**Circular Heegaard Splittings of Knot Exteriors**

By

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## Abstract

Given a knot in $S^3$, we define circular Heegaard splittings for the knot exterior and varying notions of distance associated to this circular Heegaard splitting. Analogous to a theorem proven by Hartshorn in 2002 for Heegaard splittings of Haken 3-manifolds, we show that if the thin level of the circular Heegaard splitting is incompressible, and if the knot exterior contains a closed, orientable, essential surface $T$, then the distance of the circular Heegaard splitting is bounded above by twice the genus of $T$. We then consider a knot $L$ contained inside the exterior of $K$ that is in a "nice" position with respect to the circular Heegaard splitting, which we will call being in circular bridge position. Analogous to a theorem proven by Bachman and Schleimer in 2005, we generalize our previous results by showing that if the link complement contains a compact, orientable, connected, properly-embedded surface $T$ disjoint from the boundary of the exterior of $K$, then circular distance of the circular Heegaard splitting is bounded above by twice the genus of $T$ plus the number of boundary components of $T$. Surprisingly, the requirement that the thin level be incompressible can be dropped. We give applications of our results and present some questions for further research.

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CHAPTER 1

# Introduction

In this paper, we consider a knot $K$ in $S^3$. We first define a notion of a circular Heegaard splittings for the exterior $E(K)$ of $K$, which can be seen as a generalization of a fiber surface for a fibered knot. We then define a notion of circular distance of circular Heegaard splittings, which can be seen as a generalization of translation distance for a fiber surface. Analogous to a theorem of Hartshorn in 2002 [**9**], we show that if the exterior of $K$ contains a closed, orientable, essential surface $T$, then under certain hypotheses, the distance of circular Heegaard splitting is bounded above by twice the genus of the surface. After, we define a notion of circular bridge position for a knot $L$ with respect to the circular Heegaard splitting. We prove that if the exterior of the link $K \cup L$ contains a compact, properly-embedded, orientable, connected, essential surface whose boundary, if non-empty, is disjoint from $\partial E(K)$ (and hence is on $\partial E(L)$), then under certain hypotheses, the thick distance of the circular Heegaard splitting is bounded above by twice the genus of the surface plus the number of boundary components of the surface. We also provide some applications of our results.

We first introduce some terminology and then provide an outline of the thesis. A handlebody is a 3-manifold that is homeomorphic to a closed regular neighborhood of a finite graph in $S^3$. If $M$ is a closed, orientable, connected 3-manifold, then a Heegaard splitting for $M$ is a surface $\Sigma$ such that $\Sigma$ is the boundary of two handlebodies $H_1, H_2 \subseteq M$ such that $H_1 \cup H_2 = M$ and $H_1 \cap H_2 = \Sigma$. We say that $\Sigma$ is reducible if there exists an essential simple closed curve on $\Sigma$ that is the boundary of a disk in both $H_1$ and $H_2$. We say that $\Sigma$ is strongly irreducible if for any pair of disks $D_1 \subseteq H_1$ and $D_2 \subseteq H_2$ whose boundaries are essential in $\Sigma$, we have $\partial D_1 \cap \partial D_2 \neq \emptyset$. A Heegaard splitting that is not strongly irreducible is said to be weakly reducible, and a Heegaard splitting that is not reducible is said to be irreducible.

In his 1898 thesis [**11**], Poul Heegaard defined the notion of a Heegaard splitting. In 1987, Casson and Gordon [**3**] generalized this notion to compact, orientable manifolds with boundary.

To do this, they defined a generalization of a handlebody, called a compression body. They also proved that a weakly reducible Heegaard splitting is either reducible or the manifold contains an incompressible surface of positive genus.

In his 2001 paper, Hempel [**12**] defined a notion of distance of Heegaard splittings that unifies notions of complexity of a Heegaard splitting; see also [**19**]. We will provide a brief definition here. First, if $\Sigma$ is a surface of genus $g \in \mathbb{N}$ with $g > 1$, then the curve complex for $\Sigma$ is the simplicial complex where each $n$-cell corresponds to a set of $n+1$ distinct isotopy classes of disjoint essential simple closed curves. The vertices of this curve complex correspond to essential simple closed curves in $\Sigma$, and edges correspond to a pair of non-isotopic essential simple closed curves that can be isotoped to be disjoint. The 1-skeleton of this complex can be equipped with a graph metric, where each edge is given length 1. If $M$ is a closed, orientable, connected 3-manifold and $\Sigma$ is a Heegaard splitting for $M$ that cuts $M$ into handlebodies $H_1$ and $H_2$, then the distance of $\Sigma$ is the minimum distance in the curve complex between vertices representing a curve $\alpha_1 \subseteq \Sigma$ that bounds a disk in $H_1$ and a curve $\alpha_2 \subseteq \Sigma$ that bounds a disk in $H_2$. For example, a reducible Heegaard splitting is distance 0, and a strongly irreducible Heegaard splitting is distance at least 2. See Figure 1.1.

A 3-manifold $M$ is Haken if $M$ is irreducible and contains an incompressible surface of genus $g \geq 1$. In 2002, Hartshorn [**9**] proved that if a closed, orientable, connected 3-manifold $M$ is Haken containing a closed, orientable, connected, essential surface of genus $g$, then the distance of any Heegaard splitting for $M$ is bounded above by $2g$. This can be seen as a generalization of Casson and Gordon's result.

Recall the definition for a knot to be in bridge position with respect to a Heegaard splitting; see [**2**]. Let $M$ be a closed, orientable, connected 3-manifold, let $\Sigma$ be a Heegaard splitting for $M$ such that $\Sigma$ cuts $M$ into handlebodies $H_1$ and $H_2$, and let $K$ be a knot for $M$. We say that $K$ is in bridge position with respect to $\Sigma$ if $K$ meets $H_1$ and $H_2$ in a collection of trivial arcs. In 2005, Bachman and Schleimer [**2**] generalized Hartshorn's work by showing that if $K$ is a knot in a closed, orientable, connected 3-manifold $M$ such that $K$ is in bridge position with respect to a Heegaard surface for $M$, and if the knot complement contains a compact, orientable, connected, essential, properly-embedded surface of genus $g$ with $b$ boundary components, then the distance of the Heegaard splitting with respect to $K$ is bounded above by $2g + b$.

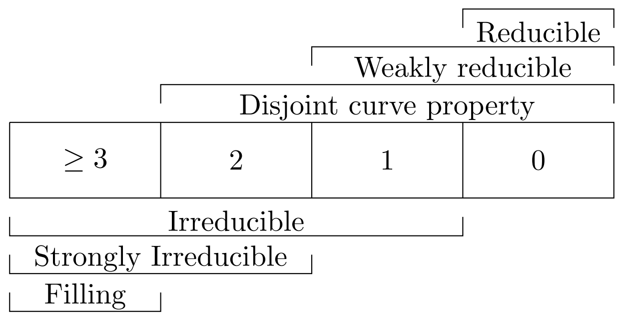


FIGURE 1.1. Complexities of a Heegaard splitting and their relationship in terms of Hempel distance.

Note that a Heegaard splitting $\Sigma$ of a closed, orientable, connected 3-manifold $M$ induces a self-indexing Morse function $h\colon M \to [0,1]$ with $\Sigma = h^{-1}(1/2)$. In this paper, we will investigate when the interval $[0,1]$ is replaced with the circle $S^1$ in the case of knot exteriors and follow a path of results similar to those outlined above in this new context.

In chapter 2, we define the notion of a circular Heegaard splitting of a knot exterior, circular distance (denoted by $cd(\cdot,\cdot)$), and thick distance (denoted by $td(\cdot)$). We then prove an analog of Hartshorn's result stated as follows:

**1. Theorem.** *Let $K$ be a knot in $S^3$, and suppose $(F,S)$ is a circular Heegaard splitting for $E(K)$ with $F$ incompressible and $F$ and $S$ non-parallel. Suppose also that the exterior $E(K)$ of $K$ contains a closed, orientable, connected, essential surface of genus $g$. Then*

$$cd(F,S) \leq 2g.$$

Whereas circular distance $cd(F,S)$ can be seen as a measure of complexity of the pair $(F,S)$, thick distance $td(S)$ can be seen as a measure of the thick surface $S$ only. As a corollary, we have a similar result for the thick distance: $td(S) \leq 2g$. We then provide applications of this theorem. All work in done in Chapter 2 is joint work with Kevin Lamb; see [**14**].

In chapter 3, we define what it means for a knot to be in bridge position with respect to a circular Heegaard splitting and prove an analog of Bachman and Schleimer's result in this setting.

**2. Theorem.** *Let $K$ be a knot in $S^3$ and suppose that $\mathcal{D} = (F, S)$ is a circular Heegaard splitting for $E(K)$. Suppose that $L$ is a knot in $E(K)$ that is in circular bridge position with respect to $(F, S)$. Suppose also that $E(K)$ contains a compact, orientable, connected, properly-embedded, essential surface of genus $g$ with $b$ boundary components which is disjoint from $\partial E(K)$, but which may have boundary on $\partial E(L)$. Then*

$$td(L, \mathcal{D}) \leq 2g + b$$

Note that Theorem 2 is a generalization of the corollary to Theorem 1 (i.e. when $L = \emptyset$). Surprisingly, the hypothesis that $F$ is incompressible is dropped. Informally, this is because there is more flexibility in how the essential surface surface sits with respect to the circular Heegaard splitting. We then provide an application and some questions for further research.

Before we start, we introduce some notation used throughout this paper. If $X$ is a subset of a 3-manifold $M$, then $\eta(X; M)$ denotes an open regular neighborhood of $X$ in $M$, or just $\eta(X)$ when the ambient space $M$ is understood. We let $E(X, M) = M - \eta(X, M)$, or just $E(X)$ when the ambient space $M$ is understood. Also, we let $\mathring{X}$ denote the interior of $X$ and $\overline{X}$ denote the closure of $X$ in $M$. Finally, if $X \subseteq M$, then $|X|$ denotes the number of connected components of $X$.

CHAPTER 2

# Circular Heegaard Splittings of Knot Exteriors

We start by stating the main theorem of this chapter.

**3. Theorem.** *Let $K$ be a knot in $S^3$. Suppose that $(F, S)$ is a circular Heegaard splitting for $E(K)$ with $F$ incompressible in $E(K)$ and $F$ and $S$ non-parallel, and suppose that $E(K)$ contains a closed, connected, orientable, essential surface $T$. If $g(T)$ is the genus of $T$, then*

$$cd(F, S) \leq 2g(T),$$

Before proving this theorem, we will establish some background and definitions. This chapter is organized as follows:

(1) We first define a restricted notion of $\partial$-compressions in a compression body with non-empty vertical boundary, called a $\partial^*$-compression.
(2) We then define the notions of circular Heegaard splittings and circular distance of circular Heegaard splittings. One can think of circular Heegaard splittings as generalizations of fiber surfaces of knots, where in this new context the surface may have critical points as it sweeps out $E(K)$. Accordingly, circular distance can be thought of as a generalization of translation distance. In this context, circular distance is computed by minimizing the distances between curves on $F \cup S$ that bound essential disks or essential spanning annuli in one component of $E(K) - (F \cup S)$ and curves on $F \cup S$ that bound essential disks or essential spanning annuli in the other component of $E(K) - (F \cup S)$.
(3) W then describe how performing $\partial^*$-compressions of $T$ across $F \cup S$ affects $T \cap (F \cup S)$.
(4) We then establish the existence of a sequence of $\partial^*$-compressions of $T$ across $F \cup S$ so that $T \cap (F \cup S)$ starts by bounding an essential disk or an essential spanning annulus in one component of $E(K) - (F \cup S)$ and ends by bounding an essential disk or an essential spanning annulus in the other component of $E(K) - (F \cup S)$.

(5) We prove our main theorem by showing that this sequence of $\partial^*$-compressions places a bound on the circular distance of $(F, S)$.

(6) We finish this chapter with applications to uniqueness of Seifert surfaces and essential tangle decompositions.

## 2.1. $\partial^*$-Compressions

Recall that a compression body $W$ is a cobordism rel $\partial$ between surfaces $\partial_- W$ and $\partial_+ W$ such that $W$ is homeomorphic to $(\partial_- W \times I) \cup (0\text{-handles}) \cup (1\text{-handles})$ and $\partial_- W$ contains no 2-sphere components; see [**3**]. We will denote $\partial_v W = \partial(\partial_- W) \times I$, so $\partial W = \partial_- W \cup \partial_v W \cup \partial_+ W$. If the genera of $\partial_+ W$ and $\partial_- W$ are $g$ and $g'$, respectively, and each have $b$ boundary components, then we call $W$ a $(g, g', b)$-compression body. See Figure 2.1.

Note that if $b > 0$ and $\partial_- W$ is connected, then a $(g, g', b)$-compression body $W$ is homeomorphic to a handlebody of genus $(g - g') + 2g' + (b - 1) = g + g' + b - 1$. As in [**9**], we will eventually perform $\partial$-compressions across the boundary of $W$; however, we wish to avoid performing $\partial$-compressions through $\partial_v W$, since $\partial_v W$ will represent the boundary of the knot exterior in our setting. We present a restricted definition of $\partial$-compressions for compression bodies.

**4. Definition.** Let $W$ be a compression body, and let $T$ be a compact, properly-embedded surface in $W$. We say that $T$ is $\partial^*$-compressible if there exists an essential arc $\alpha$ in $T$ and an arc $\beta$ in $\partial W$ such that

(1) $\partial \alpha = \alpha \cap \beta = \partial \beta$,

(2) $\alpha \cup \beta$ bounds a disk $D$ in $W$ with $D \cap T = \alpha$ and $D \cap \partial W = \beta$, and

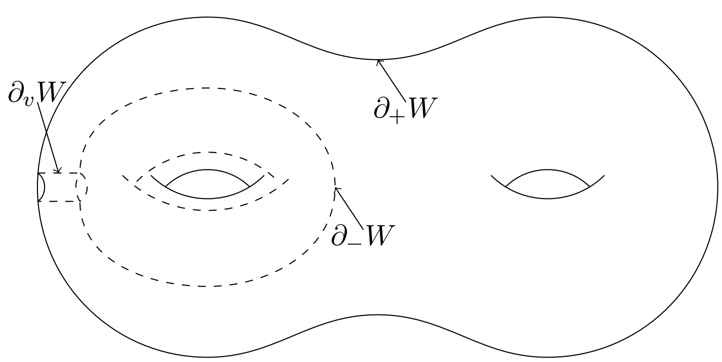


FIGURE 2.1. A $(2, 1, 1)$-compression body.

(3) $\beta \subseteq \partial_+ W$ or $\beta \subseteq \partial_- W$.

We say that $T$ is $\partial^*$-compressible through $\partial_+ W$ or $\partial_- W$, respectively, and we call $D$ a $\partial^*$-compressing disk for $T$.

When performing an isotopy of a compression body $W$, we wish to keep $\partial_v W$ distinct from $\partial_+ W$ and $\partial_- W$ for a similar reason. So, we present a modified definition of isotopy in our setting. Here, "inv" is short for "invariant".

**5. Definition.** Let $W$ be a compression body. An isotopy inv $\partial_v W$ is an isotopy $\{f_t\}_{t \in I}$ of $W$ such that for all $t \in I$, we have $f_t(x) \in \partial_v W$ if and only if $x \in \partial_v W$.

**6. Definition.** Let $W$ be a compression body, and let $T$ be a properly-embedded surface in $W$. We say that $T$ is $\partial^*$-parallel if $T$ is parallel to $\partial W - \partial_v W$.

**7. Definition.** Let $W$ be a compression body. A spanning annulus in $W$ is an annulus with one boundary component in $\partial_- W$ and one boundary component in $\partial_+ W$. We say that a spanning annulus is essential if each boundary component is essential in $\partial_- W$ and $\partial_+ W$.

An example of a spanning annulus is given in Figure 2.2. As an importantly class of examples, note that a spanning annulus can be viewed as $\alpha \times I \subseteq \partial_- W \times I$ for some curve $\alpha \subseteq \partial_- W$. In particular, this implies that if $g(\partial_+ W) > g(\partial_- W)$, then every spanning annulus is disjoint from some disk $D$ in $W$ with $D \cap \partial W = \partial D \subseteq \partial_+ W$ and $\partial D$ essential in $\partial_+ W$.

## 2.2. Circular Heegaard Splittings and Circular Distance

We now give the definition of a circular Heegaard splitting. The following definition can also be found in [**5**]. For a more general setting dealing with circular handle decompositions, see Manjarrez-Gutierrez's paper [**15**].

**8. Definition.** Let $K$ be a knot in $S^3$. Suppose $F$ and $S$ are Seifert surfaces for $K$. We will abuse notation and use $F$ and $S$ to denote $F \cap E(K)$ and $S \cap E(K)$, respectively. We say $(F, S)$ is a circular Heegaard splitting for $E(K)$ if

(1) $\mathring{F} \cap \mathring{S} = \emptyset$,

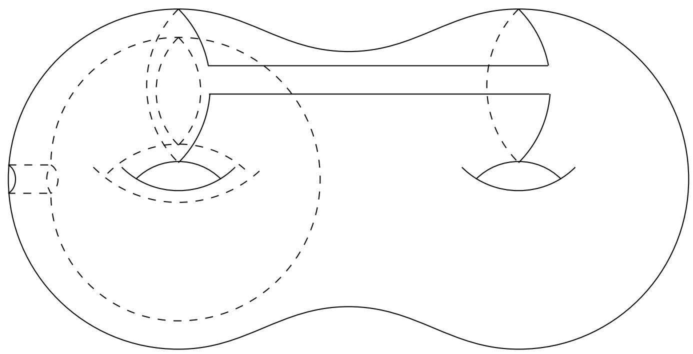

FIGURE 2.2. The two curves together bound an essential spanning annulus in the compression body.

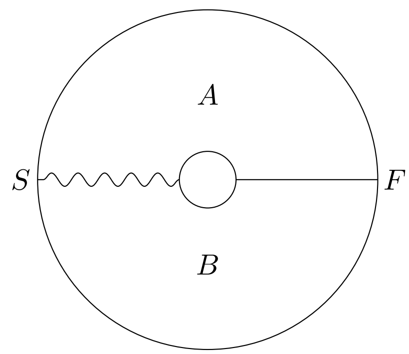


FIGURE 2.3. A diagram for a circular Heegaard splitting. This picture is similar to Figure 6 of [**15**].

(2) $F \cup S$ splits $E(K)$ into compression bodies $A$ and $B$ such that $\partial_+ A = S = \partial_+ B$ and $\partial_- A = F = \partial_- B$, and

(3) $F \cap \overline{\eta(K)}$ and $S \cap \overline{\eta(K)}$ are annuli joining $K$ to an essential simple closed curve of $\partial\eta(K)$.

See Figure 2.3.

Before defining distance of a circular Heegaard splitting, we first recall the definition of the curve complex of a surface; see [**12**] for more details. Let $S$ be a compact, orientable, connected, surface. We construct an abstract simplicial complex $\mathcal{C}(S)$ as follows: for all $k \in \mathbb{N} \cup \{0\}$, a set $\{[\alpha_0], \cdots, [\alpha_k]\}$ of isotopy classes of essential simple closed curves in $S$ represents a $k$-simplex in $\mathcal{C}(S)$ if for all $i, j \in \{0, \cdots, k\}$ with $i \neq j$, we have $[\alpha_i]$ and $[\alpha_j]$ are distinct and there exist $\alpha'_i \in [\alpha_i]$ and $\alpha'_j \in [\alpha_j]$ such that $\alpha'_i \cap \alpha'_j = \emptyset$. See Figure 2.4.

**9. Definition.** We call $\mathcal{C}(S)$ the <u>curve complex of $S$</u>.

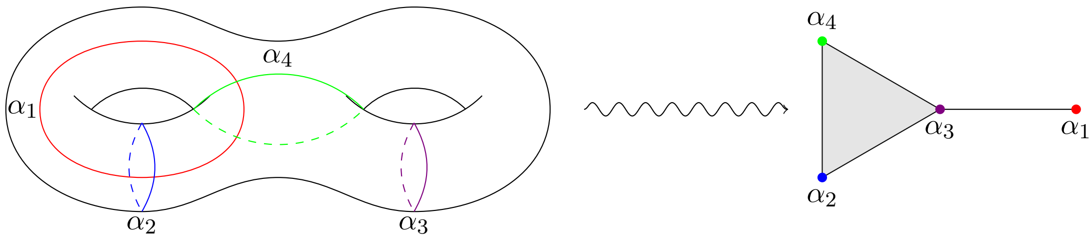


FIGURE 2.4. A portion of $\mathcal{C}(S)$.

