Homework groups:

You will complete each of seven homework assignments as part of a three- or four-person group. Group members are assigned randomly from your section and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (the writer), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines.

Homework groups don’t work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus is not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert your TA. That person will not receive credit.

Homework guidelines for writers:

(Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, hand homework in to your TA during section the day it is due. Homework is available via Smartsite Thursday night after section, and is due the following week in section (unless stated otherwise). At the discretion of the TA grading the homework, late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting. On the top left corner, please put the course information, homework number and date, e.g.:

   MAT/BIS27A
   HW 5
   Due: February 21, 2019.

On the top right corner, please put your group number, the names of your group members, with the writer at the top and clearly indicated. Also indicate any non-participating group members, e.g.:

   Group 3
   Jaromir Jagr (writer)
   Taylor “Tay-tay” Swift
   Serena Williams
   James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. CITE YOUR HELP. At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture, office hours and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to
finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it,” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. **Box your answers.**

6. Grading and regrading. We have a reasonable grading a regrading policy, see the syllabus.

**DUE: February 21, 2019. To be handed in at the beginning of lab.**
The topics of this homework are: Independence, basis and dimension; the four subspaces; and orthogonality.

These topics are covered in §3.4-4.1 of Strang.

Problems 1-8 are all or nothing; there is no partial credit available. Make sure you check your answers carefully, since you will receive no credit even for minor errors. Together, these 8 problems are worth 40 points (five points each).

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1. (Problem 23, page 177) \( U \) comes from \( A \) by subtracting row 1 from row 3:

\[
A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

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2. (Problem 2, page 190) Find bases and dimensions for the four subspaces associated with \( A \) and \( B \):

\[
A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}
\]

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3. (Problem 3, page 190) Find a basis for each of the four subspaces associated with \( A \):

\[
A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

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4. (Problem 5, page 190) If \( V \) is the subspace spanned by \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \), find a matrix \( A \) that has \( V \) as its row space. Find a matrix \( B \) that has \( V \) as its nullspace. Multiply \( AB \).
5. (Problem 11, page 191). $A$ is an $m$ by $n$ matrix of rank $r$. Suppose there are right sides $b$ for which $Ax = b$ has no solution.

a) What are all inequalities ($<$ or $\leq$) that must be true between $m$, $n$ and $r$?
b) How do you know that $A^T y = 0$ has solutions other than $y = 0$?

6. (Problem 6, page 202). This system of equations $Ax = b$ has no solution (they lead to 0=1):

\begin{align*}
x + 2y + 2z &= 5 \\
2x + 2y + 3z &= 5 \\
3x + 4y + 5z &= 9
\end{align*}

Find numbers, $y_1$, $y_2$, $y_3$ to multiply the equations so that, when added, they give $0 = 1$. You have found a vector $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in which subspace? Its dot product $y^T b = 1$, so the system of equations cannot have a solution.

7. (Problem 7, page 202) Every system with no solution is like the one in problem 6. There are numbers, $y_1, y_2, \ldots, y_m$ that multiply the $m$ equations so they add up to 0=1. This is called Fredholm’s Alternative:

Exactly one of these problems has a solution
1) $Ax = b$, or  
2) $A^T y = 0$, with $y^T b = 1$.
If $b$ is not in the column space of $A$, it is not orthogonal to the nullspace of $A^T$.

Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers $y_1$, $y_2$, $y_3$ chosen so that the equations add up to $0 = 1$.

8. (Problem 10, page 203) Suppose $A$ is a symmetric matrix ($A = A^T$).
a) Why is its column space perpendicular to its nullspace?
b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these “eigenvectors” $x$ and $z$? Symmetric matrices have perpendicular eigenvectors, $x^T z = 0$. 

3
Problems 9, 10 and 11 have partial credit. Together, these 3 problems are worth 60 points (20 points each).

9. **Age-structured populations.** In the first week of class, I introduced two biological examples. One of them was the ion channel model that we’ve discussed at length. The other was a population model that we have not talked about since then. We will now look at that model.

In the model, we were considering a population of ground squirrels with a life-span of three years. Then, we could divide the total population into three sub-populations: 1) 0-1 year-olds, \( n_{0-1} \); 2) 1-2 year-olds, \( n_{1-2} \); and 3) 2-3 year-olds, \( n_{2-3} \).

It takes 1 year to reach sexual maturity, so the number of 0-1 year-olds in the next year will be the reproductive rate of the 1-2 year-olds (\( r_1 \)) times the number of 1-2 year-olds, and the reproductive rate of the 2-3 year-olds (\( r_2 \)) times the number of 2-3 year-olds:

\[
\begin{align*}
n_{0-1}^{i+1} &= r_1 n_{1-2}^i + r_2 n_{2-3}^i,
\end{align*}
\]

where the subscript refers to the year (i.e., the current year is \( i \), so the next year is \( i + 1 \)).

