Midterm 1
MAT17C, Spring 2014, Walcott

Note: There are EIGHT questions. Points for each question are shown in a table below and also in parentheses just after each question (and part). Be sure to budget your time accordingly.

Student ID ________________________________

Name ________________________________

Section TA/time/number ________________________________

- Please write your name at the top of each page (in case the staple fails).
- Please ensure you have 10 pages (including this page, the equation page and scrap paper at the end).
- Please box your answers.
- Scrap paper is provided at the end of the exam; if you need more, just ask.
- Calculators, books, etc. are not allowed; you may only have a pen or pencil/eraser.
- Partial credit is available for all questions, but only if you show your work and it is legible.
- Read every question carefully and completely

April 30, 2014; 2:10 pm – 3:00 pm

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Potentially Useful Equations

For these equations, unless otherwise stated, assume there is a function of two variables \( f(x, y) \). The vector \( x \) is

\[
x = \begin{bmatrix} x \\ y \end{bmatrix}
\]

the notation \( f(x) = f(x, y) \),

the vector \( f(x) \) is

\[
f(x) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}
\]

\[
\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}
\]

\[
f(x, y) \approx f(x^*, y^*) + \frac{\partial f}{\partial x} \bigg|_{x^*, y^*} (x - x^*) + \frac{\partial f}{\partial y} \bigg|_{x^*, y^*} (y - y^*) = f(x^*) + \nabla f|_{x^*} \cdot (x - x^*)
\]

\[
f(x) \approx f(x^*) + J|_{x^*} \cdot (x - x^*)
\]

where the Jacobian matrix is

\[
J \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}
\]

\[
\nabla f \equiv \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}
\]

The slope in the direction of a unit vector \( \hat{e} \) is

\[
\nabla f \cdot \hat{e}
\]

At all critical points \( x_c \),

\[
\nabla f|_{x_c} = 0
\]

\[
\int_c^d \int_a^b f(x, y)dx dy = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} f(x_i, y_j) \Delta x \Delta y
\]

where \( N_x = (b - a)/\Delta x \), \( N_y = (d - c)/\Delta y \). \( x_i \) and \( y_j \) are the points at which the function is evaluated. They will typically be something like \( x_i = a + i \Delta x \) and \( y_j = c + j \Delta y \).

\[
\int_c^d \int_a^b f(x, y)dx dy = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} g(r, \theta) r dr d\theta
\]

The bounds on the integrals and the conversion between \( f(x, y) \) and \( g(r, \theta) \) depend on the problem.
1. (8 pts.) Suppose that the gradient of $z = f(x, y)$ at a particular point, $x^*$ is

$$\nabla f|_{x^*} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Given that

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

what is the slope in the direction of the vector $v$?

2. (8 pts.) Linearize the function

$$g(x, y) = x^2 \ln(y)$$

about $x = 1$, $y = 1$ and use your answer to estimate $g(1.1, 1.05)$. 
3. (8 pts.) Use a double integral to find the volume of the shape pictured below.
4. (8 pts.) Find the gradient of the following function.

\[ z = f(x, y) = x^2 e^{-xy} \]

Use your expression for the gradient to find the steepest slope at \( x = 1, \ y = 2 \).

5. (8 pts.) Consider the following function

\[ z = f(x, y) = y - x^2 \]

sketch contours at \( z = 0, \ z = 1 \) and \( z = 2 \). Be sure to label your axes, and indicate which contour is which.
6. (10 pts.) Look at the contour plot below for $z = f(x, y)$. Lines of constant altitude ($z$) are shown, the lines at $z = 5, 10$ and 15 are labeled and drawn in bold.

![Contour Plot]

a) (4 points) On the four labeled points a, b, c and d, draw an arrow in the direction of the gradient vector, $\nabla f$ (don’t worry about getting the length right).

b) (3 points) Which one of the gradient vectors would be the longest (circle one): a b c d. Why? (No points will be awarded without an explanation)

c) (3 points) Which one of the gradient vectors would be the shortest (circle one): a b c d. Why? (No points will be awarded without an explanation)
7. (25 pts.) SPECIATION Suppose you’re studying a population of animals. There are two inherited traits – say a tail (trait $a$) and antlers (trait $b$) that affect their probability of reproducing, $F$ (see sketch, below).

In the population, the proportion of animals with a tail is $a$ (so the proportion without a tail is $(1 - a)$), and the proportion of animals with antlers is $b$ (so the proportion without antlers is $(1 - b)$).

Animals that have antlers and a tail ($a$ and $b$) reproduce together successfully, similarly animals that have neither antlers nor a tail reproduce together successfully. The probability of reproducing is therefore

$$F = (1 - a)(1 - b) + ab$$

a. (7 points) Find $\nabla F$. 
b. (8 points) Use your answer to part a. to find the critical point (there is only one).

c. (6 points) Given the contour plot above,

i. is the critical point from part b a max/min/saddle? Explain.

ii. At what point(s) does the fitness, $F$, have a maximum?

iii. At what point(s) does the fitness, $F$, have a minimum?
d. (4 points). A more realistic model has a more complex contour plot (below), but the general features remain the same.

Explain what your population would look like after many generations (HINT: there are two possible correct answers). Explain your answer in a few sentences.
8. **(25 pts.)** HOW IS A PENDULUM LIKE A SPRING? A particle on a spring is good model for various things (such as a human or animal running). Here you will both see why it’s a good model.

![Diagram of Particle on a Spring and Pendulum](https://example.com/diagram.png)

The position, $x$, and velocity, $v$, of a particle on a spring obey the following differential equations, where $k$ and $m$ are positive constants (the spring constant and the particle’s mass, respectively).

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -\frac{k}{m}x
\end{align*}
\]  

(1)

a. **(6 points)** Let the vector $x = \begin{bmatrix} x \\ v \end{bmatrix}$ and $\frac{dx}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix}$. Rewrite the two equations (Eq. 1) as

\[
\frac{dx}{dt} = Mx
\]

Explicitly write out the matrix $M$ in terms of the constants $k$ and $m$. 

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b. (10 points) The angle $\theta$ and angular velocity $\omega$ of a pendulum are given by two coupled equations, where $g$ and $\ell$ are positive constants.

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -\frac{g}{\ell}\sin(\theta)$$

(2)

Linearize these equations about $\theta = 0, \omega = 0$.

HINT: Define a vector $x = \left[ \begin{array}{c} \theta \\ \omega \end{array} \right]$ and $dx/dt = \left[ \begin{array}{c} d\theta/dt \\ d\omega/dt \end{array} \right]$. Then, write out Eq. 2 as $dx/dt = f(x)$ and perform your linearization of $f$ about $x = 0$.

c. (3 points) Compare your answer from part b. to your answer from part a. They should be nearly identical. What is the “stiffness” of the pendulum? What is the “mass” of the pendulum? Explain your answers.
d. (6 points) You have now shown that a pendulum behaves like a particle on a spring under some conditions. What are those conditions?