This worksheet examines the solutions to systems of linear differential equations. Specifically, we will see that the solutions can be written in terms of the eigenvalues and eigenvectors of the matrix of coefficients and that the solution form has a very useful geometric interpretation. In Part I, last week, you considered an “uncoupled” system of differential equations (i.e., there was no interaction between the variables), and now, in Part II, we will extend those concepts to “coupled” system of differential equations.

**PART II - COUPLED SYSTEM**

[warm-up] Find the eigenvalues and eigenvectors of the matrix

\[ A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}. \]
The system in the Part I was “uncoupled.” That is, $x$ did not affect the way $y$ changed, and the change in $x$ was not affected by $y$. However, in general (as discussed in class), there could be interaction between $x$ and $y$ such that the rate of change of $x$ depends on both $x$ and $y$, and similarly, the rate of change of $y$ depends on both $x$ and $y$. Such a “coupled” (linear) system could be described by

\[
\frac{dx}{dt} = ax + by \\
\frac{dy}{dt} = cx + dy,
\]

or in matrix/vector form,

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]
The result we found for the uncoupled system can be generalized. In fact, we can always write the general form of the solution to a system of differential equations like this one as

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},
\]

where \(\lambda_1\) and \(\lambda_2\) are the eigenvalues of \(A\) and \(\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}\) and \(\vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}\) are the eigenvectors of \(A\).

Compare this solution to the solution for the uncoupled system.

1. Fill in the blanks above.

Let’s now try to solve a particular linear system. We’ll use R to help.

2. Suppose we have the system

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix},
\]

where \(A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}\)

3. Show that the eigenvalues and eigenvectors of \(A\) are

\[
\lambda_1 = -4 \quad \vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\lambda_2 = -2 \quad \vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

Note that the eigenvectors can be multiplied by any constant value.
[You can compute the eigenvalues and eigenvectors using R by the following commands
> A = matrix(c(-3,-1,-1,-3),2,2,byrow=T)
> eigen(A)
]

4. What’s the general solution?

5. Find a specific solution for the initial conditions \( x(0) = 4, \ y(0) = 0 \).

Repeat the analysis that you perform for the uncoupled system on this coupled system.

6. Compute values of \( x(t) \) and \( y(t) \) for \( t = 0, 0.2, 0.4, 0.6, \ldots, 1.2 \), and record it in a table.

7. Use the information in the table to plot \( x(t) \) and \( y(t) \).

8. Plot the trajectory \((x(t), y(t))\) in the \( xy\) phase plane. Label them with the corresponding \( t \) value.

9. Include the trajectories for the two components of the solution, \( c_1 e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \) and \( c_2 e^{\lambda_2 t} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \), in the \( xy\) phase plane. Label each point you plot with the corresponding value of \( t \).

10. Compare the points on \( \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \), \( c_1 e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \) and \( c_2 e^{\lambda_2 t} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \) for various values of \( t \). What is the relationship between the points for particular values of \( t \)? Think about the form of the solution to answer this question.

To see an animation of the exercise that you just did, download and run the program curve2.R from my website.