We will abuse notation and identify an essential simple closed curve with its isotopy class. We equip the 1-skeleton $\mathcal{C}^1(S)$ with the graph metric, so each edge in $\mathcal{C}^1(S)$ has length 1. For essential simple closed curves $\alpha$ and $\beta$ in $S$, we let $d_{\mathcal{C}(S)}(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in $\mathcal{C}^1(S)$. If $\chi(S) > -2$, then for any vertices $\alpha, \beta \in \mathcal{C}^1(S)$, we define $d_{\mathcal{C}(S)}(\alpha, \beta)$ to be 0. For any vertex $\alpha \in \mathcal{C}^1(S)$, we define $d_{\mathcal{C}(S)}(\alpha, \emptyset)$ to be 0. We also define $d_{\mathcal{C}(S)}(\emptyset, \emptyset)$ to be 0. Lastly, note that if $\varphi\colon S \to S$ is a homeomorphism, then $\varphi$ induces a distance-preserving graph isomorphism on $\mathcal{C}^1(S)$.

We are now ready to define circular distance. Let $K$ be a knot in $S^3$. Suppose $(F, S)$ is a circular Heegaard splitting for $E(K)$ separating $E(K)$ into compression bodies $A$ and $B$. Let $\Pi_A$ and $\Pi_B$ denote the set of all essential spanning annuli and essential disks with boundary on $S$ in $A$ and in $B$, respectively.

**10. Definition.** The <u>circular distance</u> of $(F, S)$, $cd(F, S)$ is defined as follows:

$$cd(F, S) = \min\{d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B)\colon P_A \in \Pi_A,\ P_B \in \Pi_B\},$$

where

$$\partial_F P_A = \partial P_A \cap F, \quad \partial_S P_A = \partial P_A \cap S, \quad \partial_F P_B = \partial P_B \cap F, \quad \text{and} \quad \partial_S P_B = \partial P_B \cap S.$$

Since it will be useful to focus on $S$ instead of $(F, S)$ at some points, we introduce the notion of thick distance for a circular Heegaard splitting.

**11. Definition.** The <u>thick distance</u> of $(F, S)$ is

$$td(S) = \min\{d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B)\colon P_A \in \Pi_A,\ P_B \in \Pi_B\},$$

Also, note that $S$ can be viewed as a Heegaard splitting for $E(K) - \eta(F)$, so we may consider the usual distance of a Heegaard splitting. Let $\Delta_A$ and $\Delta_B$ denote the set of all essential disks with boundary in $S$ and contained in $A$ and in $B$, respectively.

**12. Definition.** The Hempel distance for $S$ is

$$d(S) = \min\{d_{\mathcal{C}(S)}(\partial P_A, \partial P_B)\colon P_A \in \Delta_A,\, P_B \in \Delta_B\}.$$

Note that $\Delta_A = \emptyset$ if and only if $\Delta_B = \emptyset$. In this case, the knot $K$ is a fibered knot, and we define $d(S) = 0$.

These three notions of distance are similar in definition. The following theorem is a consequence of this similarity. It states that when $F$ and $S$ are non-parallel, one cannot make the absolute difference between any two different distances arbitrarily large.

**13. Theorem.** *Let $K$ be a knot in $S^3$. Suppose $(F, S)$ is a circular Heegaard splitting for $E(K)$.*
*a) We have*

$$0 \leq td(S) \leq cd(F, S).$$

*b) If $F$ is not isotopic to $S$ (e.g. $F$ is not a fiber surface), then*

$$\max\{0, d(S) - 2\} \leq td(S) \leq cd(F, S) \leq d(S) \leq td(S) + 2.$$

Proof. a) We have $0 \leq td(S)$ since distance in the curve complex is non-negative. Suppose $P_A \in \Pi_A$ and $P_B \in \Pi_B$ realize $cd(F, S)$, so that $cd(F, S) = d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B)$. Then

$$\begin{aligned} cd(F, S) &= d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B) \\ &\geq d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) \\ &\geq td(S). \end{aligned}$$

b) Suppose $F$ is not isotopic to $S$. As before, we have $td(S) \leq cd(F, S)$. Since $\Delta_A \subseteq P_A$ and $\Delta_B \subseteq P_B$, we have $d(S) \geq cd(F, S)$.

Suppose that $P_A \in \Pi_A$ and $P_B \in \Pi_B$ realize $td(S)$. Since $F$ is not isotopic to $S$, then there exist disks $D_A \in \Delta_A$ and $D_B \in \Delta_B$ whose boundaries are essential in $S$ such that $d_{\mathcal{C}(S)}(\partial_S P_A, \partial D_A) \leq 1$ and $d_{\mathcal{C}(S)}(\partial_S P_B, \partial D_B) \leq 1$, so

$$\begin{aligned} d(S) &\leq d_{\mathcal{C}(S)}(\partial D_A, \partial D_B) \\ &\leq d_{\mathcal{C}(S)}(\partial D_A, \partial_S P_A) + d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + d_{\mathcal{C}(S)}(\partial_S P_B, \partial D_B) \\ &\leq d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + 2 \\ &= td(S) + 2. \end{aligned}$$

Since $0 \leq td(S)$, the above inequality gives $td(S) \geq \max\{0, d(S) - 2\}$. □

Recall that if $K$ is a fibered knot with fiber surface $F$, then there exists an orientation-preserving diffeomorphism $\varphi\colon F \to F$ fixing $\partial F$, called the monodromy for $K$, such that $E(K)$ is homeomorphic to $F \times [0, 2\pi]/\sim$, where $\sim$ is the equivalence relation on $F \times [0, 2\pi]$ generated by $(x, 0) \sim (\varphi(x), 2\pi)$ for all $x \in F$. The translation distance of $\varphi$ is defined by

$$d(\varphi) = \min\left\{d_{\mathcal{C}(F)}(\alpha, \varphi(\alpha))\colon \alpha \in \mathcal{C}^0(F)\right\}.$$

We may realize translation distance by considering essential spanning annuli. To see this, suppose $\alpha \in \mathcal{C}^0(F)$ realizes $d(\varphi)$. In $F \times [0, 2\pi]/\sim$, let

$$S = F \times \{\pi\}, \quad A = F \times [0, \pi], \quad B = F \times [\pi, 2\pi], \quad P_A = \alpha \times [0, \pi], \quad \text{and} \quad P_B = \alpha \times [\pi, 2\pi].$$

Then $\partial_S P_A = \partial_S P_B$ so $d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) = 0$. Also, we have $\partial_F P_A = \alpha \times \{0\}$ and $\partial_F P_B = \alpha \times \{2\pi\} = \varphi^{-1}(\alpha) \times \{0\}$, so

$$\begin{aligned} d(\varphi) &= d_{\mathcal{C}(F)}(\alpha, \varphi(\alpha)) \\ &= d_{\mathcal{C}(F)}(\alpha, \varphi^{-1}(\alpha)) \\ &= d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B). \end{aligned}$$

So, translation distance can be interpreted as the distance between two spanning annuli with common boundary on $S$. With this in mind, we will show that if $S$ and $F$ are parallel to each

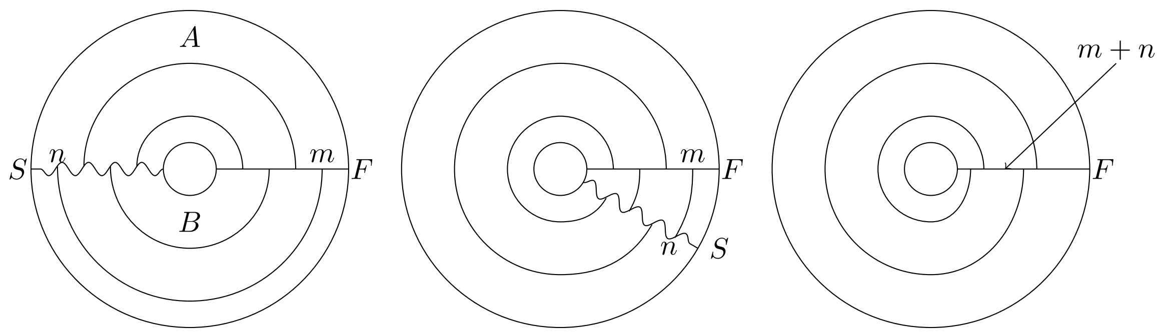


FIGURE 2.5. Start with two essential spanning annuli $P_A$ and $P_B$ that realize $cd(F,S) = m+n$, where $m = d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B)$ and $n = d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B)$. Since $F$ and $S$ are parallel, we may isotope $S$ into $F$. The two essential spanning annuli turn into a single annulus which we use to compute translation distance.

other (i.e. when $K$ is a fibered knot), then circular distance coincides with translation distance. See Figure 2.5 for an outline of a proof.

**14. Theorem.** *Let $K$ be a fibered knot with fiber surface $F$, and let $S$ be a parallel copy of $F$ in $E(K)$ with $F \cap S = \emptyset$. Let $\varphi$ be the monodromy for $K$. Then*

$$cd(F,S) = d(\varphi).$$

PROOF. The remark above shows that $d(\varphi) \geq cd(F,S)$, so it remains to show that $d(\varphi) \leq cd(F,S)$. We write $E(K) = A \cup B$, where $A \cap B = F \cup S$. Let $P_A \in \Pi_A$ and $P_B \in \Pi_B$ realize $cd(F,S)$. Let $\alpha \times \{0\} = \partial_F P_A$, so $\partial_F P_A = \alpha \times \{0\} = \varphi(\alpha) \times \{2\pi\}$.

Note that $A$ is homeomorphic to $S \times [0,\pi]$. So, there exists a map $p\colon A \to S$, which is the composition of such a homeomorphism with projection onto the first factor, such that $p[\alpha]$ is isotopic in $S$ to $\partial_S P_A$ and $p[\partial_F P_B]$ is isotopic in $S$ to $\partial_S P_B$. Note $p|_F\colon F \to S$ is a homeomorphism. Then

$$\begin{aligned} d(\varphi) &\leq d_{\mathcal{C}(F)}(\alpha, \varphi(\alpha)) \\ &\leq d_{\mathcal{C}(F)}(\alpha, \partial_F P_B) + d_{\mathcal{C}(F)}(\partial_F P_B, \varphi(\alpha)) \\ &= d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) + d_{\mathcal{C}(F)}(\partial_F P_B, \partial_F P_A) \\ &= cd(F,S). \end{aligned}$$

We conclude that $d(\varphi) = cd(F,S)$. □

### 2.3. $\partial^*$-Compressions and Euler Characteristic

Recall that if $T$ is a compact, orientable, connected, properly-embedded surface in a handlebody $H$ of positive genus, and if $T$ is essential (incompressible, $\partial$-incompressible, and not $\partial$-parallel), then $T$ is a disk such that $\partial T$ is essential in $\partial H$. The following theorem, which is a slight modification of Lemma 9 in [**1**], is the analog for compression bodies.

**15. Theorem.** *Let $W$ be a compression body, and let $T$ be a compact, orientable, connected, properly-embedded surface with non-empty boundary in $W$ that intersects $\partial_+ W$ and is disjoint from $\partial_v W$, and suppose that $T$ is incompressible and $\partial^*$-incompressible. Then $T$ is either a disk or a spanning annulus.*

Proof. Note that $\pi_1(W, \partial_+ W)$ is trivial. If $T$ is not a disk and not a spanning annulus, then $\pi_1(T, T \cap \partial_+ W, *)$ is non-trivial for a basepoint $* \in \partial_+ W$. So, there exists an essential arc $\alpha \subseteq T$ and an arc $\beta \subseteq \partial_+ W$ such that $\alpha$ and $\beta$ are homotopic rel $\partial_+ W$ in $W$. An application of the Loop Theorem and an outer-most arc argument guarantees the existence of a $\partial^*$-compressing disk for $T$. □

Since the Euler characteristic of a disk is 1 and the Euler characteristic of an annulus is 0, we have the following result.

**16. Corollary.** *Let $W$ be a compression body, and let $T$ be a compact, orientable, connected, properly-embedded surface with boundary in $W$ that intersects $\partial_+ W$ and is disjoint from $\partial_v W$. If $\chi(T) \leq -1$, then $T$ is $\partial^*$-compressible.*

Now, let $K$ be a knot in $S^3$, and suppose $(F, S)$ is a circular Heegaard splitting for $K$, where $(F, S)$ cuts $E(K)$ into compression bodies $A$ and $B$. Let $T$ be a closed, orientable, connected surface in $E(K)$, and suppose that $T \cap A$ is $\partial^*$-compressible from $A$ along $S$. Let $\alpha$ be an essential compressing arc of $T \cap A$ and $\beta$ be an arc in $S$ such that $\alpha \cup \beta$ bounds a $\partial^*$-compressing disk $\Delta \subseteq A$ for $T$; see the left panel of 2.6. We will describe the process of performing a $\partial^*$-compression, which is similar to Jaco's Isotopy of Type A, described in [**13**].

To perform a $\partial^*$-compression of $T$ from $A$ along $S$, we slide $\alpha$ across $\Delta$ and past $\beta$ while keeping

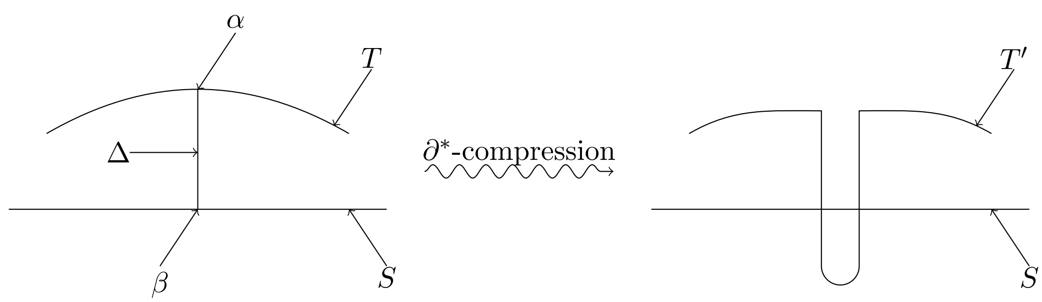


FIGURE 2.6. Jaco's isotopy of type A one dimension lower.

$\partial\alpha$ fixed. This isotopy shoves a part of $T$ through $S$ and into $B$. In $A$, this is equivalent to cutting along $\alpha$. See the right panel of Figure 2.6.

We will now describe the process of performing an annular compression, which is found in [**9**]. Suppose that $T \cap A$ contains a $\partial^*$-parallel annulus. We may perform a $\partial^*$-compression along an arc connecting the two boundary components of the annulus, and then by push the resulting disk through $S$. This two-step operation will be referred to as an annular compression of $T$ from $A$.

A $\partial^*$-compressing arc for $T \cap A$ is said to be strongly essential if it is an arc connecting two different boundary components of a $\partial^*$-parallel annulus. An elementary compression of $T$ from $A$ along $S$ is a $\partial^*$-compression of $T \cap A$ along a strongly essential arc $\alpha$ of $T \cap A$ along $S$. Similar definitions hold for $A$ replaced with $B$. We now claim that $T$ must always intersect $S$.

**17. Lemma.** *Let $K$ be a knot in $S^3$, and suppose $(F, S)$ is a circular Heegaard splitting for $E(K)$. Let $T$ be a closed, orientable, connected, essential (incompressible and not $\partial$-parallel) surface in $E(K)$. Then $T \cap S \neq \emptyset$.*

PROOF. For sake of contradiction, suppose that $T \cap S = \emptyset$. We first isotope $T$ so that $|T \cap F|$ is minimal among all such surfaces with $T \cap S = \emptyset$. We claim that every component of $T \cap A$ and $T \cap B$ is incompressible in $A$ and in $B$, respectively. To see this, suppose for sake of contradiction that some component of $T \cap A$, say, is compressible in $A$. Let $D$ be a compressing disk. Since $T$ is incompressible, there exists a disk $D' \subseteq T$ with $\partial D = \partial D'$. Note that $D' \cap S = \emptyset$ and $D' \cap F \neq \emptyset$ (we assumed that $D$ was a compressing disk for $T \cap A$). Since $E(K)$ is irreducible, then $D \cup D'$ bounds a ball in $E(K)$. Then $T' = (T - D) \cup D'$ is isotopic to $T$ and has fewer intersections with $F$ than $T$ does, which is a contradiction. So, we have $T \cap A$ is incompressible in $A$, and similarly

$T \cap B$ is incompressible in $B$.

Let $\Theta_A$ be a collection of meridian disks for $A$ so that cutting $A$ along $\Theta_A$ yields a manifold homeomorphic to $F \times I$, and define $\Theta_B$ similarly. Since each component of $T \cap A$ and $T \cap B$ is incompressible in $A$ and $B$, respectively, then we may isotope $T \cap A$ in $A$ so that $T \cap A$ is disjoint from $\Theta_A$ so that $T \subseteq A - \Theta_A$, and similarly $T \cap B \subseteq B - \Theta_B$. Note that $A - \Theta_A$ and $B - \Theta_B$ are both homeomorphic to $F \times I$. We see that a manifold homeomorphic to $F \times I$ contains a closed, orientable, essential surface that is neither horizontal (since $T \cap \partial E(K) = \emptyset$) nor vertical (since $T \cap S = \emptyset$), which is a contradiction. Hence $T \cap S \neq \emptyset$. □

Note that performing an elementary compression on a surface is equivalent to cutting the surface along an arc. With that in mind, we have the following theorem.

**18. Theorem.** *Let $T'$ be the image of $T$ after an elementary compression of $T$ from $A$ along $S$. Then $\chi(T' \cap A) = \chi(T \cap A) + 1$.*

The proofs of the following lemmas are essentially the same as those found in [**9**]. We include the proofs here for completeness. As before, we let $K$ be a knot in $S^3$, and assume that $(F, S)$ is a circular Heegaard splitting for $E(K)$ that cuts $E(K)$ into compression bodies $A$ and $B$. We also assume that $T$ is a closed, orientable, connected, essential surface in $E(K)$. By Lemma 17, we have that $T \cap S \neq \emptyset$. For each lemma, a similar result holds for $A$ replaced with $B$. Note that we do not require that $F$ is incompressible or that $F$ and $S$ are non-parallel.

**19. Lemma.** *Suppose that $T \cap A$ is incompressible in $A$. Then the image $T'$ of an elementary compression of $T$ from $A$ along $S$ also has incompressible intersection with $A$.*

Proof. Let $D$ be the disk along which the elementary compression was performed, so $D \subseteq A$ and $\partial D = \alpha \cup \beta$, where $\alpha \subseteq T \cap A$ is an arc, $\beta \subseteq S$ is an arc, and $\alpha \cap \beta = \partial\alpha = \partial\beta$. Consider a product neighborhood of $D$ in $A$; in particular, consider an embedding of $D \times I$ in $A$ so that $(\partial D) \times I = (\alpha \cup \beta) \times I$, where $\alpha \times I \subseteq T \cap A$ and $\beta \times I \subseteq S$. Then performing the elementary compression is similar to replacing $\alpha \times I$ with $D \times \partial I$, so $T' \cap A = (T \cap A) - (\alpha \times I) \cup (D \times \partial I)$.

Now, suppose there exists a disk $\Delta \subseteq A$ such that $\Delta \cap T' = \partial\Delta$, call $\partial\Delta = c$. First, using an outermost-arc argument, we isotope $c$ in $T' \cap A$ so that $c \cap (D \times \partial I) = \emptyset$. Since $A$ is irreducible,

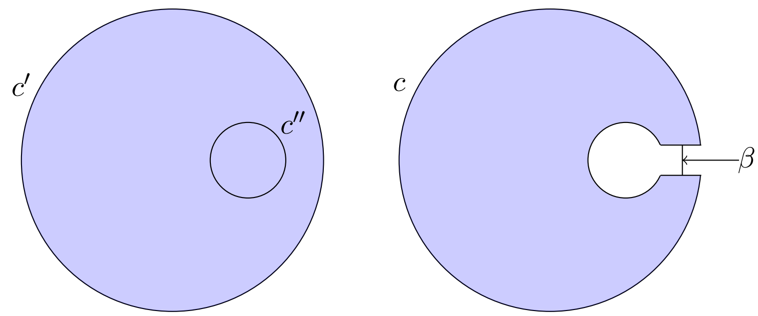


FIGURE 2.7. In Case 1 Subcase 1 of Lemma 20, the curve $c$ bounds must bound a disk, which is a contradiction.

we may use an innermost-disk argument to arrange that $\Delta \cap (D \times \partial I) = \emptyset$. So, we may view $\Delta$ as a compressing disk for $T \cap A$. Since $T \cap A$ is incompressible in $A$, then $c$ bounds a disk in $T \cap A$. Since $c \cap (D \times \partial I) = \emptyset$, then this new disk is disjoint from $\alpha \times I$. Hence, we have that $c$ bounds a disk in $T' \cap A$, and hence $T' \cap A$ is incompressible in $A$. $\square$

**20. Lemma.** *Suppose that $T \cap A$ is incompressible in $A$, that each component of $T \cap S$ is essential in $S$, and that $T'$ is the image of an elementary compression of $T$ from $A$ along $S$. Then each component of $T' \cap S$ is essential in $S$.*

PROOF. For sake of contradiction, suppose there is a component $c'$ of $T' \cap S$ that is inessential on $S$, so $c'$ bounds a disk in $S$. Then $c'$ must be a curve that is generated by the elementary compression. Let $\beta \subseteq S$ be the arc along which the elementary compression was performed. Then the endpoints of $\beta$ are either on the same or on different components of $T \cap S$.

