Homework groups:

You will complete each of seven homework assignment as part of a three- or four-person group. Group members are assigned randomly from your section and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (the writer), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines.

Homework groups don’t work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus in not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert your TA. That person will not receive credit.

Homework guidelines for writers:

(Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, hand homework in to your TA during section the day it is due. Homework is available via Smartsite Thursday night after section, and is due the following week in section (unless stated otherwise). At the discretion of the TA grading the homework, late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting. On the top left corner, please put the course information, homework number and date, e.g.:  
   MAT/BIS27A  
   HW 4  
   Due: February 14, 2019.

On the top right corner, please put your group number, the names of your group members, with the writer at the top and clearly indicated. Also indicate any non-participating group members, e.g:  

   Group 3  
   Jaromir Jagr (writer)  
   Taylor “Tay-tay” Swift  
   Serena Williams  
   James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. CITE YOUR HELP. At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture, office hours and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to
finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it,” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. Box your answers.

6. Grading and regrading. We have a reasonable grading a regrading policy, see the syllabus.

DUE: February 14, 2019. To be handed in at the beginning of lab.
The topic of this homework is independence, basis and dimension.

These topics are covered in §3.4 of Strang.

Problems 1-4 are all or nothing; there is no partial credit available. Make sure you check your answers carefully, since you will receive no credit even for minor errors. Together, these 4 problems are worth 40 points (ten points each).

1. (Problem 2, page 175) Find the largest possible number of independent vectors among

\[
v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.
\]

2. (Problem 4, page 175) If \(a, d, \) and \(f\) are non-zero in the following upper-diagonal matrix, \(U\), show that the only solution to \(Ux = 0\) is \(x = 0\). Then, \(U\) has independent columns.

\[
U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}
\]

3. (Problem 19, page 177) The columns of \(A\) are \(n\) vectors from \(\mathbb{R}^m\). If they are linearly independent, what is the rank of \(A\)? If they span \(\mathbb{R}^m\), what is the rank? If they are a basis for \(\mathbb{R}^m\), what then? The rank, \(r\), counts the number of independent columns.

4. (Problem 21, page 177) Suppose the columns of a 5x5 matrix, \(A\), are a basis for \(\mathbb{R}^5\).

a) The equation \(Ax = 0\) has only the solution \(x = 0\) because ________ .

b) If \(b\) is in \(\mathbb{R}^5\), then \(Ax = b\) is solvable because the basis vectors ________ \(\mathbb{R}^5\).

NOTE: These results tell you that \(A\) is invertible, and therefore rank 5. As you’ve seen, this means that the rows are independent (since elimination will give five pivots), so the rows, also, form a basis for \(\mathbb{R}^5\).
5. **Pendulum**

In mechanics, it is often useful to work in different coordinate systems. One can think of this as changing basis. To see how this works, consider a simple pendulum, sketched below.

![Diagram of a pendulum](image)

Figure 1:

Note that I’ve drawn a bunch of different unit vectors up at the (frictionless) hinge. The first two are \( \hat{i} \) (which points along the \( x \)-axis), and \( \hat{j} \) (which points along the \( y \)-axis, oriented downward in this picture).

We can represent the vector, \( \mathbf{x} \), the position of the mass on the end of the pendulum, as a linear combination of these two vectors: \( \mathbf{x} = x\hat{i} + y\hat{j} \).

From trigonometry, we can see that \( x = L \sin(\theta) \) and \( y = L \cos(\theta) \). Therefore,

\[
\mathbf{x} = L \sin(\theta) \hat{i} + L \cos(\theta) \hat{j}
\]

a) Notice the second set of unit vectors, \( \hat{e}_r \) and \( \hat{e}_\theta \). Write \( \mathbf{x} \) as a linear combination of the two unit vectors, \( \mathbf{x} = a\hat{e}_r + b\hat{e}_\theta \), where you must determine \( a \) and \( b \).

b) Since \( \mathbf{x} \) is the same vector, your answer from part a) must equal \( L \sin(\theta) \hat{i} + L \cos(\theta) \hat{j} \). Use this result to find an expression for \( \hat{e}_r \) as a linear combination of \( \hat{i} \) and \( \hat{j} \).