The number of 1-2 year-olds in the next year will be the 1-year survival probability (\( p_0 \)) of the 0-1 year-olds, and the number of 2-3 year-olds in the next year will be the 1-year survival probability (\( p_1 \)) of the 1-2 year-olds:

\[
\begin{align*}
n_{1-2}^{i+1} &= p_0 n_{0-1}^i, \\
n_{2-3}^{i+1} &= p_1 n_{1-2}^i.
\end{align*}
\]

a) Write this as a matrix-vector equation of the form \( \mathbf{n}_{i+1} = M \mathbf{n}_i \).

b) Show that \( \mathbf{n}_j = M^j \mathbf{n}_0 \), where \( \mathbf{n}_0 \) is the population at year 0.

c) Show that if \( \mathbf{n}_0 = a \hat{\mathbf{s}} \), where \( a \) is an arbitrary constant, then \( \mathbf{n}_j = \lambda^j \mathbf{n}_0 \).

d) Show that, in general, \( \mathbf{n}_j = a \lambda_1^j \hat{\mathbf{s}}_1 + b \lambda_2^j \hat{\mathbf{s}}_2 + c \lambda_3^j \hat{\mathbf{s}}_3 \), where \( \mathbf{n}_0 = a \hat{\mathbf{s}}_1 + b \hat{\mathbf{s}}_2 + c \hat{\mathbf{s}}_3 \).

e) Explain why, if \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \),

\[
\lim_{j \to \infty} \mathbf{n}_j = \lim_{j \to \infty} a \lambda_1^j \hat{\mathbf{s}}_1.
\]

f) Suppose you want to know the relative number of 0-1, 1-2 and 2-3 year olds in the population after a long time. How could your answer to part e) help?
10. CHEMICAL REACTIONS. Perhaps the simplest chemical reaction is radioactive decay. For this reaction, a material decays to another at a constant rate. So, if you have a number of radioactive molecules, \( n(t) \),

\[
\frac{dn}{dt} = -an \tag{1}
\]

where \( a \) is a positive constant. You may have learned that the amount of a radioactive material varies exponentially with time, \( n(t) = ce^{-at} \), where \( c \) is a constant (equal to the initial amount of material, \( n(0) \)) and \( a \) is the same constant in Eq. 1.

a) With direct calculations, verify 1) that \( n(t) = ce^{-at} \) satisfies Eq. 1 (that is, differentiate \( ce^{-at} \) with respect to time and show that it, indeed, equals \(-an\)); and 2) that \( n(0) = c \).

A more complicated chemical reaction might have the form

\[
\frac{dn_1}{dt} = a_{11}n_1 + a_{12}n_2 \\
\frac{dn_2}{dt} = a_{21}n_1 + a_{22}n_2 \tag{2}
\]

where \( a_{11}, a_{12}, a_{21} \), and \( a_{22} \) are arbitrary (positive or negative) constants.

b) Write Eq. 2 as a matrix-vector equation of the form \( \frac{dx}{dt} = Mx \). Note that \( \frac{dx}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} \).

Now, suppose that a special unit vector, \( \hat{s} \), exists such that

\[ M\hat{s} = \lambda \hat{s} \]

c) Show that if \( x = b\hat{s} \), then \( \frac{dx}{dt} \) points in the direction of \( \hat{s} \).

If you successfully complete part c), that means that if \( x \) points in the same direction as \( \hat{s} \), its “velocity” \( \frac{dx}{dt} \) also points along \( \hat{s} \). Therefore, it will not move off \( \hat{s} \), but will stay on it forever. We can then write \( x = x(t)\hat{s} \).

d) Plug the equation \( x = x(t)\hat{s} \) into the equation \( \frac{dx}{dt} = Mx \), reduce it to an equation like Eq. 1 to show that \( x(t) = x(0)e^{\lambda t} \).

e) Suppose you find two linearly independent unit vectors \( \hat{s}_1 \) and \( \hat{s}_2 \), such that \( M\hat{s}_1 = \lambda_1\hat{s}_1 \) and \( M\hat{s}_2 = \lambda_2\hat{s}_2 \). Use your answer to part d) to show that \( x = a\hat{s}_1 e^{\lambda_1 t} + b\hat{s}_2 e^{\lambda_2 t} \), for any initial concentrations \( n_1(0), n_2(0) \).

This is hard! Here’s one way to do it.

1) Show that if two functions \( f_1(t) \) and \( f_2(t) \) both satisfy \( \frac{dx}{dt} = Mx \), then \( f_1 + f_2 \) also satisfies the equation.

2) Use your answer to part d) to show that \( x = a\hat{s}_1 e^{\lambda_1 t} + b\hat{s}_2 e^{\lambda_2 t} \) will solve the equation.

3) Now, show that the solution above describes any initial concentration \( n_1(0), n_2(0) \).
11. **Worksheet.** Please turn in a completed version of worksheet 5.