<u>Case 1:</u> Suppose that $\beta$ joins the same component $c$ of $T \cap S$. Then, after the elementary compression, the component $c$ is broken into two different components: $c'$ and $c''$. Let $D' \subseteq S$ be the disk that $c'$ bounds.

<u>Subcase 1:</u> Suppose first that $c'' \subseteq D'$. Then $c''$ bounds a disk in $D'$ in by the Jordan Curve Theorem. It follows that $c$ must bound a disk in $S$, which is a contradiction to our assumption that each component of $T \cap S$ is essential in $S$. See Figure 2.7.

<u>Subcase 2:</u> Suppose now that $c'' \not\subseteq D'$, so $c'' \cap D' = \emptyset$. Let $\alpha \subseteq T \cap A$ be the arc along which the elementary compression was performed. See Figure 2.8. We claim that $\alpha$ is inessential in $T \cap A$.

Indeed, note that $\partial\beta$ separates $c$ into two components. One of the components, say $\gamma$, is such

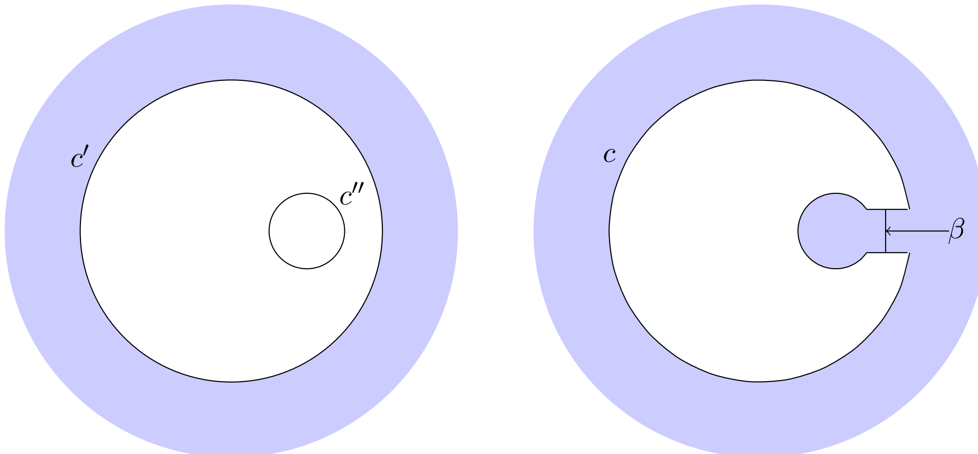


FIGURE 2.8. In Case 1 Subcase 2 of Lemma 20, the curve $c'$ bounds a disk that is disjoint from $c''$. As a result, the curve $\beta \cup \gamma$ is isotopic to $c$ and bounds a disk in $S$.

that $\beta \cup \gamma$ is isotopic to $c'$ in $S$, and hence bounds a disk $\Delta_1$ in $S$. Also, we have that $\alpha \cup \beta$ bounds a disk $\Delta_2$ in $A$. Hence, we have that $\alpha \cup \gamma$ bounds a disk $\Delta_1 \cup \Delta_2$ in $A$. We may homotope $\Delta_1$ in $S$ into $T \cap A$ while fixing $\partial\alpha$, and then homotope $\Delta_2$ in $A$ into $T \cap A$ fixing $\alpha$. This shows that $\alpha$ cobounds with a subarc of $\partial(T \cap A)$ a disk in $T \cap A$, so $\alpha$ is inessential in $T \cap A$. This is a contradiction to our assumption that the $\partial^*$-compression was an elementary compression.

Case 2: Suppose that $\beta$ joins different components of $T \cap S$, say $c_1$ and $c_2$. Also, let $D' \subseteq S$ be the disk bounded by $c'$. Then either $\beta \subseteq D'$ or $\beta \nsubseteq D'$. See Figure 2.9.

Subcase 1: Suppose that $\beta \subseteq D'$. Then each of $c_1$ and $c_2$ bounds a disk in $T \cap S$, which is a contradiction to our assumption that each component of $T \cap S$ is essential in $S$.

Subcase 2: Suppose that $\beta \nsubseteq D'$. Then $c_1 \cup c_2$ bounds an annulus in $S$. We may push $c'$ into $A$ so that it defines a compressing disk for $T' \cap A$. By Lemma 19, we have that $c'$ is inessential in $T' \cap A$, and hence bounds a disk in $T' \cap A$. So, the $\partial^*$-compression of $T \cap A$ along $\beta$ must have been the first step of an annular compression instead of an elementary compression, which is a contradiction.

Therefore, we have that each component of $T' \cap S$ is essential in $S$. □

**21. Lemma.** *Suppose that $T'$ is the image of an annular compression of $T$ along $S$. Then the collection $T' \cap S$ of simple closed curves is, up to isotopy, a proper subset of the collection $T \cap S$.*

PROOF. Since an annular compression deletes two components of $T \cap S$, the result follows. □

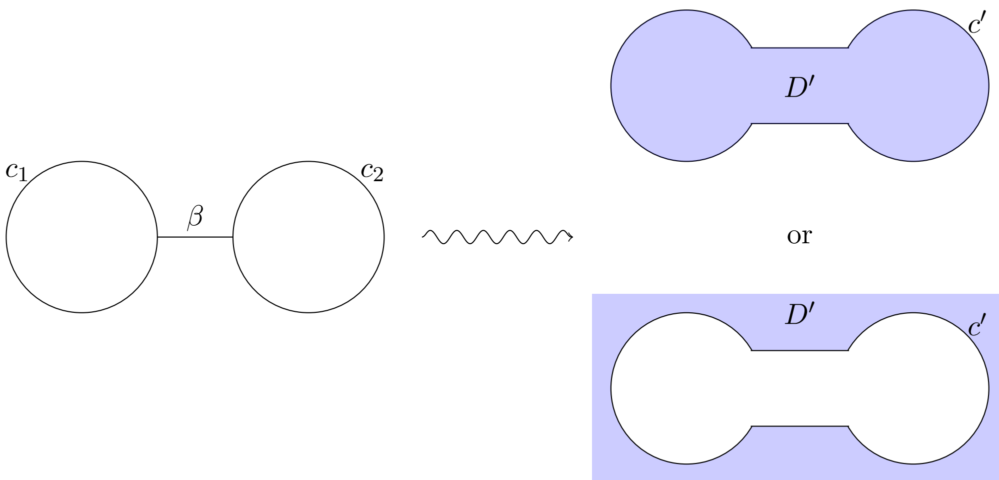


FIGURE 2.9. The two possible ways in Case 2 in which $c'$ is the boundary of a disk in $T \cap S$.

**22. Lemma.** *Suppose that $T \cap A$ is incompressible in $A$, that each component of $T \cap S$ is essential in and $S$, and that $T'$ is the image of an annular compression of $T$ from $A$ along $S$. Then $T' \cap A$ is incompressible in $A$, and each component of $T' \cap S$ is essential in $S$.*

PROOF. Recall that an annular compression moves an annular component of $T \cap A$ into $B$ and fixes the rest of $T \cap A$. Hence, if $T \cap A$ is incompressible, then so is $T' \cap A$. Also, an annular compression deletes two components of $T \cap S$ and fixes everything else. So, if each component of $T \cap S$ is essential in $S$, then each component of $T' \cap S$ is essential in $S$. □

**23. Lemma.** *Suppose that $T'$ is the image of an elementary compression of $T$ along $S$. Let $c$ and $c'$ be components of $T \cap S$ and $T' \cap S$, respectively, that are essential in $S$. Then $d_{\mathcal{C}(S)}(c, c') \leq 1$.*

PROOF. Suppose first that $c$ is not a component of $T \cap S$ that is affected by the elementary compression. Since either $c$ is isotopic to $c'$ or $c \cap c' = \emptyset$, then $d_{\mathcal{C}(S)}(c, c') \leq 1$. If $c'$ is a component of $T' \cap S$ that is not in the image of the affected components of the elementary compression, then a similar argument shows that $d_{\mathcal{C}(S)}(c, c') \leq 1$. Hence, we assume that $c$ is a component of $T \cap S$ that is affected by the elementary compression, and we assume that $c'$ is in the image of $T' \cap S$.

Let $\beta \subseteq S$ be the arc along which the elementary compression is performed. Then $\partial\beta \subseteq T \cap S$ lies on either one component or two components of $T \cap S$.

<u>Case 1</u> Suppose that $\partial\beta$ lies on two components $c_1$ and $c_2$ of $T \cap S$, so $c = c_1$ or $c = c_2$. We

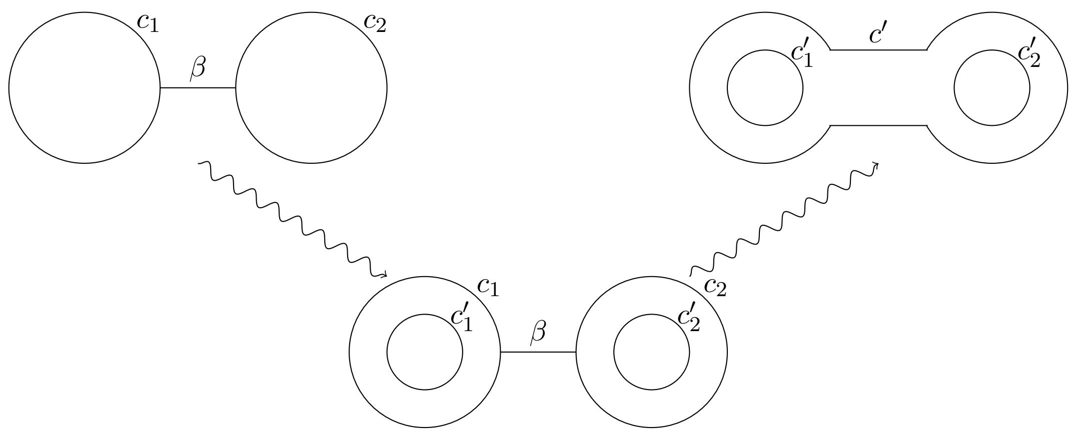


FIGURE 2.10. We first perform a small isotopy of $S$ to produce new curves $c_1'$ and $c_2'$. After, we perform the elementary compression and note that $c_1' \cap c' = c_2' \cap c' = \emptyset$.

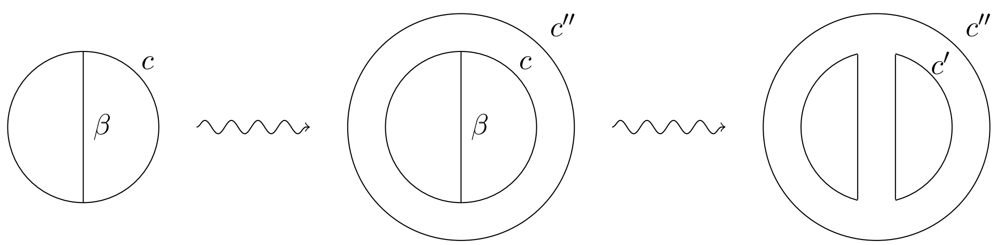


FIGURE 2.11. After performing a small isotopy and then the elementary compression, we see that $c'' \cap c' = \emptyset$.

first perform a small isotopy of $S$ to produce curves $c_1'$ and $c_2'$ in such a way that $c_i' \cap \beta = \emptyset$ and $c_i' \cap c_i = \emptyset$ for all $i \in \{1, 2\}$. See Figure 2.10. Now, note that performing this elementary compression results in the single curve $c'$. Then, after performing this elementary compression, we have $c_1' \cap c' = c_2' \cap c' = \emptyset$. It follows that $d_{\mathcal{C}(S)}(c, c') \leq 1$.

Case 2 Suppose that $\partial\beta$ lies on one component of $T \cap S$, so $\partial\beta \subseteq c$. Let $D$ be the $\partial^*$-compressing disk of this elementary compression. Then we may perform a small isotopy of $E(K)$ to produce $T''$ such that $T'' \cap T = \emptyset$ and $T'' \cap D = \emptyset$. See Figure 2.11.

Now, let $c'' \subseteq T'' \cap S$ be the image of $c$ under this isotopy. After performing the elementary compression, we see that $c'' \cap c' = \emptyset$. Since $c$ is isotopic to $c''$, we see that $d_{\mathcal{C}(S)}(c, c') \leq 1$. $\square$

## 2.4. Proof of the Main Theorem

We assume throughout that $K$ is a knot in $S^3$ and that $(F, S)$ is a circular Heegaard splitting for $E(K)$ that cuts $E(K)$ into compression bodies $A$ and $B$, that $F$ is incompressible in $E(K)$, and that $F$ and $S$ are non-parallel. We also assume that $T$ is a closed, orientable, connected, essential surface in $E(K)$.

We are almost ready to prove our main theorem. The idea is to first isotope $T$ in $E(K)$ such that exactly one of $T \cap A$ and $T \cap B$ contains either a single disk component or an essential spanning annulus, and then to establish a sequence of isotopies to obtain either a single disk component or an essential spanning annulus in the other compression body. We will show first that, under certain conditions, such a sequence of isotopies exist.

**24. Lemma.** *Suppose $T \cap B$ contains an essential disk or an essential spanning annulus, and suppose that each component of $T \cap A$ is incompressible in $A$ and each component of $T \cap S$ is essential in $S$. Then there exist $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and a sequence of isotopies*

$$T \simeq T_0 \simeq \cdots \simeq T_k \simeq \cdots \simeq T_n$$

*such that*

(1) *for all $i \in \{0, \cdots, n\}$, each component of $T_i \cap S$ is essential in $S$,*
(2) *for all $i \in \{0, \cdots, n\}$, each component of $T_i \cap A$ is incompressible in $A$,*
(3) *for all $i \in \{0, \cdots, n-1\}$, for any choice of components $c_i$ of $T_i \cap S$ and $c_{i+1}$ of $T_{i+1} \cap S$, we have $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$,*
(4) *no elementary compression of $T$ is performed along $F$,*
(5) *for all $i \in \{0, \cdots, k\}$, $T_i \cap B$ contains either essential disk components or essential annulus components,*
(6) *$T_n \cap A$ contains a single essential disk or contains an essential spanning annulus, and*
(7) *if $td(S) \geq 2$, then $k \leq n-2$, and for all $i \in \{k+1, \cdots, n-1\}$, neither $T_i \cap A$ nor $T_i \cap B$ contain any essential disk or essential spanning annulus components.*

Proof. If $T \cap A$ contains any $\partial^*$-parallel annular components, perform annular compressions to remove them and form $T_0$; else, let $T = T_0$. Now, perform an elementary compression on $T_0$ from

$A$ to form $\widehat{T}_1$. If $\widehat{T}_1$ contains any $\partial^*$-parallel annular components, perform annular compressions to remove them and form $T_1$; else, let $\widehat{T}_1 = T_1$. Continue in this way recursively for $i \in \{1, \cdots, n\}$ to form $\widehat{T}_i$ and $T_i$. Then the first and second points follow by induction from Lemmas 19, 20, and 22.

Let $i \in \{0, \cdots, n-1\}$, and let $c_i$ and $c_{i+1}$ be components of $T_i \cap S$ and $T_{i+1} \cap S$, respectively. Note that $c_{i+1}$ is a component of $\widehat{T}_{i+1} \cap S$ by Lemma 21, and so $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$ by Lemma 23, and the third point follows.

Let $n$ be the smallest integer such that $T_n \cap A$ contains an essential disk or an essential spanning annulus. Note that such an $n$ must exist; by Theorem 18, for all $i \in \{0, \cdots, n-1\}$, we have $\chi(T_{i+1} \cap A) = \chi(T_i \cap A) + 1$, so the Euler characteristic will eventually be non-negative, forcing there to be an essential spanning annulus or an essential disk component.

We claim that $T_n \cap A$ must contain either a single essential disk or contain an essential spanning annulus. Indeed, if $T_n \cap A$ does not contain an essential spanning annulus, then $T_n \cap A$ must contain exactly one essential disk, or else $\chi(T_n \cap A) \geq \chi(T_{n-1} \cap A) + 2$, which is a contradiction to Theorem 18. The fourth and sixth points now follow.

Finally, suppose that $td(S) \geq 2$. For all $i \in \{0, \cdots, n\}$, we cannot have that both $T_i \cap A$ and $T_i \cap B$ contain essential disk or essential spanning annuli at the same time because $td(S) \geq 2$ and $F$ is incompressible (so there are no essential disk components of $T_i \cap A$ or $T_i \cap B$ with boundary on $F$), so we let $k$ be the largest integer such that $T_k \cap B$ contains an essential disk or an essential spanning annulus. Then the fifth point follows by definition of $k$. Also, we must have $n > k$. If $k = n-1$, then there would exist a component $c_k$ of $T_k \cap S$ and a component $c_{k+1} = c_n$ of $T_n \cap S$ such that $c_k$ bounds an essential disk or an essential spanning annulus in $T_k \cap B$ and $c_n$ bounds an essential disk or an essential spanning annulus in $T_n \cap A$. By the third point, we have that $d_{\mathcal{C}(S)}(c_k, c_{k+1}) \leq 1$, and by definition, we would have $td(S) \leq 1$, a contradiction. Hence, we have $k \leq n-2$, and the seventh point follows. ☐

**25. Lemma.** *Suppose that each component of $T \cap S$ is essential in $S$, suppose that neither $T \cap A$ nor $T \cap B$ contain essential disk or essential spanning annulus components, and suppose that each each component of $T \cap A$ and $T \cap B$ are essential in $A$ and $B$, respectively. Then there exist $m, n \in \mathbb{N}$ and a sequence of isotopies*

$$T_{-m} \simeq \cdots \simeq T = T_0 \simeq \cdots \simeq T_n$$

*such that*

(1) *for all* $i \in \{-m, \cdots, n\}$*, each component of* $T_i \cap S$ *is essential in* $S$*,*

(2) *for all* $i \in \{-m, \cdots, n\}$*,* $T_i \cap A$ *and* $T_i \cap B$ *are essential in* $A$ *and* $B$*, respectively,*

(3) *for all* $i \in \{-m, \cdots, n-1\}$*, for any choice of components* $c_i$ *of* $T_i \cap S$ *and* $c_{i+1}$ *of* $T_{i+1} \cap S$*, we have* $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$*.*

(4) $T_n \cap A$ *contains an essential disk or essential spanning annulus, as does* $T_{-m} \cap B$*, and*

(5) *if* $td(S) \geq 2$*, then for all* $i \in \{-m+1, \cdots, n-1\}$*, neither* $T_i \cap A$ *nor* $T_i \cap B$ *contain any essential disk or essential spanning annulus components.*

Proof. For all $i \in \{1, \cdots, n\}$, define $T_i$ exactly as in the previous lemma. For all $i \in \{-1, \cdots, -m\}$, define $T_i$ exactly as in the previous lemma with the roles of $A$ and $B$ switched. The proof for each of the points is now analogous to that in the previous lemma. □

The goal now is to guarantee that the hypotheses of one of the above two lemmas hold. The following lemma establishes this.

**26. Lemma.** *If* $T$ *intersects* $S$ *and* $F$ *in a minimal number of components, then*

(1) $T \cap A$ *and* $T \cap B$ *are incompressible in* $A$ *and* $B$*, respectively,*

(2) *each component of* $T \cap S$ *and* $T \cap F$ *is a simple closed curve that is essential in* $S$ *and in* $F$*, respectively, and*

(3) *neither* $T \cap A$ *nor* $T \cap B$ *contains a* $\partial^*$*-parallel annulus.*

Proof. The proof of the first point is similar to that of Lemma 17. Now suppose for sake of contradiction that some component $c \subseteq T \cap S$ is inessential in $S$. Then there exists a disk $D \subseteq S$ with $\partial D = c$. Since $T$ is incompressible in $E(K)$, then there exists a disk $D' \subseteq T$ with $\partial D' = c$. Since $E(K)$ is irreducible, then $D \cup D'$ bounds a ball in $E(K)$. The surface $T' = (T - D') \cup D$, after a small isotopy, thus has fewer intersection with $F \cup S$ than $T$ does, which is a contradiction. So, we have that each component of $T \cap S$ is essential in $S$. A similar argument shows that each component of $T \cap F$ is essential in $F$.

