Homework groups:

You will complete each of seven homework assignments as part of a three- or four-person group. Group members are assigned randomly from your section and will remain the same for the duration of the quarter. Each group turns in one homework, and each participating group member receives the same grade on the assignment. One member of the group is responsible for writing the homework (the writer), and this writer rotates for every assignment.

Homework groups work best if: Each member of the homework group finishes (or honestly attempts) the homework independently. At some appointed time, well before the due date, the group meets and everyone compares answers. Any discrepancies are discussed until a consensus is achieved. The writer notes the group consensus and makes sure she or he understands how to do the problem. After the meeting, but before class, the writer neatly and clearly writes the homework according to the Homework guidelines.

Homework groups don’t work if: One or more of the members skips meetings; each group member does not honestly attempt the homework prior to the meeting; a consensus is not reached for each assigned problem. If a group member does not adequately participate in the homework, write a note on the homework and alert your TA. That person will not receive credit.

Homework guidelines for writers:

(Adapted from the website of Professor Andy Ruina). To get full credit, please do these things on each homework.

1. As a group writer, hand homework in to your TA during section the day it is due. Homework is available via Smartsite Thursday night after section, and is due the following week in section (unless stated otherwise). At the discretion of the TA grading the homework, late homework may or may not be accepted for reduced credit.

2. On the first page of your homework, please do the following to facilitate sorting. On the top left corner, please put the course information, homework number and date, e.g.:

   MAT/BIS27A
   HW 7
   Due: March 14, 2019.

   On the top right corner, please put your group number, the names of your group members, with the writer at the top and clearly indicated. Also indicate any non-participating group members, e.g.:

   Group 3
   Jaromir Jagr (writer)
   Taylor “Tay-tay” Swift
   Serena Williams
   James Van der Beek (did not participate)

3. Please put a staple at the top left corner. Folded interlocked corners fall apart. Paperclips fall off.

4. CITE YOUR HELP. At the top of each problem, clearly acknowledge all help you got from TAs, faculty, students or any other source (with exceptions for lecture, office hours and the text, which need not be cited). You could write, for example: “Mary Jones pointed out to me that I had forgotten to divide by three in problem 2,” or “Nadia Chow showed me how to do problem 3 from start to
finish,” or “I copied this solution word for word from Jane Lewenstein” or “I found a problem just like this one, number 9, at cheatonyourhomework.com, and copied it,” etc. You will not lose credit for getting and citing such help. Don’t violate academic integrity rules: be clear about which parts of your presentation you did not do on your own. Violations of this policy are violations of the UC Davis Code of Academic Conduct.

5. Your work should be laid out neatly enough to be read by someone who does not know how to do the problem. For most jobs, it is not sufficient to know how to do a problem, you must convince others that you know how to do it. Your job on the homework is to practice this. **Box your answers.**

6. Grading and regrading. We have a reasonable grading a regrading policy, see the syllabus.

**DUE: March 14, 2019. To be handed in at the beginning of lab.** The topics of this homework are: Eigenvalues/eigenvectors, diagonalization and symmetric matrices.

These topics are covered in §6.1, 6.2, and 6.4 of Strang.

Problems 1-8 are all or nothing; there is no partial credit available. Make sure you check your answers carefully, since you will receive no credit even for minor errors. Together, these 8 problems are worth 40 points (five points each).

1. (Problem 1, page 314) a) Factor these two matrices into $A = SAS^{-1}$:

   $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

   b) If $A = SAS^{-1}$, then $A^3 = ( ) ( ) ( )$ and $A^{-1} = ( ) ( ) ( )$.

2. (Problem 2, page 314) Suppose $A$ has $\lambda_1 = 2$ with eigenvector $s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with eigenvector $s_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $SAS^{-1}$ to find $A$. No other matrix has the same eigenvalues and eigenvectors.

3. (Problem 15, page 315) $A^k = SA^kS^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every $\lambda$ has absolute value less than ____________. Which of these matrices has $A^k \to 0$?

   $A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$

4. (Problem 16, page 316) Find $A$ and $S$ to diagonalize $A_1$ in problem 3. What is the limit of $A^k$ as $k \to \infty$? What is the limit of $SA^kS^{-1}$? In the columns of this limiting matrix, you see the ________________.

5. (Problem 29, page 317) Suppose the same $S$ diagonalizes both $A$ and $B$. They have the same eigenvectors in $A = SA_1S^{-1}$ and $B = SA_2S^{-1}$. Show that $AB = BA$.

6. (Problem 3, page 345) Write $A = X + N$, symmetric matrix $X$ plus skew-symmetric matrix $N$:

   $A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = X + N$
where \( X = X^T \) and \( N = -N^T \).

For any square matrix, \( X = \frac{1}{2}(A + A^T) \) and \( N = \) add up to \( A \).

7. (Problem 6, page 345) Find an orthogonal matrix \( Q \) that diagonalizes \( X = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix} \). What is \( \Lambda \)?

