Homework groups:

You will complete each of seven homework assignment as part of a three- or four-person group. Group members are assigned randomly from your section and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (the writer), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines.

Homework groups don’t work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus in not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert your TA. That person will not receive credit.

Homework guidelines for writers:

(Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, hand homework in to your TA during section the day it is due. Homework is available via Smartsite Thursday night after section, and is due the following week in section (unless stated otherwise). At the discretion of the TA grading the homework, late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting. On the top left corner, please put the course information, homework number and date, e.g.:

   MAT/BIS27A  
   HW 2  
   Due: January 24, 2019.

On the top right corner, please put your group number, the names of your group members, with the writer at the top and clearly indicated. Also indicate any non-participating group members, e.g.:

   Group 3
   Jaromir Jagr (writer)
   Taylor “Tay-tay” Swift
   Serena Williams
   James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. CITE YOUR HELP. At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture, office hours and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to
finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it,” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. **Box your answers.**

6. Grading and regrading. We have a reasonable grading a regrading policy, see the syllabus.

**DUE: January 24, 2019. To be handed in at the beginning of lab.**
The topic of this homework is: the row and column pictures of linear algebra, the idea of elimination and elimination with matrices.

These topics are covered in §2.1–2.3 of Strang.

Problems 1-8 are all or nothing; there is no partial credit available. Make sure you check your answers carefully, since you will receive no credit even for minor errors. Together, these 8 problems are worth 40 points (five points each).

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1 (Problem 19, page 43) What 3 by 3 matrix $E$ multiplies \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\] to give \[
\begin{bmatrix}
x \\
y \\
z + x
\end{bmatrix}
\]?

What 3 by 3 matrix $E^{-1}$ multiplies \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\] to give \[
\begin{bmatrix}
x \\
y \\
z - x
\end{bmatrix}
\]?

If you multiply \[
\begin{bmatrix}
3 \\
4 \\
5
\end{bmatrix}
\] by $E$ and then $E^{-1}$, what are the two results?

2 (Problem 26, page 44) Draw the row and column pictures for the equations $x - 2y = 0$, and $x + y = 6$. (Recall that the “row picture” would be a pair of intersecting lines; and the “column picture” would be a pair of vectors [the two columns], the sum of which equals a third vector [the right hand side]).

3 (Problem 5, page 54) Choose a right side which gives no solution and another right side which gives infinitely many solutions. For the latter case, what are two of those solutions?

\[
3x + 2y = 10 \\
6x + 4y =
\]

4 (Problem 11, page 54) A system of linear equations can’t have exactly two solutions. Why?

a) If \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\] and \[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\] are two solutions, what is another solution?

b) If 25 planes meet at two points, where else do they meet?
5. (Problem 13, page 55) Apply elimination (circle the pivots) and back substitution to solve

\[
\begin{align*}
2x - 3y &= 3 \\
4x - 5y + z &= 7 \\
2x - y - 3z &= 5
\end{align*}
\]

List the three row operations: Write each one out in the form “Subtract \(i\) times row \(j\) from row \(k\)” (where you must determine \(i\), \(j\), and \(k\)).

6. (Problem 19, page 56) Which number \(q\) makes this system singular and which right side \(t\) gives it infinitely many solutions? Find the solution that has \(z = 1\).

\[
\begin{align*}
x + 4y - 2z &= 1 \\
x + 7y - 6z &= 6 \\
3y + qz &= t
\end{align*}
\]

7. (Problem 3, page 66) Which three matrices \(E_{21}, E_{31}, E_{32}\) put \(A\) into upper triangular form, \(U\)?

\[A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U\]

Multiply those \(E\)’s to get one matrix \(M\) that does elimination: \(MA = U\).