Lastly, suppose for sake of contradiction that $T \cap A$ contains a $\partial^*$-parallel annulus. Then we may perform an annular compression to produce a surface $T'$ isotopic to $T$. But by Lemma 21, we have that $T'$ has fewer intersections with $F \cup S$ than $T$ does, which is a contradiction. Hence, we have that $T \cap A$ contains no $\partial^*$-parallel annuli. A similar argument shows that $T \cap B$ contains no $\partial^*$-parallel annuli. □

Note that if $T$ intersects $S$ and $F$ in a minimal number of components, then by Lemma 26, the hypotheses of either Lemma 24 or Lemma 25 are satisfied. We are now ready to prove our main theorem.

**27. Theorem.** *Let $K$ be a knot in $S^3$ and suppose $(F, S)$ is a circular Heegaard splitting for $E(K)$ with $F$ incompressible and $F$ and $S$ non-parallel. Suppose $E(K)$ contains a closed, connected, orientable, essential surface $T$ of genus $g \in \mathbb{N} \cup \{0\}$. Then $cd(F, S) \leq 2g$.*

Proof. First, isotope $T$ so that $T$ intersects $F \cup S$ is a minimal number of components. We suppose first that $td(S) \geq 2$. By Lemma 26, we have that each component of $T \cap A$ and $T \cap B$ is incompressible in $A$ and $B$, respectively, that each component of $T \cap S$ and $T \cap F$ is a simple closed curve essential in $S$ and in $F$, respectively, and that neither $T \cap A$ nor $T \cap B$ contains a $\partial^*$-parallel annulus. We also have that the hypotheses of Lemma 24 or Lemma 25 hold.

If $T \cap A$ contains an essential disk or an essential spanning annulus, then by Lemma 24, after a relabeling, there is a sequence $T_0, \cdots, T_\ell$ of surfaces isotopic to $T$ such that $T_0 \cap B$ and $T_\ell \cap A$ each contain a single essential disk or contains an essential spanning annulus. Further, for all $i \in \{0, \cdots, \ell\}$, we have $T_i \cap S$ is a collection of simple closed curves that are essential in $S$, and for all $i \in \{0, \cdots, \ell - 1\}$, we have $\chi(T_{i+1} \cap A) = \chi(T_i \cap A) + 1$.

If $T \cap B$ contains an essential disk or an essential spanning annulus, by relabeling the indices, an identical sequence can be constructed.

If neither $T \cap A$ nor $T \cap B$ contains essential disks or essential spanning annuli, then by Lemma 25, after a relabeling, there is a sequence $T_0, \cdots, T_\ell$ of surfaces isotopic to $T$ such that $T_0 \cap B$ and $T_\ell \cap A$ each contain a single essential disk or contains an essential spanning annulus, and for all $i \in \{0, \cdots, \ell\}$, we have $T_i \cap S$ is a collection of simple closed curves that are essential in $S$. Further, for all $i \in \{0, \cdots, \ell - 1\}$, we have $\chi(T_{i+1} \cap A) = \chi(T_i \cap A) + 1$.

In any case, let $P_B$ be the essential disk or essential spanning annulus of $T_0 \cap B$, and let $P_A$ be the essential disk or essential spanning annulus of $T_\ell \cap A$.

Case 1: If at least one of $P_A$ and $P_B$ is a disk, then for all $i \in \{0, \cdots, \ell - 1\}$, for any choice of components $c_i$ of $T_i \cap S$ and $c_{i+1}$ of $T_{i+1} \cap S$, we have $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$. So, letting $c_0 = \partial_S P_B$ and $c_\ell = \partial_S P_A$, by the triangle inequality, we have $d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) \leq \ell$. We also have $d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B) = 0$.

Since $T \cap S$ is a collection of simple closed curves, then

$$\chi(T \cap (A \cap B)) = \chi(T \cap (F \cup S)) = 0.$$

So, we have

$$\chi(T) = \chi(T \cap A) + \chi(T \cap B).$$

Since $T_0$ is isotopic to $T$, we have

$$\chi(T) = \chi(T_0 \cap A) + \chi(T_0 \cap B).$$

Since $T_0 \cap B$ contains a single essential disk or essential spanning annulus, we have $\chi(T_0 \cap B) \leq 1$. Since $\chi(T) = 2 - 2g$, we have

$$1 - 2g \leq \chi(T_0 \cap A).$$

Since $\chi(T_{i+1} \cap A) = \chi(T_i \cap A) + 1$ for all $i \in \{0, \cdots, \ell - 1\}$, we see that $T_\ell \cap A$ must have an essential disk or an essential spanning annulus after at most $2g$ elementary compressions. Since $\ell$ is the least integer with this property, we have

$$cd(F, S) \leq d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) \leq \ell \leq 2g.$$

Case 2: If both of $P_A$ and $P_B$ are essential spanning annuli, then similar to above, we have $d_{\mathcal{C}(S)}(\partial_S P_A, \partial_S P_B) \leq \ell$. Since $T$ is an embedded surface and no $\partial^*$-compressions are performed across $F$, we have $d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B) \leq 1$, so $cd(F, S) \leq \ell + 1$.

Now, since $T \cap (F \cup S)$ is a collection of simple closed curves, then

$$\chi(T \cap (A \cap B)) = \chi(T \cap (F \cup S)) = 0.$$

So, we have

$$\chi(T) = \chi(T \cap A) + \chi(T \cap B).$$

Since $T_0$ is isotopic to $T$, we have

$$\chi(T) = \chi(T_0 \cap A) + \chi(T_0 \cap B).$$

Since $T_0 \cap B$ contains essential spanning annuli and no essential disks, we have $\chi(T_0 \cap B) \leq 0$. Since $\chi(T) = 2 - 2g$, we have

$$2 - 2g \leq \chi(T_0 \cap A).$$

Since $\chi(T_{i+1} \cap A) = \chi(T_i \cap A) + 1$ for all $i \in \{0, \cdots, \ell - 1\}$, we see that $T_\ell \cap A$ must have an essential spanning annulus after at most $2g - 2$ elementary compressions. Since $\ell$ is the least integer with this property, we have

$$cd(F, S) \leq \ell + 1 \leq 2g - 1.$$

Finally, suppose that $td(S) \leq 1$. Since $F$ and $S$ are non-parallel, by Theorem 13, we have $cd(F, S) \leq 3$. If $g > 1$, then $cd(F, S) \leq 2g$ and we're done. If $g = 1$ and $cd(F, S) \leq 2$, then $cd(F, S) \leq 2g$, and we're done. So, suppose $g = 1$ and $cd(F, S) = 3$. We wish to arrive at a contradiction.

Note first that $td(S) = 1$ and $d(S) = 3$. We first suppose that $T \cap B$ contains an essential disk or an essential spanning annulus. Then by Lemma 24, there exist $n, k \in \mathbb{N} \cup \{0\}$ and a sequence of isotopies

$$T \simeq T_0 \simeq \cdots \simeq T_k \simeq \cdots \simeq T_n$$

such that the first six points of Lemma 24 hold. To show that the seventh point holds as well, note that $cd(F, S) = 3$ implies that for all $i \in \{0, \cdots, n\}$, we have $T_i \cap A$ and $T_i \cap B$ cannot contain essential disks and essential spanning annuli at the same time, so that $n > k$.

If $k = n - 1$, then there exists a component $c_k$ of $T_k \cap S$ and a component $c_{k+1} = c_n$ of $T_n \cap S$ such that $c_k$ bounds an essential disk or an essential spanning annulus $P_B$ in $T_k \cap B$ and $c_n$ bounds an essential disk or an essential spanning annulus $P_A$ in $T_n \cap A$, and $d_{\mathcal{C}(S)}(c_k, c_{k+1}) \leq 1$. Since $td(S) = 1$, then $d_{\mathcal{C}(S)}(c_k, c_{k+1}) = 1$. If $P_A$ and $P_B$ are both disks, then $d(S) = 1$, a contradiction. If $P_A$ is a disk and $P_B$ is an annulus, then $cd(F, S) = 1$, a contradiction. If $P_A$ and $P_B$ are both annuli, note $d_{\mathcal{C}(F)}(\partial_F P_A, \partial_F P_B) \leq 1$ and so $cd(F, S) \leq 2$, another contradiction. We conclude that $k \leq n - 2$.

If none of $T \cap A$ and $T \cap B$ contain an essential disk or an essential annulus, then we may recreate the argument of Lemma 25 to create a similar sequence

$$T \simeq T_0 \simeq \cdots \simeq T_k \simeq \cdots \simeq T_n$$

with similar properties. We now recreate the arguments as in Case 1 of this proof to conclude that $cd(F, S) \leq 2$, a contradiction, so the case $td(S) = 1$ with $g = 1$ and $cd(F, S) = 3$ is not possible. □

**28. Corollary.** *Let $K$ be a knot in $S^3$, and suppose $(F, S)$ is a circular Heegaard splitting for $E(K)$ with $F$ incompressible and $F$ and $S$ non-parallel. Suppose $E(K)$ contains a closed, connected, orientable, essential surface $T$ of genus $g \in \mathbb{N} \cup \{0\}$. Then $td(S) \leq 2g$.*

Proof. This follows from the fact that $td(S) \leq cd(F, S)$. □

Note that the only time that we explicitly used the fact that $F$ is incompressible was in the last paragraph of the proof of Lemma 24. In that proof, if we drop the condition that $F$ is incompressible, and if there are no essential spanning annulus components, then we can still guarantee that there is a single essential disk component of $T \cap A$ or $T \cap B$ with boundary on $S$, but we have no control over the number of essential disk components of $T \cap A$ or $T \cap B$ with boundary on $F$, and hence we have no bound on $\chi(T \cap A)$ or $\chi(T \cap B)$. So, the hypothesis that $F$ is incompressible cannot be dropped in Theorem 27.

### 2.5. Circular Distance and Seifert Surfaces

It turns out that a similar argument holds if we replace $T$ with a Seifert surface for $K$. We assume throughout that $K$ is a knot in $S^3$, that $(F, S)$ is a circular Heegaard splitting for $E(K)$, $F$ incompressible, and that $F$ and $S$ are non-parallel. We also assume that $F'$ is an incompressible Seifert surface for $K$ that is disjoint from and non-isotopic to $F$. We will first replicate the theorems and lemmas from the previous section. Then we will use this to prove that if $F$ is of minimal genus and if $d(S)$ is sufficiently large, then $F$ is the unique minimal genus Seifert surface for $K$ up to isotopy. As before, we will show that $F'$ and $S$ must intersect.

**29. Lemma.** *We have $F' \cap S \neq \emptyset$.*

Proof. For the sake of contradiction, suppose that $F' \cap S = \emptyset$. Since $F' \cap F = \emptyset$, then assume without loss of generality that $F' \subseteq A$. Let $\Delta$ be a collection of meridian disks for $A$ so that cutting $A$ along $\Delta$ yields a manifold homeomorphic to $F \times I$. Since $F'$ is incompressible and $\partial F' \subseteq \partial_v A$, then we may use an innermost-disk argument so that $F' \cap \Delta = \emptyset$. We see that $F'$ is an incompressible surface contained in a manifold that is homeomorphic to $F \times I$ and that is disjoint from $F \cup S$, and hence must be horizontal i.e. isotopic to $F$, which is a contradiction. Hence, we have $F' \cap S = \emptyset$. $\square$

**30. Lemma.** *Suppose that $F' \cap S$ is a collection of simple closed curves that are essential in $S$. If $F' \cap A$ contains an essential disk, and if each component of $F' \cap A$ is incompressible in $A$, then there exist $n, k \in \mathbb{N}$ and a sequence of isotopies*

$$F' \simeq F'_0 \simeq \cdots F'_k \simeq \cdots \simeq F'_n$$

*such that*

(1) *for all $i \in \{0, \cdots, n\}$, each component of $F'_i \cap S$ is essential in $S$,*
(2) *for all $i \in \{0, \cdots, n\}$, each component of $F'_i \cap A$ is incompressible in $A$,*
(3) *for all $i \in \{0, \cdots, n-1\}$, for any choice of components $c_i$ of $F_i \cap S$ and $c_{i+1}$ of $F_{i+1} \cap S$, we have $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$,*
(4) *for all $i \in \{0, \cdots, k\}$, $F'_i \cap A$ contains an essential disk component,*
(5) *$F'_n \cap B$ contains a single essential disk, and*
(6) *if $d(S) \geq 2$, then $k \leq n-2$ and for all $i \in \{k+1, \cdots, n-1\}$, neither $F'_i \cap A$ nor $F'_i \cap B$ contain any essential disk components.*

Proof. If $F' \cap B$ contains any $\partial^*$-parallel annuli, perform annular compressions to remove them and form $F'_0$; else, let $F'_0 = F'$. Since $d(S) \geq 2$ and $F'_0 \cap F = \emptyset$, we see that $F'_0 \cap B$ contains no essential disks an no essential spanning annuli, and hence is $\partial^*$-compressible. Perform an elementary compression to form $\widehat{F}'_1$. We may continue this process inductively, and we let $m$ be the smallest integer such that either $F'_m \cap B$ contains an essential disk or $\chi(F'_m \cap B) = 0$ and $F'_m \cap B$ contains no essential disks. Note that such an $m$ must exist; by Theorem 18, for all $i \in \{0, \cdots, m-1\}$, we have $\chi(F_{i+1} \cap A) = \chi(F_i \cap A) + 1$, so the Euler characteristic will eventually be non-negative.

Case 1: If $F'_m \cap B$ contains an essential disk, note that $\chi(F'_m \cap B) = \chi(F'_{m-1} \cap B) + 1$ by Theorem 18 and by construction. If $F'_m \cap B$ contains more than one essential disk, then we would have $\chi(F'_m \cap B) \geq \chi(F'_{m-1} \cap B) + 2$, a contradiction. In this case, we let $m = n$.

Case 2: If $\chi(F'_m \cap B) = 0$ and $F'_m \cap B$ contains no essential disks, then $F'_m \cap B$ must be a collection of annuli, none of which are $\partial^*$-parallel by construction. For sake of contradiction, suppose that all of these annuli intersect $\partial_v B$, since $F'_m \cap F = \emptyset$, these annuli must intersect $S$. So, we may isotope $F'_m$ to lie completely in $A$, which is a contradiction to Lemma 29.

So, we may choose an annulus in $F'_m \cap B$ that is disjoint from $\partial_v B$. Since this annulus is disjoint from $F$, it must be $\partial^*$-compressible via an elementary compression since $F'_m \cap B$ contains no $\partial^*$-parallel annuli. Performing this $\partial^*$-compression gives $\widehat{F}'_{m+1}$. Form $F'_{m+1}$ by performing annular compressions to all $\partial^*$-parallel annuli in $\widehat{F}'_{m+1} \cap B$, and let $n = m + 1$. Then $\chi(F'_n \cap B) = 1$, and hence $F'_n \cap B$ must contain a single disk component. This establishes the fifth point. We also have the sequence of isotopies

$$F' \simeq F'_0 \simeq \cdots \simeq F'_n,$$

and the first three points follow inductively from Lemmas 19, 20, and 22.

Now, suppose that $d(S) \geq 2$. For all $i \in \{0, \cdots, n\}$, we cannot have that both $F'_i \cap A$ and $F'_i \cap B$ contain essential disk components at the same time because $d(S) \geq 2$, so we let $k$ be the largest integer such that $F'_k \cap A$ contains an essential disk. Then the fourth point follows from the definition of $k$. Also, we have $n > k$. If $k = n - 1$, then there would exist a component $c_k$ of $F'_k \cap S$ and a component $c_{k+1} = c_n$ of $F'_n \cap S$ such that $c_k$ bounds an essential disk in $F'_k \cap A$ and $c_n$ bounds an essential disk in $F'_n \cap B$, so then $d_{\mathcal{C}(S)}(c_k, c_{k+1}) \leq 1$ by the third point, and so $d(S) \leq 1$, a contradiction. Hence, we have $k \leq n - 2$ and the sixth point follows. ☐

**31. Lemma.** *Suppose that each component of $F' \cap S$ is essential in $S$. If neither $F' \cap A$ nor $F' \cap B$ contain essential disk or essential spanning annulus components, and if each component of $F' \cap A$ and $F' \cap B$ is essential in $A$ and $B$, respectively, then there exists $m, n \in \mathbb{N}$ and a sequence of isotopies*

$$F'_{-m} \simeq \cdots \simeq F' = F'_0 \simeq \cdots \simeq F'_n$$

*such that*

(1) *for all* $i \in \{-m, \cdots, n\}$*, each component of* $F'_i \cap S$ *is essential in* $S$*,*

(2) *for all* $i \in \{-m, \cdots, n\}$*, each component of* $F'_i \cap A$ *and* $F'_i \cap B$ *are essential in* $A$ *and* $B$*, respectively.*

(3) *for all* $i \in \{-m, \cdots, n-1\}$*, for any choice of components* $c_i$ *of* $F'_i \cap S$ *and* $c_{i+1}$ *of* $F'_{i+1} \cap S$*, we have* $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$*,*

(4) $F'_n \cap A$ *contains an essential disk, as does* $F_{-m} \cap B$*, and*

(5) *if* $d(S) \geq 2$*, then for all* $i \in \{-m+1, \cdots, n-1\}$*, neither* $F'_i \cap A$ *nor* $F'_i \cap B$ *contain any essential disks.*

Proof. For all $i \in \{1, \cdots, n\}$, define $F_i$ exactly as in the previous lemma. For all $i \in \{-1, \cdots, -m\}$, define $F_i$ exactly as in the previous lemma with the roles of $A$ and $B$ switched. The proof for each of the points is now analogous to that of the previous lemma. □

**32. Lemma.** *If* $F'$ *intersects* $S$ *in a minimal number of components, then*

(1) $F' \cap A$ *and* $F' \cap B$ *are incompressible in* $A$ *and in* $B$*, respectively,*

(2) *each component of* $F' \cap S$ *is a simple closed curve that is essential in* $S$*, and*

(3) *neither* $F' \cap A$ *nor* $F' \cap B$ *contains a* $\partial^*$*-parallel annulus.*

Proof. The proof is the same as that of Lemma 26. □

**33. Theorem.** *Let* $K$ *be a knot in* $S^3$*, and let* $(F, S)$ *be a circular Heegaard splitting for* $K$ *with* $F$ *incompressible and* $F$ *and* $S$ *non-parallel. Let* $F'$ *be an incompressible Seifert surface for* $K$ *of genus* $g$ *that is disjoint from and non-isotopic to* $F$*. Then* $d(S) \leq 2g + 1$*.*

Proof. If $d(S) \leq 1$, then the theorem follows. So, we assume $d(S) \geq 2$. First, isotope $F'$ so that $F'$ intersects $S$ in a minimal number of components and $F' \cap F = \emptyset$. The conclusions of Lemma 32 thus hold.

If $F' \cap A$ contains an essential disk, then by Lemma 30, after a relabeling, there is a sequence $F'_0, \cdots, F'_\ell$ of Seifert surfaces isotopic to $F'$ such that $F'_0 \cap B$ and $F'_\ell \cap A$ each contain a single essential disk. Further, for all $i \in \{0, \cdots, \ell\}$, we have $F'_i \cap S$ is a collection of simple closed curves that are essential in $S$, and for all $i \in \{0, \cdots, \ell - 1\}$, we have $\chi(F'_{i+1} \cap A) = \chi(F'_i \cap A) + 1$.

If $F' \cap B$ contains an essential disk, by relabeling the indices, an identical sequence can be constructed.

If neither $F' \cap A$ nor $F' \cap B$ contain an essential disk, then by Lemma 31, after a relabeling, there is a sequence $F'_0, \cdots, F'_\ell$ of Seifert surfaces isotopic to $F'$ such that $F'_0 \cap A$ and $F'_\ell \cap B$ each contain a single essential disk, and for all $i \in \{0, \cdots, \ell\}$, we have $F'_i \cap S$ is a collection of simple closed curves that are essential in $S$. Further, for all $i \in \{0, \cdots, \ell - 1\}$, we have $\chi(F_{i+1} \cap A) = \chi(F_i \cap A) + 1$.

In any case, let $c_B$ be the boundary of the essential disk of $F'_0 \cap B$, and let $c_A$ be the boundary of the essential disk of $F'_\ell \cap A$. For all $i \in \{0, \cdots, \ell - 1\}$, for any choice of components $c_i$ of $F'_i \cap S$ and $c_{i+1}$ of $F'_{i+1} \cap S$, we have $d_{\mathcal{C}(S)}(c_i, c_{i+1}) \leq 1$. So, letting $c_0 = c_B$ and $c_\ell = c_A$, by the triangle inequality, we have $d_{\mathcal{C}(S)}(c_A, c_B) \leq \ell$.