c) Write \( \hat{e}_\theta \) as a linear combination of \( \hat{i} \) and \( \hat{j} \). To do so, note that the dot product of \( \hat{e}_\theta \) and \( \hat{i} \) is the cosine of the angle between them (since they are both unit vectors):

\[
\hat{e}_\theta = a\hat{i} + b\hat{j} \\
(\hat{e}_\theta) \cdot \hat{i} = (a\hat{i} + b\hat{j}) \cdot \hat{i} \\
\cos(\theta) = a\hat{i} \cdot \hat{i} + b\hat{j} \cdot \hat{i} = a
\]

Use a similar method to find \( b \).
The unit vectors \( \hat{i} \) and \( \hat{j} \) form a basis for \( \mathbb{R}^2 \). The unit vectors \( \hat{e}_r \) and \( \hat{e}_\theta \) also form a basis for \( \mathbb{R}^2 \).

Therefore, any vector \( \mathbf{v} \) can be written as a linear combination of each:

\[
\mathbf{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta = v_x \hat{i} + v_y \hat{j}
\]

where \( v_r \) and \( v_\theta \) are the components of \( \mathbf{v} \) in the basis \( \hat{e}_r, \hat{e}_\theta \). Similarly, \( v_x \) and \( v_y \) are the components of \( \mathbf{v} \) in the basis \( \hat{i}, \hat{j} \).

Note that the basis \( \hat{i}, \hat{j} \) has the very special property: in that basis, the components of \( \mathbf{v} \) are precisely the entries of \( \mathbf{v} \), that is

\[
\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_x \hat{i} + v_y \hat{j}
\]

d) Use this fact to rewrite the equation

\[
\mathbf{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta = v_x \hat{i} + v_y \hat{j}
\]
as a matrix vector equation of the form \( \mathbf{A} \mathbf{x} = \mathbf{b} \), where \( \mathbf{x} = \begin{bmatrix} v_\theta \\ v_r \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \).

Your answer to part d) allows you to change basis. That is, given a vector in the basis \( \hat{e}_r, \hat{e}_\theta \), if you multiply the components by the matrix \( \mathbf{A} \), you will find the components of the vector in the basis \( \hat{i}, \hat{j} \). If you like, you can check this by multiplying \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) by \( \mathbf{A} \), which should give you your answer to part c) and multiplying \( \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) by \( \mathbf{A} \), which should give you your answer to part b).

e) Suppose you want to transform from the \( \hat{i}, \hat{j} \) basis into the \( \hat{e}_r, \hat{e}_\theta \) basis. Then, you’d need to invert \( \mathbf{A} \), so that \( \mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \). You may recognize the matrix \( \mathbf{A} \) – it is a rotation matrix, that rotates vectors counterclockwise by an angle \( \theta \) (take a look at Fig. 1 to see if this makes sense). Therefore, \( \mathbf{A}^{-1} \) must be the same as \( \mathbf{A} \), but with \( -\theta \) replacing every \( \theta \).

i. Show, by direct multiplication, that this is true (i.e. \( \mathbf{A}(\theta) \cdot \mathbf{A}(-\theta) = \mathbf{I} \)).

ii. Transform the components of the position of the mass, \( \mathbf{x} = \begin{bmatrix} L \sin(\theta) \\ L \cos(\theta) \end{bmatrix} \), into the \( \hat{e}_r, \hat{e}_\theta \) basis. You should recover your answer to part a).

Here are a couple (maybe) interesting observations.

1) You should find that \( \mathbf{A}^{-1} = \mathbf{A}^T \). This is a property that you saw before, in problem 10 of homework 1. It arises when an invertible matrix is made of vectors that are all of length 1, and are mutually perpendicular (those vectors form an orthonormal basis – you will learn more about this later).

2) The use of all of this may seem unclear. It becomes more obvious in more complicated problems, but you can begin to see it in a simple pendulum. In particular, when you solve Newton’s equations, you get an expression for \( \theta(t) \). If you want to, say, make a movie of the pendulum, you need to plot the position of the mass – it’s easiest to work in the \( \hat{i}, \hat{j} \) basis (think of how you’d plot the position of the mass in Matlab). However, as you saw in part a), it’s easiest to describe the position of the mass in the \( \hat{e}_r, \hat{e}_\theta \) basis. So, you just multiply by \( \mathbf{A} \), the rotation matrix, which is easy to calculate if you know \( \theta \).