groups don’t work if:** One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus is not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert the TA. That person will not receive credit.

**Homework guidelines for writers:** (Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, you must hand homework in by the end of class Wednesday, the day it is due. Homework is available on my website Wednesday evenings, and is due the following week in class (unless stated otherwise). Late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting:
   - On the top left corner, please put the course information, homework number and date, e.g.:
     
     MAT207C
     HW 2
     Due April 17, 2019.
   - On the top right corner, please put the names of your group members, with the writer at the top and clearly indicated. Non-participating group members should also be indicated, e.g.:
     
     Jaromir Jagr (writer)
     Sarah Jessica Parker
     Michelle Wie
     James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. **CITE YOUR HELP.** At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it.” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. **Box your answers.**

6. Grading and regrading. We have a reasonable grading and regrading policy (see syllabus).
DUE: Wednesday, April 17, 2019. To be handed to me by the end of class.
The topics of this homework are 1. Order symbols, asymptotic approximations and asymptotic expansions; 2. Asymptotic solution of algebraic equations (covered only briefly); 3. Asymptotic solution of ODEs, regular perturbations; 4. Matched asymptotic expansions (we just started this topic)
These are covered in §1.3–2.2 in Holmes.


Assuming \( f \sim a_1 \varepsilon^\alpha + a_2 \varepsilon^\beta + \cdots \), find \( \alpha, \beta \) (with \( \alpha < \beta \)) and non-zero \( a_1, a_2 \) for the following functions:
\[
\begin{align*}
f &= \frac{1}{1 - \varepsilon^2} \\
f &= \left(1 + \frac{1}{\cos(\varepsilon)}\right)^{3/2} \\
f &= \sinh \left( \sqrt{1 + \varepsilon x} \right), \text{ for } 0 < x < \infty
\end{align*}
\]

In class we talked about a special case of projectile motion, where a ball is thrown directly upward in the presence of drag. Here, you’ll consider a more realistic case.
Consider a ball moving in 2D. It’s position is given by the vector \( \mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} \). It’s motion is governed by Newton’s equations:
\[
m \frac{d^2 \mathbf{x}}{dt^2} = -mg \mathbf{e}_y - c \left( \frac{dx}{dt} \cdot \frac{dx}{dt} \right) \mathbf{e}_v,
\]
here, \( \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is a unit vector pointing in the y-direction and \( \mathbf{e}_v = \frac{1}{||d\mathbf{x}/dt||} \frac{d\mathbf{x}}{dt} \) is a unit vector pointing in the direction of \( \frac{d\mathbf{x}}{dt} \).
For simplicity, let’s assume that \( \mathbf{x}(0) = \mathbf{0} \) and \( \left. \frac{d\mathbf{x}}{dt} \right|_{t=0} = \frac{dy}{dt} \bigg|_{t=0} = v_0 \).

a) Non-dimensionalize the ODE, as we did in class, and write the non-dimensional equation with the parameter \( \varepsilon = \frac{cv_0^2}{mg} \).
b) Show that, unsurprisingly, the lowest order expansion gives you standard projectile motion: \( x = v_0 t, \ y = v_0 t - \frac{1}{2} gt^2 \).
c) Find the next term in the expansion. (Note: You’ll have to solve some ugly integrals here. I recommend using a symbolic integral solver. There are some on-line ones that can help you).
d) Solve the differential equation numerically using Matlab, for \( \varepsilon = 0.1 \). Compare this solution to your two approximations. [Here, I just want you to check your work – generate a plot showing the numerical solution and the two approximations.]

Duffing’s equation describes a damped mass, moving in 1-D, on a non-linear spring. It is

\[ m \frac{d^2x}{dt^2} = -kx - Kx^3 - b \frac{dx}{dt} \]

Assume that the initial conditions are \( x(0) = 0 \) and \( \frac{dx}{dt} \bigg|_{t=0} = v_0 \).

a) There are several ways to non-dimensionalize this problem. Perform your non-dimensionalization so that you end up with two parameters: \( \alpha = \frac{b}{\sqrt{mk}} \) and \( \varepsilon = \frac{Kv_0^2m}{k^2} \).

(Note: The book asks you to end up with slightly different parameters, but I think this is a typo).

b) Find a two-term expansion of the solution in the case where the spring is weakly non-linear (\( \varepsilon \) is small). To make your life easier, assume that \( \alpha = 1 \).

c) Solve the differential equation numerically using Matlab, for \( \varepsilon = 0 \). Compare this solution to your two approximations. [Here, I just want you to check your work – generate a plot showing the numerical solution and the approximation.]

4. Pendulum, small angle approximation. A pendulum, released from rest at a small angle, \( a \), obeys the following ODE

\[ \frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin(\theta) = 0 \]

with initial conditions \( \theta(0) = a \) and \( \frac{d\theta}{dt} \bigg|_{t=0} = 0 \). Here, \( g \) is the gravitational constant and \( \ell \) is the length of the pendulum and \( \theta \) is the angle of the pendulum from vertical.

a) Non-dimensionalize these equations, using the initial condition \( \theta(0) = a \) as a “characteristic angle scale.”

b) Find a three-term expansion of the solution in the case where the initial displacement is small (i.e. \( a = \varepsilon \ll 1 \)).

(Note, you will have to solve three differential equations; if your expansion is \( \theta = \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 \), you have to solve an ODE for each \( \theta_i \), \( i = 0, 1, 2 \). The ODE for \( i = 2 \) is a pain to solve).

c) Solve the differential equation numerically using Matlab, for \( \varepsilon = 0.1 \). Compare this solution to your two approximations. [Here, I just want you to check your work – generate a plot showing the numerical solution and the approximation.]
5. (Problem 1 on page 57 from Holmes, 1995) Friedrichs’ (1942) model problem for a boundary layer in a viscous fluid is

\[
\varepsilon \frac{d^2 y}{dx^2} = a - \frac{dy}{dx}
\]

for 0 < x < 1 and y(0) = 0, y(1) = 1, and a is a given positive constant.

a) After finding the first term of the inner and outer expansions, derive a composite expansion for the solution to this problem.

b) Derive a two-term composite expansion of the solution of this problem.

c) Solve the differential equation numerically using Matlab for your choice of \( \varepsilon \) and a (be sure to list your choices) and compare this numerical solution to your approximations.