Since $F' \cap S$ is a collection of simple closed curves, then

$$\chi(T \cap (A \cap B)) = \chi(T \cap (F \cup S)) = 0.$$

So, we have

$$\chi(F') = \chi(F' \cap A) + \chi(F' \cap B).$$

Since $F'_0$ is isotopic to $F'$, we have

$$\chi(F') = \chi(F'_0 \cap A) + \chi(F'_0 \cap B).$$

Since $F'_0 \cap B$ contains a single essential disk, we have $\chi(F'_0 \cap B) \leq 1$. Since $\chi(F') = 1 - 2g$, we have

$$-2g \leq \chi(F'_0 \cap A).$$

Since $\chi(F'_{i+1} \cap A) = \chi(F'_i \cap a) + 1$ for all $i \in \{0, \cdots, \ell - 1\}$, we see that $F'_i \cap A$ must have an essential disk after at most $2g + 1$ elementary compressions. Since $\ell$ is the least integer with this property, we have

$$d(S) \leq d_{\mathcal{C}(S)}(c_A, c_B) \leq \ell \leq 2g + 1,$$

as desired. □

Before providing an application of this theorem, we recall Proposition 5 of [**17**], which has been restated for our purposes.

**34. Lemma.** *Let $K$ be a knot in $S^3$, and let $F, F''$ be two non-isotopic, minimal genus Seifert surfaces for $K$ such that $F \cap F'' \neq \emptyset$. Then there exists a minimal genus Seifert surface $F'$ for $K$ that is disjoint from and non-isotopic to $F$ and $F''$.*

As a result of Theorem 33 and Lemma 34, we have the following result.

**35. Corollary.** *Let $K$ be a knot in $S^3$, and let $(F, S)$ be a circular Heegaard splitting for $E(K)$ with $F$ and $S$ non-parallel such that $F$ is a minimal genus $g$ Seifert surface for $K$. If $d(S) > 2g+1$, then $F$ is the unique minimal genus Seifert surface for $K$.*

PROOF. Suppose $d(S) > 2g(F) + 1$. For sake of contradiction, suppose that $F''$ is another minimal genus Seifert surface for $K$ that is non-isotopic to $F$. By Lemma 34, there exists a minimal genus Seifert surface $F'$ for $K$ that is disjoint from and non-isotopic to $F$. By Theorem 33, we have that $d(S) \leq 2g + 1$, which is a contradiction. □

## 2.6. Essential Tangle Decompositions

One application of Theorem 27 is to tangles and tangle decompositions. Our goal is to show that circular distance can be used to detect parallel strands in an essential tangle decomposition. We first recall the definition of a tangle.

**36. Definition.** Let $n \in \mathbb{N}$. An <u>$n$-string tangle</u> is a pair $(B, T)$, where $B$ is a 3-ball and $T$ is a collection of $n$ mutually disjoint, properly-embedded arcs in $B$ called strings. If $T = \{t_1, \cdots, t_n\}$ and $i \in \{1, \cdots, n\}$, then a meridian $m(t_i)$ of $t_i$ is a <u>meridian</u> of the annulus $\partial\eta(t_i) - \partial B$.

We will abuse notation and use $T$ to denote $\bigcup_{i=1}^{n} t_i$ as well. See Figure 2.12. We are mainly interested in essential and parallel tangles. We make these definitions precise below.

**37. Definition.** Using the notation above, we say that $(B, T)$ is <u>essential</u> if $\partial B - \eta(T)$ is incompressible and $\partial$-incompressible in $B - \eta(T)$. If $t_i, t_j \in T$ are distinct, we say that $t_i$ and $t_j$ are <u>parallel</u> if there exists a properly-embedded disk $D$ in $E(T)$ such that $|\partial D \cap m(t_i)| = 1$ and $|\partial D \cap m(t_j)| = 1$.

We note that if $(B, T)$ is an essential 2-string tangle, say $T = \{t_1, t_2\}$, and if $t_1$ and $t_2$ are parallel, then $B - T$ contains an essential torus. To see this, let $D$ be a disk realizing that $t_1$ and

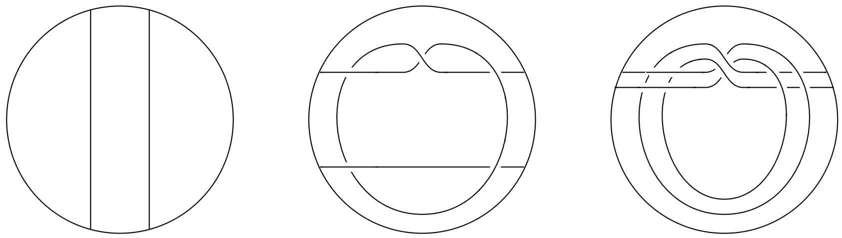

FIGURE 2.12. Three examples of tangles. The left-most tangle is said to be inessential and parallel. The middle tangle is said to be essential. The right-most tangle is said to be essential and parallel.

$t_2$ are parallel. Then, form $\partial(B - \eta(D))$. The resulting surface, call it $G$, is a torus. To show that $G$ is incompressible in $E(T)$, notice that $G$ separates $B$ into two components: $X$ and $X'$, where $X$, say, contains $\partial B$. Note that $X'$ is homeomorphic to a cube with knotted hole, and hence the boundary is incompressible. If $\Delta$ is a compressing disk for $G$ in $X$, then this disk must be disjoint from a meridian of $\partial\eta(D)$, so $\Delta$ is contained in a manifold homeomorphic to $\Sigma \times I$, where $\Sigma$ is a 2-punctured sphere. But $\partial D$ lies completely in $\Sigma \times \{0\}$, say, and cannot bound a disk in $\Sigma \times I$ unless it bounds a disk in $\Sigma \times \{0\}$, which is a contradiction. This proves the following lemma.

**38. Lemma.** *Let $(B, T)$ be an essential 2-string tangle. If the strings in $T$ are parallel, then $B - T$ contains an essential torus.*

**39. Definition.** Let $K$ be a knot in $S^3$. An essential $n$-string tangle decomposition for $K$ is a 2-sphere $\Sigma$ in $S^3$ that intersects $K$ transversely in $2n$ points such that $(B, T)$ and $(B', T')$ are essential $n$-tangles, where $B, B'$ are the 3-balls in $S^3$ with boundary $\Sigma$ and $T = K \cap B$ and $T' = K \cap B'$. We denote $(S^3, K) = (B, T) \cup_\partial (B', T')$.

When $n = 1$, this sphere is known as a decomposing sphere. When $n = 2$, this sphere is known as a Conway sphere. See Figure 2.13. Note that $\Sigma \cap E(K)$ is essential in $E(K)$.

Recall that a knot $K$ is composite if $K$ admits an essential 1-string tangle decomposition. In this case, the knot exterior $E(K)$ admits an essential torus called a swallow-follow torus. The following is a result of Theorem 27.

**40. Corollary.** *Let $K$ be a knot in $S^3$. If $K$ is composite, then $cd(F, S) \leq 2$ for any circular Heegaard splitting $(F, S)$ for $E(K)$ with $F$ incompressible.*

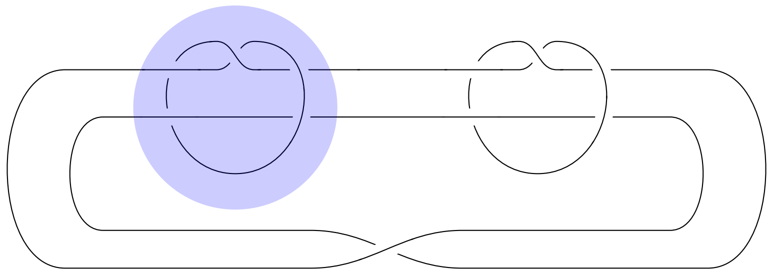

FIGURE 2.13. The shaded region represents a Conway sphere for the knot. Note that we may successively tube this four-punctured sphere along annular subsets of the boundary of the knot in four different ways to obtain a knotted surface of genus 2. At least one way of tubing the sphere gives an essential genus 2 surface.

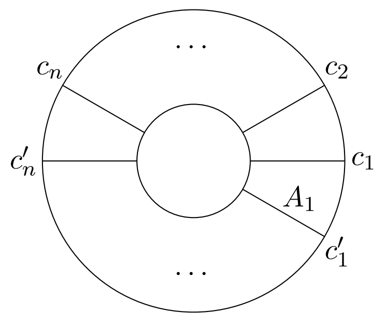


FIGURE 2.14. The curves $c_1, c_2, \cdots, c_n, c'_n, \cdots, c'_1$ lying successively on $\partial E(K)$.

Before continuing, we recall what it means to tube a surface, as described in [**20**]. Suppose that $T_0$ is a compact, properly-embedded surface in $E(K)$ such that for some $n \in \mathbb{N}$, we have

$$T_0 \cap \partial E(K) = c_1 \cup \cdots \cup c_n \cup c'_n \cup \cdots \cup c'_1$$

lying successively on $\partial E(K)$. See Figure 2.14. Let $A_1$ be the annulus in $\partial E(K)$ bounded by $c_1 \cup c'_1$ and disjoint from $c_2$. For all $i \in \{2, \cdots, n\}$, let $A_i$ be the annulus in $\partial E(K)$ bounded by $c_i \cup c'_i$ and containing $A_1$, let

$$T'_i = T_{i-1} \cup A_i,$$

and let $T_i$ be obtained by pushing the $A_i$ part of $T'_i$ off of $\partial E(K)$.

**41. Definition.** The surfaces $T_1, \cdots, T_n$ are said to be obtained from $T_0$ by <u>successively tubing through $A_1$</u>.

The following theorem, which is proved in [**20**], states that if an initial tubing of an incompressible surface remains incompressible, then the surface resulting from successive tubing remains incompressible.

**42. Lemma.** *Suppose $T_0$ is a connected, separating, properly-embedded, incompressible surface in $E(K)$, say $T_0$ separates $E(K)$ into $M_0'$ and $M_0''$. Let $A_1$ be an annulus component of $\partial M_0'$. If $T_1' = T_0 \cup A_1$ is incompressible in $M_0'$, then the surfaces $T_1, \cdots, T_n$ obtained from $T_0$ by successively tubing through $A_1$ are all incompressible in $E(K)$.*

The following theorem, which is Theorem 1 of [**10**], establishes that if an essential two-string tangle is not parallel, then tubing along one of the strands yields an incompressible surface.

**43. Lemma.** *Let $(B, T)$ be an essential 2-string tangle, where $T = \{t_1, t_2\}$. Then precisely one of the following holds:*

(1) *$t_1$ and $t_2$ are parallel, or*
(2) *there exists $i \in \{1, 2\}$ such that $F_i = \partial B - \eta(T) \cup \partial\eta(t_i)$ is incompressible in $E(T)$.*

We may now use Lemma 43 to provide the existence of an essential surface in a knot exterior with a two-string tangle decomposition using Lemma 38 and Lemma 42.

**44. Theorem.** *Let $K$ be a knot in $S^3$, and suppose $(S^3, K) = (B, T) \cup_\partial (B', T')$ is an essential 2-string tangle decomposition.*

(1) *If at least one of $T$ and $T'$ is parallel, then $E(K)$ contains an essential torus.*
(2) *If none of $T$ and $T'$ is parallel, then $E(K)$ contains a closed, orientable, connected, essential surface of genus 2.*

Proof. Let $\Sigma = \partial B = \partial B'$. Suppose that at least one of $T$ and $T'$ contains a pair of parallel strings, say $T$. Then $B - T$ contains an essential torus $G$ as constructed in Lemma 38. Note that $G$ separates $E(K)$ into $X$ and $X'$, where $K \subseteq X'$, say.

Note that $X$ is homeomorphic to a cube with knotted hole, so that $\partial X = G$ is incompressible in $X$. So, suppose that $D \subseteq X'$ is a compressing disk for $G$. Since $\Sigma$ is incompressible in $E(K)$, by an inner-most disk argument, we may isotope $D$ so that $D \cap \Sigma = \emptyset$. It follows that $D \subseteq B - T$. But this shows that $G$ is compressible in $B - T$, which is a contradiction, so that $G$ is incompressible in $E(K)$.

Now suppose that none of $T$ and $T'$ contain a pair of parallel strings. Then by Lemma 43 and Lemma 42, it follows immediately that $E(K)$ contains a closed, orientable, essential surface of genus 2. $\square$

As a result, Theorem 27 gives the following corollary.

**45. Corollary.** *Let $K$ be a knot in $S^3$, and suppose $(S^3, K) = (B, T) \cup_\partial (B', T')$ is an essential 2-string tangle decomposition. Let $(F, S)$ an a circular Heegaard splitting for $E(K)$ such that $F$ is incompressible.*

(1) *If at least one of $T$ and $T'$ is parallel, then $cd(F, S) \leq 2$.*

(2) *If none of $T$ and $T'$ are parallel, then $cd(F, S) \leq 4$.*

CHAPTER 3

# Circular Bridge Position

We start by stating the main theorem of this chapter. The definition of circular bridge position is postponed to Section 3.1; roughly, it means that a link component L is in a "nice" position relative to a circular Heegaard splitting $(F, S)$ for a knot $K$.

**46. Theorem.** *Let $K$ be a knot in $S^3$, let $\mathcal{D} = (F, S)$ be a circular Heegaard splitting for $K$, and let $L$ be a knot in $E(K)$ that is in circular bridge position with respect to $\mathcal{D}$. Suppose $T$ is a properly-embedded, connected, orientable, essential surface in $E(K \cup L)$ with $T \cap \partial E(K) = \emptyset$ (so $T \cap \partial E(K \cup L) = T \cap \partial E(L)$). Then*

$$td(L, \mathcal{D}) \leq 2g(T) + |\partial T|.$$

Note that this appears to be a generalization of Corollary 27 in the spirit of [**2**]. That is, we will consider knots that are in circular bridge position with respect to a circular Heegaard splitting of a knot, define a notion of distance in this new setting, and prove an analog of Corollary 27. Surprisingly, the hypothesis that the thin level is incompressible can be dropped.

Before proving this theorem, we will establish some background and definitions. In this chapter, we will be dealing with a link $K \cup L$ in $S^3$, a circular Heegaard splitting for one component $K$ of the link, and assuming that the other component $L$ of the link is in a "nice" position with respect to the circular Heegaard splitting. This chapter is organized as follows:

(1) We first generalize the notion of a compressing disk to a cut surface. This in turn yields different definitions for essential surfaces. These definitions are given in [**2**] and repeated here for convenience.
(2) We then define what it means for $L$ to be in circular bridge position with respect to the circular Heegaard splitting and what thick distance means in this setting.

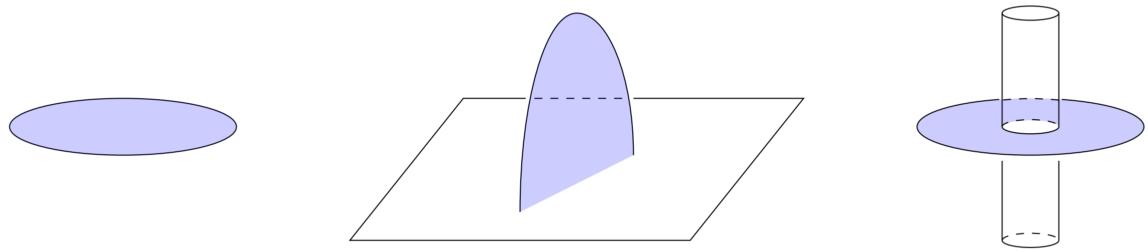

FIGURE 3.1. Disk, bigon, and meridional annulus cut surfaces.

(3) We deal with the thick distance equal to 0 case separately before proving the main theorem, since assuming positive distance grants us access to some basic and necessary tools.
(4) We assume that the surface is also in a "nice" position, called standard position, and then move on to prove our main theorem.
(5) We then briefly discuss an application to Bing doubles.
(6) We finish this chapter with a discussion of further research.

## 3.1. Cut Surfaces and Distance

We assume in this section that $K \cup L$ is a two-component link in $S^3$, and we suppose $\mathcal{D} = (F, S)$ is a circular Heegaard splitting for $E(K)$. We denote by $A$ and $B$ the compression bodies formed by $\mathcal{D}$. We also let $T$ be a compact, orientable, connected, properly-embedded surface in $E(K \cup L)$ with $T \cap \partial E(K) = \emptyset$. The following terminology will be useful.

**47. Definition.** A cut surface $E$ is a surface in $E(K \cup L)$ disjoint from $\partial E(K)$ that is either

(1) a disk such that $E \cap \partial E(K \cup L) = \emptyset$,
(2) a bigon such that $E \cap \partial E(L)$ is an arc, or
(3) an annulus with exactly one meridional boundary component on $\partial E(L)$.

In each case, we say that the closure of $\gamma = \partial E - \partial E(L)$ in $E(K \cup L)$ bounds a cut surface. See Figure 3.1.

We also need to generalize what it means for a surface to be essential in this new setting.

**48. Definition.** We say that a properly-embedded simple curve in $T$ is inessential if it bounds a cut subsurface of $T$ or is boundary-parallel, and essential otherwise. If an essential simple curve $\gamma$

of $T$ bounds a cut surface $E$ such that $E \cap T = \gamma$, then we say that $E$ is a compression for $T$ and $\gamma$ bounds a compression for $T$.

So, for example, simple closed curves that bound disks in $T$, arcs that cobound bigons with subarcs of $\partial T$, or $\partial$-parallel simple closed curves in $T$ are inessential. We will use this definition to define what it means for a surface to be essential in this new setting.

**49. Definition.** We say that $T$ is essential if no essential simple curves on $T$ bound compressions for $T$ in $E(K \cup L)$ and $\partial T$ is not null-homotopic in $\partial E(L)$. We say a 2-sphere $T$ is essential in $E(K \cup L)$ if $T$ does not bound a 3-ball.

Note that this notion of essential is different from that of Chapter 2 in a subtle way. In particular, if $L$ admits an essential 2-string decomposition where none of the tangles contain a pair of parallel strings, then the genus-two surface obtained by successive tubing (see Section 2.5) is not essential in $E(K \cup L)$. This is because a meridian of one of the annuli along which we tubed is essential in the surface, but it bounds a meridional annulus compression for the surface.

We will now define what it means for the knot $L$ to be in circular bridge position with respect to the circular Heegaard splitting $\mathcal{D}$.

**50. Definition.** We say that $L$ is in circular bridge position with respect to $\mathcal{D}$ if $L \cap A$ is a collection of arcs such that each arc either bounds a bigon with $\partial_+ A$ or is isotopic inv $\partial_v A$ to $\{x_0\} \times I$ for some $x_0 \in \partial_- A$, and similarly for $L \cap B$.

Recall that if $W$ is a compression body, and if $\gamma \subseteq \partial_- W$ is a compact 1-manifold, then $\gamma \times I \subseteq W$ is called a product surface. In particular, a product disk in $W$ is of the form $\gamma \times I$ for some compact, properly-embedded, simple arc $\gamma$. We say that such a product disk is essential if $\gamma$ is essential in $\partial_- W$.

We also denote $X_L = X \cap E(K \cup L)$ for any subset $X \subseteq E(K)$. So, for example, we have that $S_L$ and $F_L$ have $|S \cap L|$ and $|F \cap L|$ extra boundary components compared to $S$ and $F$, respectively. We also denote

$$\partial_{S_L} P = \partial P \cap S_L$$

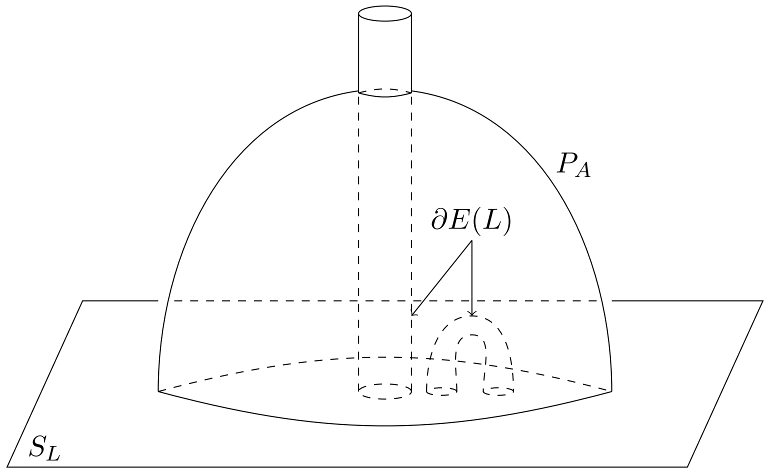


FIGURE 3.2. A meridional annulus $P_A \in \Pi_A$.

for any surface $P \subseteq A_L$ or $P \subseteq B_L$.

Last, the abstract simplicial complex $\mathcal{AC}(S_L)$ is called the arc-curve complex of $S_L$, and is defined exactly as in Definition 6, except that we allow essential properly-embedded simple arcs in addition to essential simple closed curves. We again equip $\mathcal{AC}^1(S_L)$ with the graph metric. We are now ready to define thick distance with respect to $L$.