8. (Problem 17, page 67) The parabola \(y = a + bx + cx^2\) goes through the points \((x, y) = (1, 4)\) and \((2, 8)\) and \((3, 14)\). Find and solve a matrix equation for the unknowns \((a, b, c)\).
Problems 9, 10 and 11 have partial credit. Together, these 3 problems are worth 60 points (20 points each).

9. **ION CHANNELS, REVISITED.** In class, we have been discussing a simple model for an ion channel. Ion channels are special proteins that are situated in the membranes of cells and (sometimes) allow ions to pass through the membrane. They are critical to processes like the signaling of brain cells and the contraction of heart muscle. For this reason, many drugs target ion channels to, say, make a heart attack less likely.

![Diagram of ion channel model](image)

**Figure 1:**

Figure 1 shows a diagram that represents a model for an ion channel. In the top row of the diagram, you should recognize the model we’ve seen a few times in class. The ion channel can be closed (state 1) or open (state 2). The remaining two states (states 3 and 4) represent the ion channel with drug bound — you can see the drug molecule at the upper right of the ion channel. The equations that govern this model are

\[
\begin{align*}
\frac{dn_1}{dt} &= -(k_o + k_c^+)n_1 + k_c n_2 + k_c^- n_3 \\
\frac{dn_2}{dt} &= k_o n_1 - (k_c + k_o^+) n_2 + k_o^- n_4 \\
\frac{dn_3}{dt} &= k_c^+ n_1 - (k_c^- + k_{do}) n_3 + k_{dc} n_4 \\
\frac{dn_4}{dt} &= k_o^+ n_2 + k_{do} n_3 - (k_o^- + k_{dc}) n_4
\end{align*}
\]

(1)

a) As we saw in class, when the system reaches steady state, each \( \frac{dn_i}{dt} = 0 \). Write the system of equations, Eq. 1, as a matrix vector equation of the form \( Kn = 0 \). (Note, the entries of the matrix \( K \) will contain combinations of the constants \( k_o, k_c, k_o^+, \ldots \), etc.).

For the reasons discussed in class (i.e., because the number of ion channels is constant with time), the rows of the matrix \( K \) are not linearly independent. You can test this by adding them together to get zero. Therefore the matrix \( K \) is singular — it has no inverse.

b) Because \( K \) is singular, its columns are linearly dependent. Show this.

(HINT one way to do this is to notice that each column of \( K \) is a vector whose entries sum to 0. That is, each entry looks like \( c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ -c_1 - c_2 - c_3 \end{bmatrix} \).
c) It’s hard to solve for \( n \) in the equation \( Kn = 0 \), since \( K \) is singular. It is possible to write the steady-state version of Eq. 1 as an equation \( Ax = b \) by eliminating one of the equations using the constraint \( N_{tot} = n_1 + n_2 + n_3 + n_4 \). Do this.

(Hint: start by writing, say, \( n_4 = N_{tot} - n_1 - n_2 - n_3 \). Then, plug this equation into the equations for \( \frac{dn_1}{dt} \), \( \frac{dn_2}{dt} \), and \( \frac{dn_3}{dt} \), each of which you can set to zero because we’re looking for steady-state. Now, convert these equations to a 3x3 matrix \( A \) times the vector \( x \), whose entries are \( n_1 \), \( n_2 \) and \( n_3 \).

Let \( N_{tot} = 3000 \), and use the following numbers for the rate constants: \( k_o = 2 \), \( k_c = 1 \), \( k_c^+ = 8 \), \( k_c^- = 1 \), \( k_{do} = 1 \), \( k_{dc} = 4 \), \( k_o^+ = 2 \), and \( k_o^- = 2 \) (let us not worry about units, for simplicity).

d) Using elimination, solve your equation in part c) to find the steady-state values of \( n_1 \), \( n_2 \), \( n_3 \) and \( n_4 \). When writing your solution, write the elimination matrix for each step that turns the matrix \( A \) into an upper diagonal matrix. Note that, since the vector \( x \) from part c) has length 3 (i.e., it’s entries are \( n_1 \), \( n_2 \) and \( n_3 \)), you’ll have to use the constraint \( N_{tot} = n_1 + n_2 + n_3 + n_4 \).

e) Suppose that the constants for part d) come from a particular drug. The purpose of this drug is to decrease the flow of ions through the ion channel by increasing the probability that the ion channels are closed. Does the drug work?

