Midterm 2
MAT17C, Winter 2015, Walcott

Note: There are SEVEN questions. Points for each question are shown in a table below and also in parentheses just after each question (and part). Be sure to budget your time accordingly.

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Potentially Useful Equations

For these equations, unless otherwise stated, assume there is a function of two variables \( f(x, y) \). The vector \( \mathbf{x} \) is

\[
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

the notation \( f(\mathbf{x}) = f(x, y) \),

the vector \( f(\mathbf{x}) \) is

\[
f(\mathbf{x}) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}
\]

\[
f(\mathbf{x}) \approx f(\mathbf{x}^*) + J_{|\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*)
\]

where the Jacobian matrix is

\[
J = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix}
\]

The solution to the system of ODEs

\[
\frac{dx}{dt} = M\mathbf{x}
\]

where the matrix \( M \) has eigenvectors \( s_1 \) and \( s_2 \) with associated eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is

\[
\mathbf{x} = a s_1 e^{\lambda_1 t} + b s_2 e^{\lambda_2 t}
\]

the constants \( a \) and \( b \) are determined by the initial condition \( \mathbf{x}(0) = a s_1 + b s_2 \).

If the real parts of every eigenvalue of the matrix \( M \) are less than zero, then all trajectories head toward \( \mathbf{x} = 0 \). This point is then said to be stable.

Eigenvalues real, positive – Unstable Node.
Eigenvalues real, negative – Stable Node.
Eigenvalues real, one positive, one negative – Saddle (unstable).
Eigenvalues imaginary, with negative real part – Stable Spiral.
Eigenvalues imaginary, with positive real part – Unstable Spiral.
Eigenvalues imaginary, with no real part – Center (trajectories form closed paths)

Linearizing about a fixed point, \( \mathbf{x}^* \), we get

\[
\frac{d\mathbf{x}}{dt} = J_{|\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*)
\]

or, if \( \mathbf{u} = \mathbf{x} - \mathbf{x}^* \),

\[
\frac{d\mathbf{u}}{dt} = J_{|\mathbf{x}^*} \mathbf{u}
\]

Euler’s formula is

\[
e^{ait} = \cos(at) + i \sin(at)
\]

where \( a \) is a constant and \( i = \sqrt{-1} \).
1. (10 pts.) Consider the linear ODE
\[
\frac{dx}{dt} = Mx
\]
where \( x = \begin{bmatrix} x \\ y \end{bmatrix} \). Suppose that \( M \) has the following eigenvalue/vector pairs:
\[
\lambda_1 = -1, \ s_1 = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix} \quad \lambda_2 = -10, \ s_2 = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}
\]
On the axes provided below, sketch a trajectory \( x(t) \) starting from each of the four initial conditions that are indicated with an “o.” To get you started, I’ve sketched the first one.

![Axes with sketched trajectories]

2. (8 pts.) The following linear ODEs
\[
\frac{dx}{dt} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} x
\]
have a center at \( x = 0 \), meaning that every trajectory describes an ellipse.

Do trajectories go clock-wise or counter-clockwise? For full credit, you must clearly show all calculations that lead you to your answer.
3. (10 pts.) For the following linear ODEs

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x - ay 
\end{align*}
\]

For what values of \( a \) is the fixed point \( x = 0, y = 0 \) a stable node?

4. (10 pts.) For the following linear ODEs

\[
\begin{align*}
\frac{dx}{dt} &= 2x + y \\
\frac{dy}{dt} &= -x - y 
\end{align*}
\]

Sketch the flow direction (i.e. the vector \( \frac{dx}{dt} \)) at the four points indicated with an “o.”
5. (12 pts.) For the following flow field, sketch the two eigendirections.

Classify the fixed point \((x = 0, y = 0)\) as one of the following: Unstable node, Stable node, Saddle, Stable spiral, Unstable spiral, Center.
6. (25 pts.) Lotka-Volterra Predator/Prey This model describes a population of foxes, \( F(t) \), and rabbits, \( R(t) \). The foxes prey on the rabbits, while the rabbits eat grass. The population is described with the following non-linear equations:

\[
\frac{dR}{dt} = R - aRF \\
\frac{dF}{dt} = aRF - bF
\]  \( (1) \)

where \( a \) and \( b \) are positive constants.

a. (6 pts.) Equations 1 have a single fixed point with \( R > 0 \) and \( F > 0 \). In terms of \( a \) and \( b \), find this fixed point.
When $a = 2$ and $b = 2$, Equations 1 are

$$\frac{dR}{dt} = R - 2RF$$
$$\frac{dF}{dt} = 2RF - 2F$$

These equations have the fixed point $F = 1/2$, $R = 1$.

b. (12 pts.) Linearize these equations about the fixed point AND classify the fixed point as one of the following: Unstable node, Stable node, Saddle, Stable spiral, Unstable spiral, Center.
C. (7 pts.) On the following axes, sketch $R(t)$ and $F(t)$ starting from the given initial condition (indicated with an “o” on both plots)
7. **(25 pts.) Modeling an Epidemic**

A population is divided into three categories: those who are susceptible to the disease, $S(t)$, those who are infected with the disease $I(t)$, and those who have recovered from the disease, $R(t)$. These latter are resistant to infection for life.

The fraction of the population that is susceptible, $S(t)$, and infected, $I(t)$, is given by these non-linear ODEs.

$$\frac{dS}{dt} = 1 - S - 3SI$$

$$\frac{dI}{dt} = 3SI - 2I$$

(Note, the proportion of the population that is resistant ($R(t)$) can be found from the equation $S + I + R = 1$ – which says that everyone in the population is either susceptible, infected or resistant).

**a.** (12 pts.) There are two fixed points, $S = 1, I = 0$ (the disease is eradicated) and $S = 2/3, I = 1/6$ (the disease persists). Determine the stability (stable/unstable) and categorize (Unstable node, Stable node, Saddle, Stable spiral, Unstable spiral, Center), EACH of the two fixed points.
b. (7 pts.) Sketch a phase portrait near the fixed point $S = 2/3, I = 1/6$. (If it’s a node or saddle, sketch the eigendirections and ONE representative trajectory; if it’s a spiral or center, draw ONE representative trajectory).

c. (6 pts.) Suppose that a few infected individuals are introduced to an otherwise susceptible population. Based on your answers to parts a and b, briefly describe in words how the disease progresses.

(Your answer should address questions like: what fraction of the population is sick and what population is susceptible after a long time? does the disease die out or not?)