**51. Definition.** The thick distance of $L$ with respect to $\mathcal{D}$ is defined to be

$$td(L, \mathcal{D}) = \min\{d_{\mathcal{AC}(S_L)}(\partial_{S_L} P_A, \partial_{S_L} P_B) \colon P_A \in \Pi_A, P_B \in \Pi_B\},$$

where $\Pi_A$ is the set of all compressions, essential spanning annuli, and essential product disks in $A_L$, and similarly for $\Pi_B$. See Figure 3.2.

## 3.2. Distance-Zero Case

Throughout, we again assume that $K \cup L$ is a two-component link in $S^3$ and that $\mathcal{D} = (F, S)$ is a circular Heegaard splitting for $E(K)$. Also, we denote by $A$ and $B$ the compression bodies formed by $\mathcal{D}$, and we will assume that $L$ is in circular bridge position with respect to $\mathcal{D}$.

In Chapter 2, we dealt with the distance less than or equal to 2 case separately. In this chapter, we deal with the distance 0 case separately and assume positive distance for the rest of the chapter, since positive distance implies many desired and basic consequences.

**52. Lemma.** *Let $D$ be a compressing disk for $F_L$. Then $D \cap S_L \neq \emptyset$.*

Proof. For sake of contradiction, suppose $D \cap S_L = \emptyset$. Without loss of generality, suppose $D \subseteq A_L$. Let $\Delta$ be a basis of disks and bigons for $A_L$, so that cutting $A_L$ along $\Delta$ yields a manifold homeomorphic to $F_L \times [0, \pi]$. Since $A_L$ is irreducible, we may use an innermost-disk argument to isotope the interior of $D$ so that $D \cap \Delta = \emptyset$. But this implies that $F_L \times \{0\}$ is compressible in $F_L \times [0, \pi]$, which is a contradiction. So, we have $D \cap S_L \neq \emptyset$. □

**53. Lemma.** *Suppose that $T$ is a compact, orientable, connected, properly-embedded surface in $A_L$ that intersects $S_L$ more than once and is disjoint from $\partial E(K)$. Suppose also that $T$ is incompressible and is not a disk and not a spanning annulus. Then $T$ is $\partial^*$-compressible through $S_L$ via an arc connecting two different boundary components of $T$.*

Proof. Note first that $\pi_1(A_L, S_L)$ is trivial. Since $T$ is not a disk and not a spanning annulus, then $\pi(T, T \cap S_L, *)$ is non-trivial for some basepoint $* \in S_L$. So, let $\alpha$ be an essential arc in $T$ such that $\partial\alpha \subseteq S_L$ lies on different boundary components of $T$. Then there exists an arc $\beta \subseteq S_L$ such that $\alpha$ and $\beta$ are homotopic rel $S_L$ in $A_L$. An application of the Loop Theorem and an innermost-disk argument provides the existence of a $\partial^*$-compressing disk for $T$. □

**54. Lemma.** *Suppose $E(K \cup L)$ contains an essential 2-sphere or meridional disk that is disjoint from $\partial E(K)$ (i.e. a disk whose boundary is a meridian of $\partial E(L)$). Then $td(L, \mathcal{D}) = 0$.*

Proof. Choose an essential 2-sphere or meridional disk disjoint from $\partial E(K)$ that intersects $F_L \cup S_L$ minimally and call it $\Sigma$.

<u>Case 1:</u> Suppose $\Sigma \cap F_L = \emptyset$. We claim that $|\Sigma \cap S_L| = 1$.

Indeed, suppose first that $|\Sigma \cap S_L| = 0$. Without loss of generality, suppose $\Sigma \subseteq A_L$. If $\Sigma$ is a sphere, we arrive at a contradiction since $A_L$ is irreducible. If $\Sigma$ is a meridional disk, let $\Delta$ be a system of disks and bigons for $A_L$, so cutting $A_L$ along $\Delta$ yields a manifold homeomorphic to $F_L \times [0, \pi]$. Since $A_L$ is irreducible, we may isotope $\Sigma$ so that $\Sigma \cap \Delta = \emptyset$. This implies that some component of $\partial F_L \times [0, \pi]$ is compressible in $F_L \times [0, \pi]$, which is a contradiction.

Now, suppose $|\Sigma \cap S_L| > 1$. Then the closure in $\Sigma$ of some component of $\Sigma - S_L$ is not a disk, call it $\Delta$. Since $\Delta \cap F_L = \emptyset$, then $\Delta$ is not an essential spanning annulus. So, by Lemma 53, we have $\Delta$ is

$\partial^*$-compressible through $S_L$ via an arc connecting two different components of $\Delta \cap S_L$. Performing this $\partial^*$-compression yields a sphere or meridional disk $\Sigma'$ such that $|\Sigma' \cap (F_L \cup S_L)| < |\Sigma \cap (F_L \cup S_L)|$, a contradiction to $\Sigma$ intersecting $F_L \cup S_L$ minimally.

So, we have $|\Sigma \cap S_L| = 1$. We see that $\Sigma \cap S_L$ bounds an essential disk or an essential meridional annulus in one compression body and an essential disk in the other compression body, so that $td(L, \mathcal{D}) = 0$.

Case 2: Suppose $\Sigma \cap F_L \neq \emptyset$. Let $c$ be an innermost-curve of $\Sigma \cap F_L$ in $\Sigma$ bounding a disk $\Delta$ in $\Sigma$, so $\Delta \cap F_L = \partial\Delta$. By Lemma 52, we have $\Delta \cap S_L \neq \emptyset$. We claim that $|\Delta \cap S_L| = 1$.

Indeed, suppose for sake of contradiction that $|\Delta \cap S_L| > 1$. Then $\Delta$ is $\partial^*$-compressible through $S_L$ via an arc connecting two different components of $\Delta \cap S_L$ by Lemma 53. Performing this $\partial^*$-compression yields a sphere or meridional disk $\Sigma'$ such that $|\Sigma' \cap (F_L \cup S_L)| < |\Sigma \cap (F_L \cup S_L)|$, a contradiction to $\Sigma$ intersecting $F_L \cup S_L$ minimally.

So, we have $|\Delta \cap S_L| = 1$. We see that $\Delta \cap S_L$ bounds an essential spanning annulus in one compression body and an essential disk in the other compression body, so that $td(L, \mathcal{D}) = 0$. □

**55. Lemma.** *Let $\gamma$ be a simple curve in $E(K \cup L)$ such that $\gamma$ bounds two cut surfaces $E$ and $E'$ with $E \cap E' = \gamma$. Then either $E$ and $E'$ are both bigons, both annuli, or both disks, or $td(L, \mathcal{D}) = 0$.*

PROOF. Note that if $E$ and $E'$ are cut surfaces of different type, then $E \cup E'$ is a meridional disk, and the result now follows from Lemma 54. □

**56. Lemma.** *Let $\Sigma$ be a compact, properly-embedded surface in $E(K \cup L)$ disjoint from $\partial E(K)$, and suppose $\gamma \subseteq \Sigma$ is an essential curve that bounds a cut surface $E$ in $E(K \cup L)$. Then either there is a curve $\gamma' \subseteq E \cap \Sigma$ that bounds a compression for $\Sigma$ or $td(L, \mathcal{D}) = 0$.*

PROOF. Suppose that $td(L, \mathcal{D}) > 0$. Let $\Lambda$ be the set of all curves on $E \cap \Sigma$ that are essential on $\Sigma$, let $E'$ be the closure in $E$ of a component of $E - \Lambda$ that is a cut surface, say that this cut surface is bounded by $\gamma' \subseteq E' \cap \Sigma$.

Now, let $\Theta$ denote the set of all cut surfaces bounded by $\gamma'$ such that the only curve of intersection with $\Sigma$ that is essential on $\Sigma$ is $\gamma'$. Note $E' \in \Theta$ so $\Theta \neq \emptyset$. So, we choose a cut surface $E^*$ in $\Theta$ of minimal intersection with $\Sigma$.

We claim that $E^* \cap \Sigma = \gamma'$. For sake of contradiction, suppose $E^* \cap \Sigma \neq \gamma'$. Let $E''$ be a cut surface component of $E^* - \Sigma$, and let $\gamma'' = E'' \cap \Sigma$. Then $\gamma''$ is inessential on $\Sigma$ by definition of $\Theta$ and hence bounds a cut surface $\Delta \subseteq \Sigma$, so $\Delta \cap \Sigma = \gamma''$. By Lemma 55, we obtain a new cut surface by replacing $E''$ with $\Delta$ pushed off of $\Sigma$, reducing the number of intersections of $E^*$ with $\Sigma$. But this is a contradiction to our hypothesis that $|E^* \cap \Sigma|$ is minimal. So, we have $E^* \cap \Sigma = \gamma'$ so that $\gamma'$ bounds a compression for $\Sigma$. □

**57. Lemma.** *Let $\Sigma$ be a compact, properly-embedded, essential surface in $E(K \cup L)$ that is disjoint from $\partial E(K)$. If we surger $\Sigma$ along a disk or bigon cut surface, then either one of the remaining components is essential or $td(L, \mathcal{D}) = 0$.*

Proof. Suppose $td(L, \mathcal{D}) > 0$, and suppose that we perform surgery on $\Sigma$ along a disk or bigon cut surface $E'$ with $E' \cap \Sigma = \gamma$. Note that since $\Sigma$ is essential, then $\gamma$ bounds a cut surface $E \subseteq \Sigma$. Also note that $E$ and $E'$ are both homeomorphic to a disk. Now, when we perform surgery on $\Sigma$ along $E'$, we obtain two surfaces: one isotopic to $E \cup E'$, and another isotopic to $\Sigma' = (\Sigma - E) \cup E'$.

For sake of contradiction, suppose that $\Sigma'$ is not essential. Then there exists a simple curve $\gamma' \subseteq \Sigma'$ that bounds a compression $C$ for $\Sigma'$. Since $E'$ is homeomorphic to a disk, we may isotope $\gamma'$ in $\Sigma'$ to be disjoint from $E' \subseteq \Sigma'$. But now $\gamma' \subseteq \Sigma$ bounds a cut surface $C$ for $\Sigma$. Since $td(L, \mathcal{D}) > 0$, then by Lemma 56, there is a curve on $\Sigma$ that bounds a compression for $\Sigma$, which is a contradiction to $\Sigma$ being essential. □

**58. Lemma.** *Let $\Sigma$ be a compact, properly-embedded, essential surface in $E(K \cup L)$ disjoint from $\partial E(K)$. If we surger $\Sigma$ along a disk, bigon, or meridional annulus cut surface, then either one of the remaining components is essential or $td(L, \mathcal{D}) = 0$.*

Proof. Let $\gamma \subseteq \Sigma$ be the curve bounding the cut surface $E'$ along which we surger $\Sigma$. Since $\Sigma$ is essential, then $\gamma$ bounds a cut surface $E \subseteq \Sigma$. Note that surgering $\Sigma$ along $E$ produces two surfaces: one isotopic to $E \cup E'$ and another isotopic to $\Sigma' = (\Sigma - E) \cup E'$.

If $td(L, \mathcal{D}) = 0$, then we're done. So, suppose that $td(L, \mathcal{D}) > 0$. By Lemma 57, we may assume that $E'$ is a meridional annulus. So, by Lemma 55, we have that $E$ is a meridional annulus. Note that $E \cup E'$ is then an annulus with core loop $\gamma$. If $E \cup E'$ is essential, then we're done. So, suppose that $E \cup E'$ is not essential. Since $\gamma$ is not essential in $E \cup E'$, then $E \cup E'$ admits a compressing bigon $B$. If we surger $E \cup E'$ along this bigon, we obtain a disk $D$ with $\partial D \subseteq \partial E(L)$ that bounds

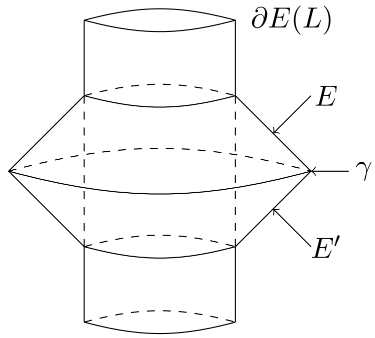


FIGURE 3.3. Informally, the solid torus is either the "obvious" interior solid torus bounded by the torus in the above picture, or else the solid torus is the exterior, in which case the picture above may be inaccurate, as there may be some non-trivial topology in the "interior" of the torus above.

a disk $D' \subseteq \partial E(L)$. Since $td(L, \mathcal{D}) > 0$, by Lemma 54, we have that $D \cup D'$ is not essential in $E(K \cup L)$. Hence, we have that $E \cup E'$ cobounds a solid torus $V$ with an annular subset of $\partial E(L)$. If $\Sigma \cap V = \emptyset$, then $\Sigma'$ is properly isotopic to $\Sigma$ an hence is essential in $E(K \cup L)$. See Figure 3.3.

For sake of contradiction, suppose that $\Sigma \cap V \neq \emptyset$ (informally, suppose $V$ is not the "obvious" solid torus bounded by $\partial V$). Then $\Sigma$ may be properly isotoped so that $\Sigma \subseteq V$. If $\Sigma \cap B = \emptyset$, then $\Sigma$ is an essential surface contained inside of a 3-ball, which is a contradiction. We conclude that $\Sigma \cap B \neq \emptyset$. The closure in $B$ of some component of $B - \Sigma$ is either a disk cut surface or a bigon cut surface for $\Sigma$. By Lemma 57, if we surger $\Sigma$ along this cut surface, we then obtain a new essential surface with fewer intersections with $B$. Continuing in this fashion, we obtain an essential surface contained in a 3-ball, which is a contradiction. Hence, we have $\Sigma \cap V = \emptyset$, and so $\Sigma'$ is properly isotopic to $\Sigma$.

We conclude that either $td(L, \mathcal{D}) = 0$, or else the properly-embedded annulus $E \cup E'$ with meridional boundary components on $\partial E(L)$ is essential in $E(K \cup L)$ or $\Sigma'$ is essential in $E(K \cup L)$ and is properly isotopic to $\Sigma$. □

## 3.3. Proof of the Main Theorem

As before, we assume that $K \cup L$ is a two-component link in $S^3$. We also assume that $\mathcal{D} = (F, S)$ is a circular Heegaard splitting for $K$. Next, we assume that $L$ is in circular bridge position with respect to $\mathcal{D}$ and that $td(L, \mathcal{D}) > 0$. The setup we describe is similar to that of [**2**].

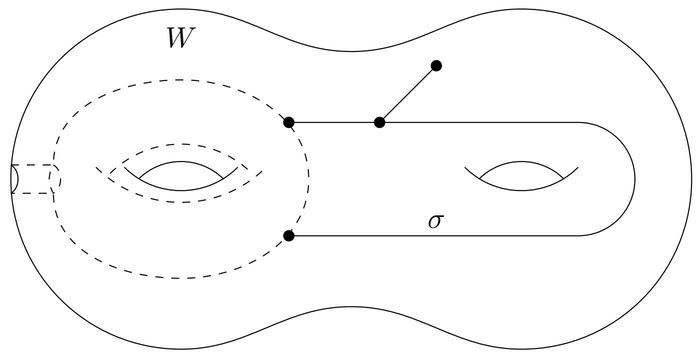


FIGURE 3.4. The graph $\sigma$, consisting of 4 vertices and 3 edges, is a spine for $W$.

Recall that a spine $\sigma$ for a compression body $W$ is a graph such that $\sigma \cap \partial W = \sigma \cap \partial_- W$ consists of valence-one vertices and $W$ deformation retracts onto $\partial_- W \cup \sigma$; see Figure 3.4. Note that spines are not unique. We let $\sigma_A \subseteq A$ and $\sigma_B \subseteq B$ denote spines for $A$ and $B$, respectively.

If $L \cap A$ consists of a single product arc, we insist on isotoping the midpoint of this segment through $S$, so that $L \cap A$ now consists of a product arc and an arc that bounds a bigon with $S$, and $L \cap B$ has an additional arc that bounds a bigon with $S$; this is needed for Case 2 of Lemma 61. Note that $L$ is still in circular bridge position after such an isotopy. We also insist on isotoping $L$ so that every arc of $L \cap A$ and $L \cap B$ that bounds a bigon with $S$ intersects $\sigma_A$ and $\sigma_B$, respectively.

We now introduce an ordering on $S^1$. We first identify $S^1 = [0, 2\pi]/\sim$, where $\sim$ is the equivalence relation on $[0, 2\pi]$ generated by $0 \sim 2\pi$. Let $t_1, t_2 \in S^1$. We write $t_1 \preceq t_2$ if the least non-negative residue modulo $2\pi$ of $t_1$ is no greater than that of $t_2$. So, for example, we have $\frac{13\pi}{6} \preceq \frac{\pi}{2}$, since the least non-negative residue of $\frac{13\pi}{6}$ is $\frac{\pi}{6}$ and $\frac{\pi}{6} \leq \frac{\pi}{2}$. Note that $\preceq$ is an ordering on $S^1$, so we will make free use of interval notation $[t_1, t_2] = \{t \in S^1 \colon t_1 \preceq t \preceq t_2\}$. Also, when $0 \preceq t_1 \preceq t_2 \prec 2\pi$, and when each element of $[t_1, t_2]$ is expressed in the least non-negative residue modulo $2\pi$, then $\preceq$ is the same as the usual ordering on $[t_1, t_2] \subseteq \mathbb{R}$, so the least upper-bound property holds in that case.

We also introduce a Morse function $h \colon E(K) \to S^1$ such that $h^{-1}(0) = F \cup \sigma_A$, and for all $t \in S^1 - \{0\}$, we have $h^{-1}(t)$ is a properly-embedded surface in $E(K)$ that is parallel to but disjoint from $S$, except $h^{-1}(\pi) = S$. We denote $S(t) = h^{-1}(t) \cap E(K \cup L)$. We also denote $W(t)$ to be the closure of $h^{-1}[0, t] \cap E(K \cup L)$ in $E(K \cup L)$ and $W'(t)$ to be the closure of $E(K \cup L) - W(t)$ in $E(K \cup L)$.

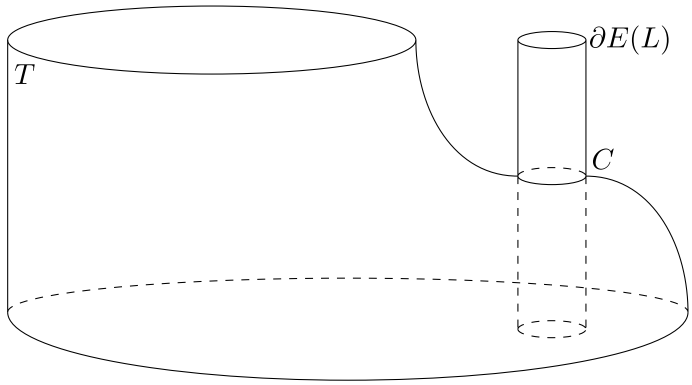


FIGURE 3.5. A portion of $T$ in the region between $S(t-\varepsilon)$ and $S(t+\varepsilon)$ that contains $C$.

Let $\varepsilon_0 > 0$ such that $\varepsilon_0$ is greater than the radius of $\eta(K)$, but small enough so that each component of $T \cap S(\varepsilon_0)$ and $T \cap S(2\pi - \varepsilon_0)$ bounds a compression or cobounds with with some curve in $F$ an essential spanning annulus or an essential product disk in $W(\varepsilon_0)$ and $W'(2\pi - \varepsilon_0)$, respectively.

Since for all $t \in [\varepsilon_0, 2\pi - \varepsilon_0]$, we have that $S(t)$ is parallel to $S_L$, then $\bigcup_{t=\varepsilon_0}^{1-\varepsilon_0} S(t)$ is homeomorphic to $S_L \times [\varepsilon_0, 2\pi - \varepsilon_0]$. We let $p$ denote the composition between such a homeomorphism and a projection onto the first factor. In particular, if $\gamma$ is a properly-embedded simple curve on $S(t)$, then $p[\gamma]$ is a properly-embedded simple curve on $S$.

Lastly, we will assume that $T$ has been isotoped into a certain position, called <u>standard position</u>. First, isotope $T$ to intersect $F_L \cup S_L$ transversely. Near the boundary of $T$, we assume:

(1) If $T$ has meridional boundary components, then they are level: for every component $C$ of $\partial T$, there exists $t \in (\varepsilon_0, 1 - \varepsilon_0)$ such that $C \subseteq S(t)$. We consider $t$ a critical value of $T$ in this case.
(2) If $T$ has non-meridional boundary components, then for each component $\gamma$ of $\partial T - S(t)$, the endpoints of $\gamma$ lie on different components of $\partial S(t)$.

Both of these conditions are possible since $\partial T$ is not null-homotopic on $\partial E(K \cup L)$. In the interior of $T$, we assume:

(1) All critical points of $h|_T$ are maxima, saddles, or minima.
(2) The values of any two critical points of $T$ are distinct.