8. (Problem 16, page 346) This matrix \( M \) is antisymmetric and also \( \) add up to \( A \). Then, all of its eigenvalues are pure imaginary and they also have \( |\lambda| = 1 \). (\( ||Ms|| = ||\lambda s|| \) for every \( s \), so \( ||\lambda s|| = ||s|| \) for eigenvectors.) Find all four eigenvalues from the trace of \( M \):

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 \\
-1 & 1 & 0 & -1 \\
-1 & -1 & 1 & 0
\end{bmatrix}
\]

\( m = \frac{1}{\sqrt{3}} \) can only have eigenvalues \( i \) or \( -i \).
Problems 9 and 10 have partial credit. Together, these 2 problems are worth 60 points (30 points each).

Both problem 9 and 10 focus on age structured populations, which you’ve now seen a few times. Problem 9 explains the solution in terms of matrix diagonalization, and asks you to make connections to your work on Homework 5. Problem 10 is aimed at giving you an intuition for eigenvalues and eigenvectors, and their role in this problem. It includes a programming component, and asks you to type in code.

Recall... Here is a reminder of what you saw on HW 5: We were considering a population of ground squirrels with a life-span of three years. We divided the total population into three sub-populations: 1) 0-1 year-olds, \( n_{0-1} \); 2) 1-2 year-olds, \( n_{1-2} \); and 3) 2-3 year-olds, \( n_{2-3} \).

It takes 1 year to reach sexual maturity, so the number of 0-1 year-olds in the next year will be the reproductive rate of the 1-2 year-olds (\( r_1 \)) times the number of 1-2 year-olds, and the reproductive rate of the 2-3 year-olds (\( r_2 \)) times the number of 2-3 year-olds:

\[
n_{0-1}^{i+1} = r_1 n_{1-2}^i + r_2 n_{2-3}^i
\]

The number of 1-2 year-olds in the next year will be the 1-year survival probability (\( p_0 \)) of the 0-1 year-olds, and the number of 2-3 year-olds in the next year will be the 1-year survival probability (\( p_1 \)) of the 1-2 year-olds:

\[
\begin{align*}
  n_{1-2}^{i+1} & = p_0 n_{0-1}^i \\
  n_{2-3}^{i+1} & = p_1 n_{1-2}^i
\end{align*}
\]

where the subscript refers to the year (i.e., the current year is \( i \), so the next year is \( i + 1 \)).

This system of equations can be written as

\[
n_{i+1} = \begin{bmatrix} 0 & r_1 & r_2 \\ p_0 & 0 & 0 \\ 0 & p_1 & 0 \end{bmatrix} n_i = M n_i
\]

and, since this equation says that multiplication by the matrix \( M \) advances the population one year, it must be that

\[
n_j = M^j n_0
\]

where \( n_0 \) is the population at year 0 – that is, by multiplying the initial population by \( M \) \( j \) times, we age the population by \( j \) years.

Then, by writing the initial condition as a linear combination of the eigenvectors of \( M \), \( n_0 = a \hat{s}_1 + b \hat{s}_2 + c \hat{s}_3 \), we get the important solution

\[
n_j = a \lambda_1^j \hat{s}_1 + b \lambda_2^j \hat{s}_2 + c \lambda_3^j \hat{s}_3
\]

The importance of this solution is that if \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \), then

\[
\lim_{j \to \infty} n_j \to \lim_{j \to \infty} a \lambda_1^j \hat{s}_1
\]

This means that, after a long time, the relative number of 0-1, 1-2 and 2-3 year olds in the population is given by the relative values of the entries of the eigenvector, \( s_1 \). That is, if \( s_1 = \begin{bmatrix} s_{11} \\ s_{21} \\ s_{31} \end{bmatrix} \), then the proportion of the population that is between 0 and 1 years is \( s_{11}/(s_{11} + s_{21} + s_{31}) \).
9. Here, you will see how diagonalization can help you efficiently find the long-time behavior of such a population.

Suppose that, for a particular population, \( r_1 = 4, r_2 = 2, p_0 = 0.9, p_1 = 0.4 \).

a) Interpret each of those parameters in the context of the problem (i.e., \( r_1 = 4 \) means that 1-2 year old squirrels have, on average, two offspring).

b) Use Matlab to find the eigenvalues and eigenvectors of the matrix \( M \), repeated below with the given values for the parameters:

\[
M = \begin{bmatrix}
0 & 4 & 2 \\
0.9 & 0 & 0 \\
0 & 0.4 & 0 \\
\end{bmatrix}
\]

To do so, type in the following commands:

\[
>> [V,D]=eig([0 4 2;0.9 0 0;0 0.4 0])
\]

This command will return a 3x3 matrix \( V \), whose columns are the eigenvectors \( V = [s_1 \; s_2 \; s_3] \). Note that each eigenvector is normalized, i.e., its length is 1. It will also return a 3x3 matrix \( D \), whose diagonal entries are \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), starting from the upper left and ending at the lower right.

c) Suppose that \( n_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \). I.e., suppose that at year 0, we start with 100 newborn squirrels. What is the population after 100 years?