(Hint: The number of closed channels is the number of channels in states 1 and 3 \( n_1 + n_3 \) and the number of open channels is \( n_2 + n_4 \). In the absence of drug, we get the system that we analyzed in class, where there are twice as many open channels as closed channels – meaning that there are 2000 open channels and 1000 closed channels).
10. **Mechanical equilibrium.** Consider the mechanical system, pictured in Figure 2, where three weights that slide frictionlessly on a pole are suspended by a series of four springs. Each spring \((i = 1, 2, 3, 4)\) has a rest length \(\ell_i\) and a spring constant \(k_i\), so that the spring generates a force \(F = k_i(y_i - \ell_i)\), where \(y_i\) is the extension of the spring.

\[
\begin{align*}
0 &= -k_1(y_1 - \ell_1) + k_2(y_2 - \ell_2) - m_1 g \\
0 &= -k_2(y_2 - \ell_2) + k_3(y_3 - \ell_3) - m_2 g \\
0 &= -k_3(y_3 - \ell_3) + k_4(y_4 - \ell_4) - m_3 g
\end{align*}
\] (2)

In addition to these three equations, we have the constraint that \(y_1 + y_2 + y_3 + y_4 = L\).

a) Using the constraint equation, eliminate the variable \(y_4\) from the system of equations, Eq. 2. Then, write the system of linear equations as \(Kx = b\), where \(x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\). Note that the constants \(k_i, \ell_i (i = 1, 2, 3, 4)\), \(m_j (j = 1, 2, 3)\), and \(g\) will appear in \(K\) and \(b\).

b) Show that, if each spring has the same strength \((k = k_1 = k_2 = k_3 = k_4)\), and if the following relationship holds for \(i = 1, 2, 3\)

\[
\ell_{i+1} - \ell_i = -\frac{m_i g}{k}
\]

then the weights are equally distributed along the pole \((y_1 = y_2 = y_3 = y_4 = L/4)\).

To do this, in your equation \(Kx = b\), divide both sides by \(k\). Then your matrix \(K\) should contain only numbers. The vector \(b\) should also simplify considerably. Then, use elimination to solve for \(x\). When writing your solution, write the elimination matrix for each step that turns the matrix \(K\) into an upper diagonal matrix.

c) Suppose, instead, that each spring has the same strength \((k = k_1 = k_2 = k_3 = k_4)\) and the same rest length \((\ell = \ell_1 = \ell_2 = \ell_3 = \ell_4)\), and that each weight has the same mass \((m = m_1 = m_2 = m_3)\). Show that
the same series of elimination moves in part b) produces

\[
\begin{bmatrix}
    y_1 \\
    y_2 \\
    y_3 \\
    y_4
\end{bmatrix} = \begin{bmatrix}
    L/4 - 3mg/2k \\
    L/4 - mg/2k \\
    L/4 + mg/2k \\
    L/4 + 3mg/2k
\end{bmatrix}
\]

\( \quad \)

\[d) \quad \text{Do these answers make sense? Start with part c). Based on your intuition, make a sketch for where the weights should be, given that all of the masses and springs are identical. Which spring is longest, which is shortest? Is that consistent with your answer to part c? Briefly discuss.}\]

Now, consider part d). Based on your intuition, if all the weights end up equally distributed on the pole, which spring would need to be longest? Which one shortest? Is that consistent with the equation \( \ell_{i+1} - \ell_i = -\frac{m_i g}{k} \)? Briefly discuss.

11. **Worksheet.** Please turn in a completed version of worksheet 2.