Midterm 2
MAT17C, Spring 2013, Walcott

Note: There are SEVEN questions. Points for each question are shown in a table below and also in parentheses just after each question (and part). Be sure to budget your time accordingly.

STUDENT ID

NAME

SECTION TA/TIME/NUMBER

- Please write your name at the top of each page (in case the staple fails).
- Please ensure you have all 11 pages (including this page, the equation page and the scrap paper at the end).
- Please box your answers.
- Scrap paper is provided at the end of the exam; if you need more, just ask.
- Calculators, books, etc. are not allowed; you may only have a pen or pencil/eraser.
- Partial credit is available for all questions, but only if you show your work and it is legible.
- Read every question carefully and completely

May 20, 2013; 9:00 am – 9:50 am

<table>
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<tr>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
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<td>/12</td>
<td>/8</td>
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Potentially Useful Equations

For these equations, unless otherwise stated, assume there is a function of two variables $f(x, y)$. The vector $x$ is

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

the notation $f(x) = f(x, y)$, the vector $f(x)$ is

$$f(x) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

$$\int_c^d \int_a^b f(x, y) \, dx \, dy = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} f(x_i, y_j) \Delta x \Delta y$$

where $N_x = (b - a)/\Delta x$, $N_y = (d - c)/\Delta y$. $x_i$ and $y_j$ are the points at which the function is evaluated. They will typically be something like $x_i = a + i\Delta x$ and $y_j = c + j\Delta y$.

$$\int_c^d \int_a^b f(x, y) \, dx \, dy = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} g(r, \theta) r \, d\theta \, dr$$

The bounds on the integrals and the conversion between $f(x, y)$ and $g(r, \theta)$ depend on the problem.

The solution to the system of ODEs

$$\frac{dx}{dt} = Ax$$

where the matrix $A$ has eigenvectors $s_1$ and $s_2$ with associated eigenvalues $\lambda_1$ and $\lambda_2$ is

$$x = as_1 e^{\lambda_1 t} + bs_2 e^{\lambda_2 t}$$

The constants $a$ and $b$ are determined by the initial condition: $x(0) = as_1 + bs_2$.

If the real parts of every eigenvalue of the matrix $M$ are less than zero, then all trajectories head toward $x = 0$. This point is then said to be stable.

Eigenvalues real, positive – Unstable Node.
Eigenvalues real, negative – Stable Node.
Eigenvalues real, one positive, one negative – Saddle (unstable)
Eigenvalues imaginary, with negative real part – Stable Spiral
Eigenvalues imaginary, with positive real part – Unstable Spiral
Eigenvalues imaginary with no real part – Center.

$$f(x) \approx f(x^*) + J_{|x^*}, (x - x^*)$$

where the Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$
1. (12 pts.) Four phase portraits are shown below, each corresponding to a system of differential equations

\[
\frac{dx}{dt} = Mx
\]

where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). Each phase portrait came from a matrix on the list below. Write the letter corresponding to that matrix in the box at the upper left of the appropriate phase portrait. Eigenvectors and eigenvalues are listed beside each matrix.

a. \( M = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \)
   \( \lambda_1 = i\sqrt{2}, s_1 = \begin{bmatrix} 1 \\ -i\sqrt{2} \end{bmatrix} \)   \( \lambda_2 = -i\sqrt{2}, s_2 = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix} \)

b. \( M = \begin{bmatrix} -0.25 & -1 \\ 1 & 0 \end{bmatrix} \)
   \( \lambda_1 = -1 + \frac{i\sqrt{63}}{8}, s_1 = \begin{bmatrix} 1 \\ -1 - \frac{i\sqrt{63}}{8} \end{bmatrix} \)   \( \lambda_2 = -1 - \frac{i\sqrt{63}}{8}, s_2 = \begin{bmatrix} 1 \\ -1 + \frac{i\sqrt{63}}{8} \end{bmatrix} \)

c. \( M = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \)
   \( \lambda_1 = -1, s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)   \( \lambda_2 = -5, s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

d. \( M = \begin{bmatrix} 0.25 & -1 \\ 1 & 0 \end{bmatrix} \)
   \( \lambda_1 = -1 - \frac{i\sqrt{63}}{8}, s_1 = \begin{bmatrix} 1 \\ -1 + \frac{i\sqrt{63}}{8} \end{bmatrix} \)   \( \lambda_2 = -1 + \frac{i\sqrt{63}}{8}, s_2 = \begin{bmatrix} 1 \\ -1 - \frac{i\sqrt{63}}{8} \end{bmatrix} \)

e. \( M = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \)
   \( \lambda_1 = -5, s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)   \( \lambda_2 = -1, s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

f. \( M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)
   \( \lambda_1 = 1, s_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)   \( \lambda_2 = -1, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)

g. \( M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \)
   \( \lambda_1 = -1, s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)   \( \lambda_2 = 1, s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)

NOTE: axes are scaled equally, meaning that a circle would appear as a circle.
2. (8 pts.) Look at the following figure, sketched from J. D. G. Kooijman et al., Science, 322:339–342 (2011). It shows the eigenvalues of the Jacobian in the linearized equations of motion of a bicycle as a function of forward speed. The real parts are shown as solid lines, the imaginary parts as dashed lines.