**Hints:** You will need to start by writing \( n_0 \) as \( as_1 + bs_2 + cs_3 \). To solve this, write an equation like

\[
n_0 = Ax
\]

where \( x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). Then, use Matlab to solve for \( x \), by inverting the matrix \( A \). Finally, plug everything into the equation \( n_j = a\lambda_j^1 \hat{s}_1 + b\lambda_j^2 \hat{s}_2 + c\lambda_j^3 \hat{s}_3 \).

You can check your work directly with Matlab by typing

\[
>> \text{check}=[0 4 2;0.9 0 0;0 0.4 0]^{100}*[100;0;0]
\]

However, be sure to show your work as described above, don’t just write the answer.

d) Compare the the proportion of the population that is between 0 to 1, 1 to 2, and 2 to 3 years old in \( n_{100} \) to the long-time limit (i.e. \( \lim_{j \to \infty} n_j \)), calculated from the eigenvector associated with the largest eigenvalue.

e) That was a pain, right? There’s a much easier way to get the answer in part c) – use diagonalization. To see how this works:

i) Show that your answer to part c) can be obtained from \( x = S^{-1}n_0 \), where \( S \) is the eigenvector matrix.

ii) Show that \( SA^j = [\lambda_j^1 \hat{s}_1 \; \lambda_j^2 \hat{s}_2 \; \lambda_j^3 \hat{s}_3] \).
Finally, show that, because $\begin{bmatrix} \lambda_1 \hat{s}_1 & \lambda_2 \hat{s}_2 & \lambda_3 \hat{s}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \lambda_1 \hat{s}_1 + b \lambda_2 \hat{s}_2 + c \lambda_3 \hat{s}_3$, we can then just find the population after any number of years by simply writing $n_j = S \Lambda^j S^{-1} n_0$.

In fact, this is so useful that Matlab’s “eig” command directly gives you $S$ and $\Lambda$!

10. Here, you will explore some simple programs to watch how the population distribution approaches the eigenvector associated with the dominant (i.e. largest) eigenvalue.

Let’s use a simple 2x2 example, because that’s easiest to visualize. Here, there is a population with two age groups – perhaps it is a population of mice with a two year lifespan, so the first entry of $n_i$ is the number of 0-1 year olds, and the second entry is the number of 1-2 year olds at year $i$

$$n_{i+1} = \begin{bmatrix} 1.2 & 2 \\ 0.5 & 0 \end{bmatrix} n_i = M n_i$$

a) Interpret each entry of the matrix $M$ in terms of the population model (i.e., what’s the probability that a 0-1 year old mouse survives to become a 1-2 year old mouse? How many offspring, on average, does a 0-1 year old mouse have? etc.).

b) By hand, calculate the eigenvalues and eigenvectors of this matrix (the numbers will be messy). You may use Matlab to check your work but, to get full credit, you must demonstrate that you know how to calculate the eigenvalues and eigenvectors of this matrix.

c) Given that $n_0 = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$, plot the two eigenvectors along with $n_0$, $n_1$, $n_2$, $n_3$, $n_4$, $n_5$, $n_6$, and $n_7$. Note, for the eigenvectors to be visible, you’ll need to multiply them by a pretty big number. Here, is some code that does it:

```
M=[1.2 2;0.5 0];
[v,d]=eig(M);
s_1=v(:,1);s_2=v(:,2);
n_0=[100;0];
figure(1);clf;plot(5000*[0 s_1(1)],5000*[0 s_1(2)],'r',5000*[0 s_2(1)],500*[0 s_2(2)],'b');
hold on;
for ii=0:7
    n=M^n_0;
    plot(n(1),n(2),'ro');
end
```

Run the code and print the plot. On the plot, label the two eigenvectors, and indicate which points are $n_0$, $n_1$, $n_2$, etc.

d) We can repeat the process for part c), but instead of plotting $n_i$, we can plot the proportion of mice aged 0-1 ($n_1/(n_1+n_2)$) and the proportion of mice aged 1-2 ($n_1/(n_1+n_2)$) along with the long-time values ($s_{11}/(s_{11}+s_{21})$ and $s_{21}/(s_{11}+s_{21})$) Here, is some code that does it:

```
M=[1.2 2;0.5 0];
[v,d]=eig(M);
s_1=v(:,1);s_2=v(:,2);
```
\[ n_0 = [100; 0]; x = 0:10; \]

```
figure(1); clf; plot(x, (s_1(1)/sum(s_1))*ones(1,length(x)), x, (s_1(2)/sum(s_1))*ones(1,length(x))); hold on;
for ii=0:7
    n=M^ii*n_0;
    plot(ii,n(1)/sum(n), 'ro', ii,n(2)/sum(n), 'bo');
end
```

Run the code and print the plot. On the plot, label which points and lines correspond to the proportion of mice aged 0-1 and which correspond to mice aged 1-2.