This worksheet examines the solutions to systems of linear differential equations. Specifically, we will see that the solutions can be written in terms of the eigenvalues and eigenvectors of the matrix of coefficients and that the solution form has a very useful geometric interpretation. In Part I, we will first consider an “uncoupled” system of differential equations (i.e., there is no interaction between the variables), and then next week in Part II, the concepts that we develop will be extended to “coupled” system of differential equations.

PART I - UNCOUPLED SYSTEM

[warm-up] Find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
-4 & 0 \\
0 & -2
\end{bmatrix}.
\]
Consider the two (uncoupled) differential equations for $x(t)$ and $y(t)$

\[
\frac{dx}{dt} = -4x, \quad x(0) = c_1,
\]

and

\[
\frac{dy}{dt} = -2y, \quad y(0) = c_2,
\]

1. What is the solution to the differential equation for $x(t)$ with the initial condition $x(0) = c_1$?

2. What is the solution to the differential equation for $y(t)$ with the initial condition $y(0) = c_2$?
Now, consider the two differential equations as a system of differential equations

\[
\begin{align*}
\frac{dx}{dt} &= -4x \\
\frac{dy}{dt} &= -2y
\end{align*}
\]

with the initial conditions

\[
\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\]

Note that the solution to this system of differential equations can be written in vector form as

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-4t} \\ c_2 e^{-2t} \end{bmatrix}.
\]  \hspace{1cm} (1)

However, it is useful to write the solution as follows

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]  \hspace{1cm} (2)

3. Simplify the solution in form (2) to verify that it is equivalent to the solution in form (1), and make sure you understand this notation, as it will be very important later on.
We can write the system of differential equations in matrix/vector form as

\[
\frac{d}{dt} \vec{X} = A \vec{X},
\]

where \( A = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} \) is the matrix of coefficients, and \( \vec{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \).

4. Expand the system in matrix to verify that it is equivalent to the system of differential equations on previous page.

5. What are the eigenvalues of the matrix of coefficients \( A \) for the above system of differential equations? (See warm-up problem on page 1.)
6. Describe how the eigenvalues of $A$ appear in the solution. There are two vectors that appear in the solution, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Verify that these vectors are the eigenvectors of the matrix of coefficients.

Thus, the solution in form (2) shows us that we can *decompose* the solution (and trajectory) of the system of differential equations into two components. The “direction” of these components are given by the eigenvectors of *matrix of coefficients* $A$, and the rate of change of the components are determined by the corresponding eigenvalues of $A$. We will see that solutions to general systems of linear differential equations can be written in this form.
Now that we have a solution to the system of differential equations, let’s visualize it and interpret it graphically. For particular initial conditions, take $c_1 = 4$ and $c_2 = 4$.

7. Based on your solution, compute values of $x(t)$ and $y(t)$ for $t = 0, 0.25, 0.50, 0.75, ..., 2.0$, and record it in the table below.¹

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>$y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Using the information in the table, plot $x(t)$ and $y(t)$ (i.e., $x$ vs $t$, and $y$ vs $t$) on the same graph.

¹Note: We could use R/R-fiddle to help compute $x(t)$ and $y(t)$, and plot related graphs. See code in Appendix.
Using the information in the table, we can also plot the trajectory of the solution. That is, we can plot the points of \((x(t), y(t))\) as \(t\) changes in the \(xy\)-plane (called the phase plane).

**9.** Plot the \((x(t), y(t))\) points from the table that you created above in the phase plane. Label them with the corresponding \(t\) value. Draw a smooth curve through the points.

The trajectory you just plotted above could be notated by \[
\begin{bmatrix}
x(t) \\ y(t)
\end{bmatrix}.
\]

**10.** Plot the trajectories of \[
\begin{bmatrix}
x(t) \\ 0
\end{bmatrix}
\] and \[
\begin{bmatrix}
0 \\ y(t)
\end{bmatrix}
\] on your graph. Label each point you plot with the corresponding value of \(t\).
11. Compare the points $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\begin{bmatrix} x(t) \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y(t) \end{bmatrix}$ for $t = 0, .5, \text{ and } 1$. What is the relationship between the three points in each case?

To see the relationship between the three points $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\begin{bmatrix} x(t) \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y(t) \end{bmatrix}$ for general values of $t$, note that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y(t) \end{bmatrix} = x(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

That is, you have full plotted the solution trajectory, and the two components (in the direction of the eigenvectors) that add up to the full solution. It will be very useful to consider graphical representations of solutions to general systems of linear differential equations in this manner.
APPENDIX – R-CODE

To compute $x(t)$ and $y(t)$ for $t = 0, 0.25, 0.50, 0.75, ..., 2.0$, types the following com-
mands in R-fiddle and run the code.

```r
(t = seq(0.0, 2.0, 0.25)
x = function (t) {4*exp(-4*t)}
y = function (t) {4*exp(-2*t)}
x(t)
y(t)
```

Note that the first line defines a sequence of $t$ values (0.0 to 2.0 by 0.25), the second
two lines define the functions for $x(t)$ and $y(t)$, and the last two line evaluate $x(t)$
and $y(t)$ at the defined values of $t$.

You can also plot these $x(t)$ and $y(t)$ on the same graph by including the following
commends in your program

```r
plot(t, x(t), "b")
lines(t, x(t), "b", pch=25)
```

or instead plot the trajectories $(x(t), y(t))$, $(x(t), 0)$, and $(0, y(t))$ by including

```r
plot(x(t), y(t), "b")
lines(x(t), 0*y(t), "b")
lines(0*x(t), y(t), "b")
```