On the figure, indicate the speeds at which the bicycle is STABLE.

3. (8 pts.) Use a double integral to find the volume under the function \( z = \sqrt{1 - (2x)^2 - (2y)^2} \) for \( z \geq 0 \). The function is sketched below:
4. (12 pts.) In the following figure, the region $0 \leq x \leq 1, 0 \leq y \leq 1$ is divided into 64 squares, each of width 0.125. In every square, there is a box with height equal to the sum of the squares of the $x$ and $y$ values of the back corner. For example, the drawing on the right shows a box of height $h = 0.75^2 + 0.375^2$.

![Diagram of boxes and coordinates]

a. (4 pts.) Using sigma notation, write a sum that, when evaluated, would give the total volume of all of the boxes.

b. (8 pts.) Suppose that you make the squares smaller and smaller, and more and more numerous. The height of a given box remains the sum of the squares of the $x$ and $y$ values of the back corner. As the boxes get infinitely small and infinitely numerous, what value does the total volume of all the boxes approach?
5. (10 pts.) The vector \( \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \) satisfies the differential equation

\[
\frac{dx}{dt} = M \mathbf{x}
\]

The matrix \( M \) has eigenvectors \( s_1 \) and \( s_2 \) and associated eigenvalues \( \lambda_1 = -10 \) and \( \lambda_2 = -1 \). The initial condition \( \mathbf{x}_0 \) is pictured at left (\( a \) and \( b \) are constants). On the axes at right, sketch the trajectory along which \( \mathbf{x}_0 \) approaches the fixed point \( \mathbf{x} = 0 \).

In a few sentences, briefly explain your sketch.
6. (25 pts.) Pendulum. Here is a sketch of a pendulum:

The angle of the pendulum, \( \theta \), obeys the following system of non-linear ODEs

\[
\begin{align*}
\frac{d\omega}{dt} &= -\frac{g}{L} \sin \theta - b\omega \\
\frac{d\theta}{dt} &= \omega
\end{align*}
\]

where \( b \), \( g \) and \( L \) are positive constants. There are two fixed points, one when the pendulum points straight down, \( \omega = 0, \theta = 0 \), and one when the pendulum points straight up, \( \omega = 0, \theta = \pi \).

a. (7 pts.) Linearize the non-linear ODEs about the fixed point where the pendulum points straight up, \( (\omega = 0, \theta = \pi) \). Write your answer in matrix-vector form.

b. (6 pts.) Based on your linearization, is this fixed point stable for the parameters \( b = 1 \), \( g = 8 \) and \( L = 4 \)? Classify the fixed point (i.e. is it a node, saddle, spiral, center, other).
For the parameters \( b = 1 \), \( g = 8 \) and \( L = 4 \), and defining \( x = \begin{bmatrix} \omega \\ \theta \end{bmatrix} \), the linearization about the other fixed point \((\omega = 0, \theta = 0)\) is

\[
\frac{dx}{dt} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} x
\]

Recall that this is the fixed point where the pendulum points straight down.

c. (6 pts.) Is this fixed point stable? Classify the fixed point (i.e. is it a node, saddle, spiral, center, other).

d. (6 pts.) Based on the linearization, sketch a phase portrait near \((\omega = 0, \theta = 0)\).
7. (25 pts.) CHEMICAL REACTIONS. Consider the following set of chemical reactions.

\[ \text{A} + \chi \xrightarrow{k_A} 2\chi \]
\[ x + y \xrightarrow{k_y} 2y \]
\[ y \xrightarrow{k_B} \text{B} \]

The net reaction is a molecule of \( \text{A} \) turning into a molecule of \( \text{B} \). Suppose that you keep the concentration of \( \text{A} \) constant, and observe the concentration of \( \text{B} \).

Assuming \( k_A = 1 \), \( A = 1 \), \( k_y = 1 \) and \( k_B = 1 \), the system can be represented by the following equations:

\[
\frac{dx}{dt} = x - xy \\
\frac{dy}{dt} = xy - y 
\]

For these equations, there are two fixed points: \( x = 0, y = 0 \) and \( x = 1, y = 1 \)

a. (7 pts.) For the fixed point at \( x = 1 \) and \( y = 1 \), write out the linearized equations.

b. (6 pts.) Based on your linearization, classify the fixed point (i.e. is it a node, saddle, spiral, center, other).
c. (6 pts.) Based on your linearization, sketch a phase portrait near the fixed point.

d. (6 pts.) Interpret your results. Specifically, suppose you measured the rate of formation of $B$ as a function of time (this rate $dB/dt = k_B y = y$). What does the linearization predict that you will see?