(3) Suppose $T$ has a meridional boundary component $C$ with $h[C] = t$. Let $\varepsilon > 0$ be small, and let $P$ denote the closure of the component of $T - S(t \pm \varepsilon)$ that has $C$ as a boundary component. Then $P$ is a once-punctured annulus with one boundary component on $S(t+\varepsilon)$, one boundary component on $S(t - \varepsilon)$, and one boundary component $C$ on $\partial E(L)$. This uses the fact that $K \cup L$ has two components and no more. See Figure 3.5.

**59. Lemma.** *For all $t \in S^1 - \{0\}$, we have $T \cap S(t) \neq \emptyset$.*

Proof. This similar to the proof of Lemma 17. We recreate the proof here in this new setting for completeness. For the sake of contradiction, suppose that $T \cap S(t) = \emptyset$ for some $t \in S^1 - \{0\}$. We first isotope $T$ so that $|T \cap F_L|$ is minimal among all such surfaces with $T \cap S(t) = \emptyset$. We claim that each component of $T \cap W(t)$ and $T \cap W'(t)$ are incompressible in $W(t)$ and $W'(t)$, respectively. To see this, suppose instead that some component of $T \cap W(t)$, say, is compressible in $W(t)$. Let $D$ be such a compressing disk. Since $T$ is incompressible, then there exists a disk $D' \subseteq T$ with $\partial D = \partial D'$. Note that $D' \cap S(t) = \emptyset$ and $D' \cap F_L \neq \emptyset$ (we assumed that $D$ was a compressing disk for $T \cap W(t)$). Since $E(K \cup L)$ is irreducible, then $D \cup D'$ bounds a ball in $E(K \cup L)$. Then $T' = (T - D) \cup D'$ is isotopic to $T$ and has fewer intersections with $F_L$ that $T$ does, which is a contradiction. So, we have $T \cap W(t)$ is incompressible in $W(t)$, and a similar argument shows that $T \cap W'(t)$ is incompressible in $W'(t)$.

Now, let $\Theta_t$ be a collection of meridian disks and bigons for $W(t)$ so that cutting $W(t)$ along $\Theta_t$ yields a manifold homeomorphic to $F_L \times I$, and define $\Theta'_t$ similarly. Since each component of $T \cap W(t)$ and $T \cap W'(t)$ is incompressible in $W(t)$ and $W'(t)$, respectively, we may isotope $T \cap W(t)$ in $W(t)$ so that $T \cap W(t)$ is disjoint from $\Theta_t$, and similarly for $T \cap W'(t)$. We see that a manifold homeomorphic to $F_L \times I$ contains a compact, orientable, properly-embedded, essential surface that is neither horizontal (since $T \cap E(K) = \emptyset$) nor vertical (since $T \cap S(t) = \emptyset$), which is a contradiction. $\square$

**60. Definition.** We define $\underline{t_0}$ to be the supremum of the set of all $t \in [\varepsilon_0, 2\pi - \varepsilon_0]$ such that some component of $T \cap S(t)$ bounds a compression for $S(t)$ or cobounds with some simple curve in $F$ an essential spanning annulus an essential product disk. We define $\underline{t_1}$ similarly with infimum instead of supremum and $W'(t)$ instead of $W(t)$.

**61. Lemma.** *We have $\varepsilon_0 \prec t_0$.*

Proof. Let $\varepsilon \succ \varepsilon_0$ be small.

Case 1: Suppose $T$ is closed or has meridional boundary. For the sake of contradiction, suppose $T \cap (F_L \cup \sigma_A) = \emptyset$. Then, after an isotopy of $E(K \cup L)$, we have $T \subseteq W'(\varepsilon)$.

Let $\Delta$ be a basis of meridian disks and bigons for $W'(\varepsilon)$, so $W'(\varepsilon)$ cut along $\Delta$ yields a manifold $H$ homeomorphic to $F_L \times [\varepsilon, 2\pi]$. Since $T$ is essential and $td(L, \mathcal{D}) > 0$, by Lemma 58, a sequence of surgeries on $T$ will produce an essential surface, which we also call $T$, such that $T \cap \Delta = \emptyset$.

Now, note that $H$ is a handlebody. If $T$ is closed, then $H$ contains a closed incompressible surface, which is a contradiction. So, suppose $T$ has meridional boundary. We identify $H$ with $F_L \times [\varepsilon, 2\pi]$.

Let $\{\alpha_i \colon i \in \{1, \cdots, |F \cap L|\}\}$ be a collection of properly-embedded arcs in $F_L \times \{\varepsilon\}$ such that each component of $\partial E(L) \cap F$ contains exactly one endpoint of exactly one $\alpha_i$, and for all $i$, we have one boundary component of $\alpha_i$ is contained in $\partial F$, and the other boundary component of $\alpha_i$ is contained in $\partial E(L)$. See Figure 3.6. Consider $\Delta' = \bigcup_{i=1}^{|F \cap L|} (\alpha_i \times [\varepsilon, 2\pi])$. Then $\Delta'$ is a collection of product rectangles, each with one component in $\partial F$, one component on $F_L \times \{\varepsilon\}$, one component on $F_L \times \{2\pi\}$, and one component on $\partial E(L)$. Note $T \cap \Delta'$ is a collection of simple closed curves bounding disks in $\Delta'$ or bigons with endpoints in $\partial E(L)$ since $T \cap \partial F = T \cap F_L \times \{\varepsilon\} = T \cap F_L \times \{2\pi\} = \emptyset$. Hence, by Lemma 58, a sequence of surgeries on $T$ will produce an essential surface, which we again call $T$, such that $T \cap \Delta' = \emptyset$. Since cutting $F_L \times \{\varepsilon\}$ yields a disk, then cutting $H$ along $\Delta'$ yields a 3-ball. Hence, a 3-ball contains an essential surface $T$, which is a contradiction.

So, we have $T \cap (F_L \cup \sigma_A) \neq \emptyset$. Since $\varepsilon$ is small, then some component of $T \cap S(\varepsilon)$ bounds a compressing disk for $S(\varepsilon)$ if $T \cap \sigma_A \neq \emptyset$, or some component of $T \cap S(\varepsilon)$ bounds an essential spanning annulus or an essential product disk if $T \cap F_L \neq \emptyset$, so $\varepsilon_0 \preceq \varepsilon \preceq t_0$.

Case 2: Suppose $\partial T \neq \emptyset$ is non-meridional. Since $\varepsilon \succ \varepsilon_0$ is small, and since some arc of $L \cap A$ intersects $\sigma_A$, then $T \cap S(\varepsilon)$ contains an essential simple curve that bounds a bigon compression for $S(\varepsilon)$. Hence, we have $\varepsilon_0 \prec \varepsilon \preceq t_0$. $\square$

**62. Lemma.** *We have $t_0 \prec 2\pi - \varepsilon_0$.*

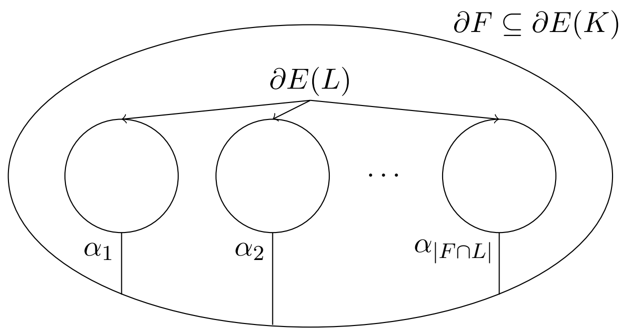


FIGURE 3.6. The arcs $\{\alpha_i \colon i \in \{1, \cdots, |F \cap L|\}\}$ in Case 1 of Lemma 61. Cutting $F_L \times \{\varepsilon\}$ along these arcs will yield a disk.

PROOF. Recall that $t_0 \preceq 2\pi - \varepsilon_0$ by definition. For sake of contradiction, suppose $t_0 = 2\pi - \varepsilon_0$. Let $\varepsilon \succ \varepsilon_0$ be small so that $2\pi - \varepsilon$ is greater than the height of the highest critical submanifold. By definition of $t_0$, we have that some component of $T \cap S(2\pi - \varepsilon)$ bounds a compression for $S(2\pi - \varepsilon)$ or cobounds with some essential simple curve in $F$ an essential spanning annulus or an essential product disk.

Now, since $T$ is in standard position and $\varepsilon > 0$ is small, we have that each component of $T \cap S(2\pi - \varepsilon)$ bounds a disk, a bigon, an essential spanning annulus, or an essential product disk in $W'(2\pi - \varepsilon)$. We see that some component of $T \cap S(2\pi - \varepsilon)$ bounds a compression or an essential spanning annulus or an essential product disk in both $W(2\pi - \varepsilon)$ and $W'(2\pi - \varepsilon)$, so that $td(L, \mathcal{D}) = 0$, which is a contradiction. □

**63. Lemma.** *If $t_0 = t_1$, then $td(L, \mathcal{D}) = 1$.*

PROOF. Suppose $t_0 = t_1$. Let $\varepsilon \succ \varepsilon_0$ be small so that there is a curve of $T \cap S(t_0 + \varepsilon)$ bounding a compression or cobounding with $F$ an essential spanning annulus or an essential product disk $W'(t_0 + \varepsilon)$ (by definition of $t_1$) and a curve of $T \cap S(t_0 - \varepsilon)$ bounding a compression or cobounding with $F$ an essential spanning annulus or an essential product disk in in $W(t_0 - \varepsilon)$. Since $S_L$ is orientable, then for $\varepsilon$ sufficiently small, we note that $p[T \cap S(t_0 + \varepsilon)]$ and $p[T \cap S(t_0 - \varepsilon)]$ can be made disjoint from each other, and so $td(L, \mathcal{D}) = 1$ since $td(L, \mathcal{D}) > 0$. □

As [**2**] states, the proof of Lemma 63 is similar to that of Lemma 4.4 of [**7**]. Since we are assuming that $td(L, \mathcal{D}) > 0$, then $T$ is neither an essential sphere nor an essential meridional disk, and so $td(L, \mathcal{D}) \leq 1 \leq 2g(T) + |\partial T|$, and our theorem still holds. Thus, we assume for the remainder of the paper that $\varepsilon_0 \prec t_0 \prec t_1 \prec 2\pi - \varepsilon_0$.

**64. Lemma.** *Let $t \in S^1$. A component of $S(t) \cap T$ that is inessential in $S(t)$ is inessential on $T$.*

Proof. Suppose $\gamma$ is a component of $S(t) \cap T$ that is inessential in $S(t)$. Then $\gamma$ bounds a cut surface in $S(t)$. If $\gamma$ were essential in $T$, then since $td(L, \mathcal{D}) > 0$, by Lemma 56, we would have that $\gamma$ bounds a compression for $T$, which is a contradiction to $T$ being essential. Hence, we must have that $\gamma$ is inessential in $T$. □

**65. Lemma.** *Suppose $t \in S^1$ is a critical value for $h|_T$. Then for $\varepsilon > 0$ sufficiently small, the distance in $\mathcal{AC}(S_L)$ between $p(S(t-\varepsilon) \cap T)$ and $p(S(t+\varepsilon) \cap T)$ is at most 1.*

Proof. The proof is the same as that of Lemma 63 unless $S(t-\varepsilon) \cap T$, say, consists of curves that are inessential in $S(t-\varepsilon)$, and hence do not appear in $\mathcal{AC}(S_L)$. By Lemma 64 we see that each component of $S(t-\varepsilon) \cap T$ is inessential in $T$ as well. Since $td(L, \mathcal{D}) > 0$, by Lemma 58, we may surger $T$ along each of these curves to produce an essential surface that is disjoint from $S(t-\varepsilon)$. This is a contradiction to Lemma 59. □

Before continuing, we introduce some notation. Let $t \in [\varepsilon_0, 2\pi - \varepsilon_0]$ be a regular value, and let $\gamma$ be a curve component of $T \cap S(t)$. We say that $\gamma$ is mutually essential if $\gamma$ is essential in both $T$ and $S(t)$. We say that $\gamma$ is mutually inessential if $\gamma$ is not essential in $T$ and not essential in $S(t)$. We say that $\gamma$ is mutual if $\gamma$ is mutually essential or mutually inessential. Last, we say that $\gamma$ is special if $\gamma$ is essential in $S(t)$ and inessential in $T$. In light of Lemma 64, these are the only possibilities for $\gamma$.

**66. Lemma.** *Let $t \in [t_0, t_1]$ be a regular value of $h|_T$. Then every curve of $S(t) \cap T$ is mutual.*

Proof. For sake of contradiction, suppose that there is a curve $\gamma$ of $S(t) \cap T$ that is not mutual. Then this curve must be essential on $S(t)$ and inessential on $T$. Then $\gamma$ bounds a cut surface $E' \subseteq T$. Since $td(L, \mathcal{D}) > 0$, then by Lemma 56, there is a compression $E$ for $S(t)$. If $E \cap F_L = \emptyset$, then $E \subseteq W(t)$ so that $t < t_0$, or $E \subseteq W'(t)$ so that $t > t_1$, each of which gives a contradiction. So, we have $E \cap F_L \neq \emptyset$. Now, if $E \cap F_L$ contains any components that are inessential in $F_L$, then

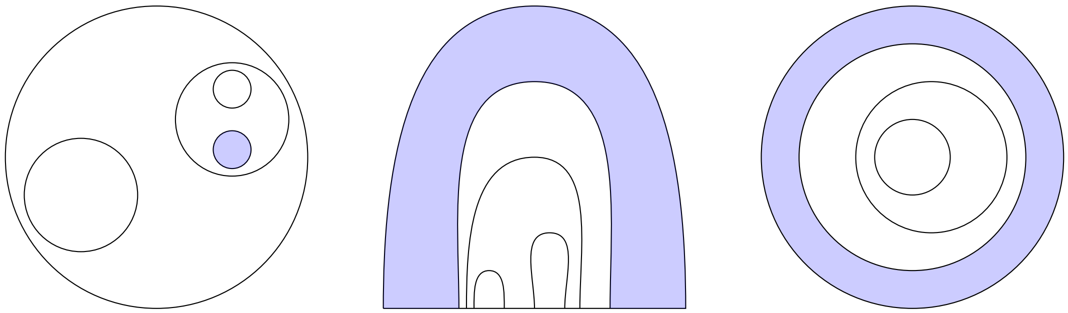

FIGURE 3.7. Each shaded region represents a cut surface discussed in each case of Lemma 66. The bottom segment of the second figure and the inner-most circle of the third figure are subsets of $\partial E(L)$.

since $E(K)$ is irreducible, we may use an innermost-disk/outermost-arc argument to isotope $E$ so that each component of $E \cap F_L$ is essential in $F_L$.

Case 1 Suppose that $E$ is a disk cut surface. Then by an innermost-disk argument and by Lemma 52, we have that $|E \cap S(t)| > 1$, a contradiction to $E$ being a compression for $S(t)$.

Case 2 Suppose $E$ is a bigon cut surface. As in Case 1, we note that $E \cap F_L$ cannot contain any closed components. We also note that the endpoints of each component of $E \cap F_L$ lie on $E - \gamma$. Thus, we may choose an arc in $E \cap F_L$ that is adjacent to $\gamma$ to obtain an essential product disk $E''$. Either $E'' \subseteq W(t)$, in which case $t \prec t_0$, or $E'' \subseteq W'(t)$, in which case $t \succ t_1$, each of which is a contradiction.

Case 3 Suppose $E$ is a meridional annulus cut surface. We note that $E \cap F_L$ contains no closed components that bound disks in $E$, else we may recreate the argument in Case 1. So, each component of $E \cap F_L$ is a curve that is parallel in $E$ to $\partial E - \partial E(L)$. So, we choose such a curve that is adjacent to $\partial E - \partial E(L)$ to obtain an essential annulus, again reaching a contradiction.

In either case, we arrive at a contradiction and conclude that each component of $S(t) \cap T$ is mutual. □

**67. Lemma.** *Let $t \in (t_0, t_1)$ be a regular value of $h|_T$. Suppose $\alpha$ is an arc component of $S(t) \cap T$ with $h[\alpha] = t$. Then $\alpha$ is mutually essential.*

PROOF. Note first that since $h[\alpha] = t$, we have $\alpha \subseteq S(t)$. For sake of contradiction, suppose $\alpha$ is not mutually essential. By Lemma 64, we have that $\alpha$ is mutually inessential. This means that $\alpha$

cobounds with some subarc $\gamma \subseteq \partial T$ a disk in $T$, and $\partial\alpha = \partial\gamma$ is contained in the same component of $\partial S(t)$. This is a contradiction to our assumption that $T$ is in standard position. □

Note that in $h^{-1}[t_0, t_1]$, the four usual types of critical points appear: minima, saddles, maxima, and meridional boundary components. As in [**2**], we introduce terminology that classifies certain critical points.

**68. Definition.** Let $t \in [t_0, t_1]$ be a critical value of $h$ corresponding to a saddle or a meridional boundary component. Let $\varepsilon > 0$ be small, and let $P$ be the component of $T \cap h^{-1}[t - \varepsilon, t + \varepsilon]$ containing the critical submanifold. We call $P$ a horizontal neighborhood in $T$ of the critical submanifold.

Now, let $\partial_\pm P = P \cap S(t \pm \varepsilon)$. We say that the critical submanifold is special if at least one component of $\partial_\pm P$ is special. Else, if the critical submanifold is not special, we say that the critical submanifold is inessential if the closure in $T$ of some component of $T - P$ is a disk, and essential otherwise.

By Lemma 64, we note that if the critical submanifold is inessential, then some component of $\partial_\pm P$ is mutually inessential (in $T$ and in $S(t \pm \varepsilon)$), and hence bounds a disk in each.

**69. Lemma.** *Let $t_* \in [t_0, t_1]$. Suppose there is a special critical submanifold at $t_*$. Then $t_* = t_0$ or $t_* = t_1$.*

Proof. Let $\varepsilon > 0$ be small. By assumption, there exists a component $\alpha$ of $T \cap S(t_* + \varepsilon)$ or of $T \cap S(t_* - \varepsilon)$ that is essential in $S(t)$ and inessential in $T$. First suppose that $\alpha \subseteq T \cap S(t_* - \varepsilon)$. By Lemma 66, we have $t_* - \varepsilon \notin [t_0, t_1]$. Since $\varepsilon > 0$ was arbitrary, we have $t_* = t_0$. Similarly, if $\alpha \subseteq T \cap S(t_* + \varepsilon)$, then $t_* = t_1$. □

**70. Lemma.** *Let $t_- \prec t_+$ be regular values in $[t_0, t_1]$ such that every saddle and every meridional boundary component of $T$ in $h^{-1}[(t_-, t_+)]$ is inessential. Then $p[T \cap S(t_-)]$ and $p[T \cap S(t_+)]$ share a vertex in $\mathcal{AC}(S_L)$.*

Proof. Let $\{t_i\}$ be the set of critical values of $h|_T$ in $[t_-, t_+]$, and for all $i$, let $r_i$ be slightly greater than $t_i$. Let $R = \{r_i\} \cup \{t_- + \varepsilon\}$.

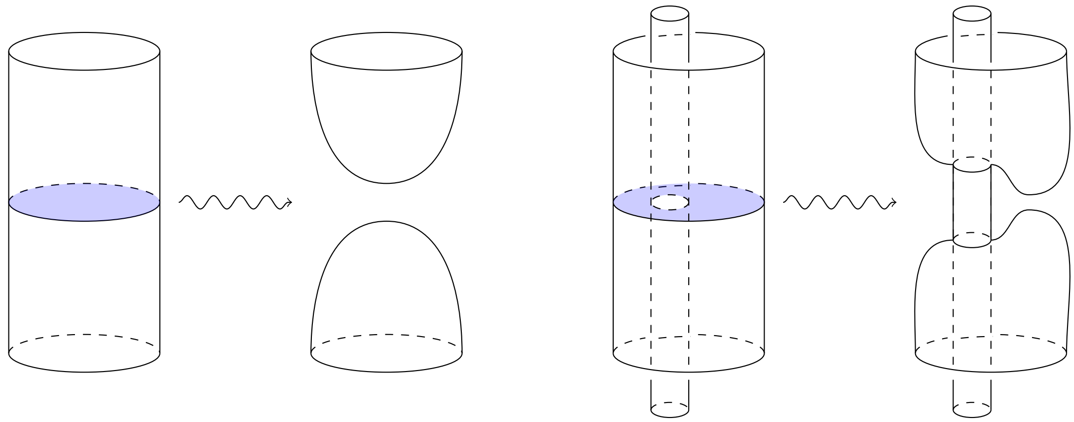

FIGURE 3.8. After performing a surgery on $T$, the resulting surface gains either two new critical points (left) or four new critical critical points (right). The maximum and minimum critical points on the far right right are to ensure the surface remains in standard position.

Now, for all $r \in R$, we will surger $T$ as follows: if $T \cap S(r)$ contains mutually inessential curves, then some such curve bounds a cut surface in $S(r)$. Surger $T$ along this curve. After these surgeries, we obtain from $T$ a surface such that for all $r \in R$, this surface meets $S(r)$ in mutually essential curves.

Now, let $M' = h^{-1}[t_-, t_+]$, and let $T'$ be the intersection of the surface constructed above with $M'$. Note that for each surgery performed on $T$, we have $h|_{T'}$ has either two or four new critical values from $h|_T$; see Figure 3.8.

Now, we say a surface $V$ in $M'$ is vertical if there exists an embedded 1-manifold $\alpha$ of $S_L$ such that $V = p^{-1}[\alpha] \cap M'$. So, a vertical surface in our case is either a disk or an annulus. We will take a short detour before continuing with the proof of Lemma 70.

**71. Lemma.** *Each component of $T'$ is either*

(1) *trivial: a sphere or a meridional annulus with both components on $\partial E(L)$,*

(2) *horizontal: properly isotopic into $S(t_-)$ or $S(t_+)$, or*

(3) *vertical: properly isotopic to a vertical surface $V$ such that $p[V]$ is essential in $S_L$.*

PROOF. Let $T''$ be a component of $T'$.

<u>Case 1:</u> Suppose that $T''$ has no critical submanifolds. Then $T''$ is isotopic to either a vertical disk or a vertical annulus. By construction of $T'$, we have that $p[T'' \cap \partial M']$ is essential in $S_L$, and so

the third point holds. Note that this is the desired case, as the existence of such a vertical surface shows that $p(T \cap S(t_-))$ and $p(T \cap S(t_+))$ share a vertex in $\mathcal{AC}(S_L)$.

Case 2: Suppose that $T''$ contains only maximum or minimum critical submanifolds and contains no meridional boundary components. Then $T''$ is a sphere, in which case the first point holds, or a disk that is parallel into $S(t_-)$ or $S(t_+)$, in which case the second point holds.

Case 3: Suppose that $T''$ contains a critical submanifold that is not a maximum or a minimum, and that is not a saddle or meridional boundary component of $T$. Then $T''$ is either a meridional annulus (the meridional boundary component not arising from $T$), in which case the first point holds, or a boundary-parallel annulus with one boundary component on $\partial E(L)$ and the other boundary component on $S(t_-)$ or $S(t_+)$, in which case the second point holds.

Case 4: Suppose that $T''$ contains a critical submanifold that is either a saddle or a meridional boundary component of $T$, say at height $t_* \in [t_-, t_+]$. Let $P$ be the corresponding horizontal neighborhood with $\partial_\pm P = P \cap S(t_* \pm \varepsilon)$, where $\varepsilon > 0$ is small. Since every critical submanifold of $T \cap M'$ is inessential, then at least one loop component of $\partial_\pm P$ bounds a disk in $T$. Since $\varepsilon > 0$ is small, we may assume that $P \subseteq T$.

Subcase 1: Suppose that this critical submanifold is a meridional boundary component of $T$. Write $\partial_\pm P = c_1 \cup c_2$, where $c_1$, say, bounds a disk $D$ in $T$ and $c_1 \subseteq S(t + \varepsilon)$. Note that $D \cup P \subseteq T$ is a meridional annulus and so $c_2$ is also inessential in $T$ since it bounds a meridional cut annulus subsurface of $T$. By Lemma 64, we see that both $c_1$ and $c_2$ are inessential in $S(t_* \pm \varepsilon)$ and in $S(t_* - \varepsilon)$, respectively, and hence bound cut subsurfaces. By Lemma 55, we see that $c_1$ bounds a disk subsurface of $S(t_* + \varepsilon)$ and $c_2$ bounds a meridional annulus subsurface of $S(t_* - \varepsilon)$. We conclude that $T''$ is a meridional annulus with both boundary components on $\partial E(L)$, and the first point follows.

Subcase 2: Suppose that $T''$ contains no meridional boundary components of $T$, and hence contains a saddle. We first suppose at least one saddle of $T''$, say the one at height $t_*$, is such that at least two components of $\partial_\pm P$ are inessential in $T$. Recall that since this critical submanifold is inessential, then the closure in $T$ of some component of $T - P$ is a disk. If two components of $P$ bound disks cut subsurfaces of $T$, call them $D_1$ and $D_2$, then the third component will bounds a disk cut subsurface $P \cup D_1 \cup D_2$ of $T$. It follows that $T''$ in this case is a sphere and the first point holds.

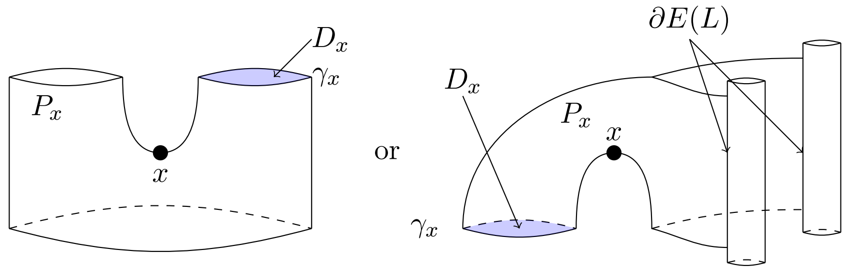


FIGURE 3.9. In a horizontal neighborhood $P_x$ of the saddle point $x$, since $\gamma_x$ bounds a disk in $T''$, we see that $T''$ is either a union of annuli or a union of disks.

If one component of $\partial P$ bounds a disk cut subsurface $D$ of $T$ and another component bounds a meridional annulus cut subsurface $E$ of $T$, then the third component bounds a meridional annulus cut subsurface $P \cup D \cup E$ of $T$, and hence $T''$ is a meridional annulus and the first point holds.

We now suppose that every saddle of $T''$ has a horizontal neighborhood with exactly one inessential boundary component. Let $x$ be a saddle point of $T''$, and let $P_x$ be a horizontal neighborhood of $x$ in $T$, so exactly one component $\gamma_x$ of $\partial_\pm P_x$ is inessential in $T$, and hence bounds a disk in $T$. By Lemma 66 and Lemma 67, we see that $\gamma_x$ must bound a disk $D_x$ in $T''$. See figure 3.9.

We see that $T''$ is either a union of disks (if $T$ has longitudinal boundary) or a union of annuli (if $T$ does not have longitudinal boundary). If $T''$ is a union of disks, then $T''$ is isotopic to a vertical disk, and the third point holds. If $T''$ is a union of annuli, then $T''$ is either isotopic to a vertical annulus, in which case the third point holds, or is an annulus that is boundary-parallel into $S(t_-)$ or $S(t_+)$, in which case the second point holds. This exhausts all possible cases. □

We now continue with our proof of Lemma 70. For sake of contradiction, suppose that no component of $T'$ intersects both $S(t_-)$ and $S(t_+)$. By Lemma 71, we note that every component of $T'$ meeting $S(t_-)$ is properly isotopic into $S(t_-)$; in particular, every such component is boundary parallel in $M'$. So, we may isotope $S(t_-)$ using these boundary parallel components to obtain a surface $S'$ such that each component of $T \cap S'$ is mutually inessential. The closure in $S'$ of some component of $S' - T$ is a cut surface, which we may use to surger $T$. By Lemma 58, we obtain an essential surface that intersects $S'$ in fewer components. We continue in this fashion to obtain an essential surface disjoint from $S'$, which is a contradiction to Lemma 59.

Hence, we must have a component $T''$ of $T'$ intersecting both $S(t_-)$ and $S(t_+)$. By Lemma 71, this component is properly isotopic in $M'$ to either a vertical disk or a vertical annulus with essential boundary components. This proves that $p[T \cap S(t_-)]$ and $p[T \cap S(t_+)]$ share a vertex in $\mathcal{AC}(S_L)$. $\square$

**72. Theorem.** *Let $K$ be a knot in $S^3$, let $\mathcal{D} = (F, S)$ be a circular Heegaard splitting for $K$, and let $L$ be a knot in $E(K)$ that is in bridge position with respect to $\mathcal{D}$. Suppose $T$ is a properly-embedded, connected, orientable, essential surface in $E(K \cup L)$ with $T \cap \partial E(K) = \emptyset$. Then*

$$td(L, \mathcal{D}) \leq 2g(T) + |\partial T|.$$

Proof. Let $\varepsilon > 0$ be small. By definition of $t_0$ and $t_1$, there exist $P_A \in \Pi_A$ and $P_B \in \Pi_B$ such that $\partial_{S_L} P_A = p[\alpha]$ and $\partial_{S_L} P_B = p[\beta]$ for some components $\alpha \subseteq T \cap S(t_0 - \varepsilon)$ and $\beta \subseteq T \cap S(t_1 + \varepsilon)$. Then $td(L, \mathcal{D}) \leq d_{\mathcal{AC}(S_L)}(\partial_{S_L} P_A, \partial_{S_L} P_B)$. By Lemma 65 and 70, we see that $d_{\mathcal{AC}(S_L)}(\partial_{S_L} P_A, \partial_{S_L} P_B)$ is bounded above by the number $e$ of essential critical submanifolds plus the number of special critical submanifolds, of which there are at most 2 by Lemma 69. So, we have $d_{\mathcal{AC}(S_L)}(\partial_{S_L} P_A, \partial_{S_L} P_B) \leq e + 2$.

Now, suppose that an essential critical submanifold occurs at $t_* \in (t_0, t_1)$. Let $P$ be its horizontal neighborhood in $T$. By definition of an essential critical submanifold, we note $\partial P - \partial T$ is essential in $T$ and hence essential in $S(t_* \pm \varepsilon)$ by Lemma 66. In all cases, note that $P$ contributes $-1$ to the Euler characteristic of $T$ (that is, if $T'$ is the closure of $T - P$ in $T$, then $\chi(T') = \chi(T) - 1$). Hence, we have $\chi(T) \leq -e$. We conclude that

$$td(L, \mathcal{D}) \leq e + 2 \leq -\chi(T) + 2 = -(2 - 2g(T) - |\partial T|) + 2 = 2g(T) + |\partial T|.$$

$\square$

## 3.4. Bing Doubles

One application of Theorem 72 is to Bing doubles. See Figure 3.10. We give a precise definition below, which is provided in [**4**].

**73. Definition.** Let $K$ be a knot in $S^3$ with standard meridian and preferred longitude $m$ and $\ell$, respectively, let $L \subseteq S^3$ be the Borromean rings arbitrarily oriented, let $L_0$ be a component of $L$, and let $m_0$ and $\ell_0$ denote a standard meridian and a preferred longitude for $L_0$, respectively. The

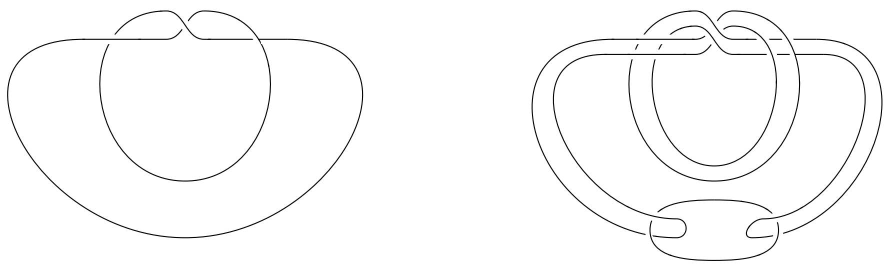

FIGURE 3.10. A Bing double of the trefoil knot.

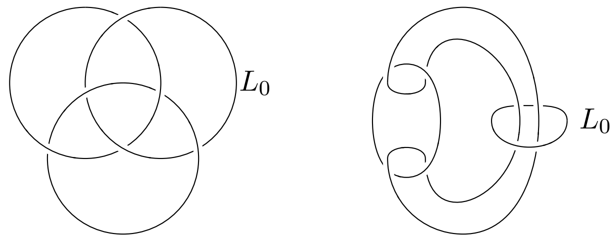


FIGURE 3.11. The Bing double of a knot can be obtained by, after isotoping the Borromean rings into the link to the right, applying Dehn surgery to $L_0$.

Bing double for $K$ is the 2-component oriented link given by the image of $L - L_0$ in $(S^3 - \eta(L_0)) \cup_\varphi (S^3 - \eta(K))$, where the pasting homeomorphism $\varphi\colon \partial(S^3 - \eta(K)) \to \partial(S^3 - \eta(L_0))$ maps $m$ onto $\ell_0$ and $\ell$ onto $m_0$. See Figure 3.11.

**74. Definition.** Let $n \in \mathbb{N}$ and $L = L_1 \cup \cdots \cup L_n \subseteq S^3$ be a link. We say that $L$ is a boundary link if there exists a set $\{T_1, \cdots, T_n\}$ such that

(1) for all $i \in \{1, \cdots, n\}$, we have $T_i$ is a Seifert surface for $L_i$, and
(2) for all $i, j \in \{1, \cdots, n\}$ with $i \neq j$, we have $T_i \cap T_j = \emptyset$.

An example is given in Figure 3.12. A constructive proof that bing doubles are boundary links is given in [**4**], which we provide below.

**75. Theorem.** *Let $K$ be a knot in $S^3$, and let $B(K)$ denote the Bing double for $K$. Then $B(K)$ is a boundary link. In fact, there exists a set $\{T_1, T_2\}$ which realizes $B(K)$ as a boundary link with $g(T_i) = 2g(K)$ for $i \in \{1, 2\}$.*

PROOF. We label the Borromean rings as $L = L_0 \cup L_1 \cup L_2$. Note that $S^3 - \eta(L_0)$ is a solid torus $V$. Now, consider a planar surface $P_1$ in $V$ whose boundary consists of $L_1$ and two copies

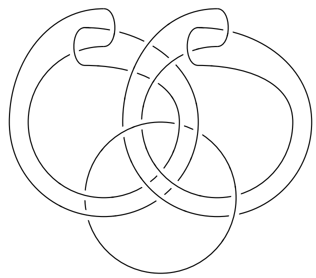

FIGURE 3.12. An example of a boundary link. There exists a set of Seifert surfaces realizing this set as a boundary link; all have genus 1. The Borromean rings from Figure 3.11 is not a boundary link. This example is from Chapter 5, Section E, Exercise 4 of [**16**].

of a preferred longitude of $V$, and consider $P_2$ similarly. The pasting homeomorphism $\psi$ ($= \varphi^{-1}$ from Definition 73) maps longitudes of $V$ onto longitudes of $\partial\eta(K)$ i.e. parallel unlinked copies of $K$. Note that these parallel unlinked copies of $K$ bound disjoint parallel Seifert surfaces. So, we paste $\psi[P_1 \cup P_2]$ with four parallel Seifert surfaces for $K$ to obtain two surfaces $T_1$ and $T_2$, which are Seifert surfaces for the two different components of $B(K)$. We see by construction that $g(T_i) = 2g(K)$ for $i \in \{1, 2\}$. □

Theorem 72 and Theorem 75 thus give the following result.

**76. Theorem.** *Let $K$ be a knot in $S^3$. If $B(K) = L \cup L'$ is a Bing double for $K$, then for any circular Heegaard splitting $\mathcal{D} = (F, S)$ of $L$, we have*

$$td(L, \mathcal{D}) \leq 4g(K) + 1$$

## 3.5. Further Research

In this section, we present some questions for the interested reader. In Chapter 2, we introduced the definition of both thick and circular distance. Circular distance can be defined similarly in this new setting.

**77. Definition.** Let $K \cup L$ be a two-component link in $S^3$, and suppose that $\mathcal{D} = (F, S)$ is a circular Heegaard splitting for $E(K)$. Suppose $L$ is in circular bridge position with respect to $\mathcal{D}$.

Then the circular distance of $\mathcal{D}$ with respect to $L$ is

$$cd(L,\mathcal{D}) = \min\{d_{\mathcal{AC}(S_L)}(\partial_{S_L}P_A, \partial_{S_L}P_B) + d_{\mathcal{AC}(F_L)}(\partial_{F_L}P_A, \partial_{F_L}P_B) \colon P_A \in \Pi_A, P_B \in \Pi_B\}.$$

We see immediately that $td(L,\mathcal{D}) \leq cd(L,\mathcal{D})$. Certainly, Theorem 72 should hold with $cd(L,\mathcal{D})$ replacing $td(L,\mathcal{D})$; however, simply replacing $td(L,\mathcal{D})$ with $cd(L,\mathcal{D})$ in the process outlined in chapter 3 does not work. In particular, the process gives no control on how the essential spanning surfaces intersect on $F_L$. Also, changing the definitions of $t_0$ and $t_1$ by, for example, imposing that some component of $T \cap W(t)$ is a compression or cobounds with $F$ an essential spanning annulus or an essential product disk (using the notation $s_0$ and $s_1$ instead, say) does not appear to work, as there is no obvious bound on the distance between the curves of $p[T \cap S(s_0 + \varepsilon)]$ and of $p[T \cap S(t_0 - \varepsilon)]$ for $\varepsilon > 0$ small. Most likely, a different approach is needed to prove Theorem 72 with $cd(L,\mathcal{D})$ replacing $td(L,\mathcal{D})$. For now, we pose this as a conjecture.

**78. Conjecture.** *Let $K$ be a knot in $S^3$, let $\mathcal{D} = (F,S)$ be a circular Heegaard splitting for $K$, and let $L$ be a knot in $E(K)$ that is in bridge position with respect to $\mathcal{D}$. Suppose $T$ is a properly-embedded, orientable, essential surface in $E(K \cup L)$ with $T \cap \partial E(K) = \emptyset$. Then*

$$cd(L,\mathcal{D}) \leq 2g(T) + |\partial T|.$$

In [**13**], Johnson proves the following fact (Theorem 3).

**79. Theorem.** *Let $(L,\pi)$ be an open book decomposition of a 3-manifold $M$ and translation distance $d$. If $\Sigma$ is a genus $g$ Heegaard splitting with $d > 4g+4$, then $\Sigma$ is a stabilization of the Heegaard splitting induced by $(L,\pi)$.*

In particular, recall that an embedded 2-sphere in $S^3$ is a genus 0 Heegaard splitting of $S^3$. Since such a Heegaard splitting cannot be a stabilization of any Heegaard splitting, the contrapositive of Theorem 79 states that the translation distance of any fibered link in $S^3$ is no greater than 4. In Theorem 14, we showed that circular distance is, in some sense, a generalization of translation distance. So, a natural question arises.

**80. Question.** *Does there exist a constant $c \in \mathbb{N} \cup \{0\}$ such that for any knot $K$ be in $S^3$ and circular Heegaard splitting $(F,S)$ for $E(K)$, we have $cd(F,S) \leq c$?*

Before we continue, we need some notation from [**18**]. If $T$ is a closed connected surface, we define the complexity of $T$ to be

$$c(T) = \begin{cases} 1 - \chi(T) & \text{if} \quad T \neq S^2, \\ 0 & \text{if} \quad T = S^2. \end{cases}$$

If $T$ is a compact connected surface, we define $c(T) = c\left(\overline{T}\right)$, where $\overline{T}$ is $T$ with its boundary capped off with disks. Finally, if $T$ is a compact surface, we define $c(T) = \sum_{i=1}^{n} c(T_i)$, where $n \in \mathbb{N}$ and $T_1, \cdots, T_n$ are the connected components of $T$.

Now, in [**15**], Manjarrez-Gutierrez defines a notion of circular thin position for a knot exterior. In particular, she considers a generalized circular Heegaard splitting $\mathcal{D} = (F_1, S_1, F_2, S_2, \cdots, F_n, S_n)$ of a knot $K$ in $S^3$, where $n \in \mathbb{N}$, and arranges the set $\{c(S_i) : i \in \{1, \cdots, n\}\}$ in non-increasing order. We say that $E(K)$ is in circular thin position with respect to $\mathcal{D}$ if $\mathcal{D}$ is minimal among all such generalized circular Heegaard splittings for $E(K)$, where we compare such sets using lexicographical ordering. A knot is said to be almost fibered if the knot is in circular thin position with respect to a circular Heegaard splitting (i.e. $n = 1$). Our first question is as follows.

**81. Question.** *Is there a relationship between circular distance and circular thin position? In particular, if $K$ is an almost fibered knot in $S^3$ with circular thin Heegaard splitting $(F, S)$, does there exist a constant $c_K \in \mathbb{N} \cup \{0\}$ such that if $cd(F, S) > c_K$, then $(F, S)$ is the unique circular thin decomposition for $K$?*

In Remark 3.6 of [**15**], Manjarrez-Gutierrez states that it is plausible to suspect that a minimal genus Seifert surface arises as part of a thin circular handle decomposition of a knot exterior. This question is of particular interest to us in the case that the circular handle decomposition has only one thin level and one thick level.

**82. Question.** *If possible, how can circular distance be used to prove that the thin level of an circular thin decomposition of an almost-fibered knot is of minimal genus?*

Certainly, the answer to this question is known to be true for fibered knots, since fiber surfaces are unique minimal genus Seifert surfaces for fibered knots. As Manjarrez-Gutierrez states, the answer is known to be true for all non-fibered knots prime knots up to 10 crossings as well; see [**8**].

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