Discrete Isometry Subgroups of Negatively Pinched Hadamard Manifolds

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To my family

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Abstract

In this dissertation, we prove two main results about the discrete isometry subgroups of negatively pinched Hadamard manifolds. The first one is to generalize Bonahon's characterization of geometrically infinite torsion-free discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ to geometrically infinite discrete isometry subgroups Γ of negatively pinched Hadamard manifolds X. We also generalize a theorem of Bishop to prove every discrete geometrically infinite isometry subgroup Γ has a set of nonconical limit points with the cardinality of the continuum.

The second main result is to prove a quantitative version of the Tits alternative for negatively pinched Hadamard manifolds X. Precisely, we prove that a nonelementary discrete torsion-free isometry subgroup of $\mathrm{Isom}(X)$ generated by two non-elliptic isometries g, f contains a free subgroup of rank 2 generated by isometries f^N, h of uniformly bounded word length. Furthermore, we show that this free subgroup is convex-cocompact when f is hyperbolic.

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CHAPTER 1

Introduction

A *Hadamard manifold* is a complete, simply connected Riemannian manifold of nonpositive curvature. A *negatively pinched Hadamard manifold* is a Hadamard manifold such that all the sectional curvatures lie between two negative constants. Hyperbolic spaces $\mathbb{H}^n$ and rank 1 symmetric spaces are examples of negatively pinched Hadamard manifolds. We study some algebraic properties of discrete isometry groups of negatively pinched Hadamard manifolds, and use them to obtain geometric properties of the manifolds in this dissertation.

1.1. Geometric finiteness

The notion of geometrically finite discrete groups was originally introduced by Ahlfors in [**1**], for subgroups of isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$ as the finiteness condition for the number of faces of a convex fundamental polyhedron. In the same paper, Ahlfors proved that the limit set of a geometrically finite subgroup of isometries of $\mathbb{H}^3$ has either zero or full Lebesgue measure in S^2. The notion of geometric finiteness turned out to be quite fruitful in the study of Kleinian groups. Alternative definitions of geometric finiteness were later given by Marden [**31**], Beardon and Maskit [**5**], and Thurston [**37**]. These definitions were further extended by Bowditch [**11**] and Ratcliffe [**36**] for isometry subgroups of higher dimensional hyperbolic spaces and, a bit later, by Bowditch [**12**] to negatively pinched Hadamard manifolds. While the original Ahlfors' definition turned out to be too limited (when used beyond the hyperbolic 3-space), other definitions of geometric finiteness were proven to be equivalent by Bowditch in [**12**].

Our work is motivated by the definition of geometric finiteness due to Beardon and Maskit [**5**] who proved

Theorem 1.1.1. *A discrete isometry subgroup* Γ *of* $\mathbb{H}^3$ *is geometrically finite if and only if every limit point of* Γ *is either a* conical limit point *or a* bounded parabolic fixed point.

This theorem was improved by Bishop in [**7**]:

THEOREM 1.1.2. *A discrete subgroup* $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ *is geometrically finite if and only if every point of* $\Lambda(\Gamma)$ *is either a conical limit point or a parabolic fixed point. Furthermore, if* $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ *is geometrically infinite,* $\Lambda(\Gamma)$ *contains a set of nonconical limit points with the cardinality of the continuum.*

The key ingredient in Bishop's proof of Theorem 1.1.2 is Bonahon's theorem[1] [**10**]:

THEOREM 1.1.3. *A discrete torsion-free subgroup* $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ *is geometrically infinite if and only if there exists a sequence of closed geodesics* λ_i *in the manifold* $M = \mathbb{H}^3/\Gamma$ *which "escapes every compact subset of* M*," i.e., for every compact subset* $K \subset M$,

$$\mathrm{card}\left(\{i : \lambda_i \cap K \neq \emptyset\}\right) < \infty.$$

According to Bishop, Bonahon's theorem also holds for groups with torsion. We extend Bonahon's proof and prove that Bonahon's theorem holds for discrete isometry subgroups of negatively pinched Hadamard manifolds X.

Bowditch generalized the notion of geometric finiteness to discrete subgroups of isometries of negatively pinched Hadamard manifolds [**12**]. From now on, we use X to denote an n-dimensional negatively pinched Hadamard manifold, $\partial_\infty X$ its visual (ideal) boundary, $\bar{X}$ the visual compactification $X \cup \partial_\infty X$, Γ a discrete subgroup of isometries of X, $\Lambda = \Lambda(\Gamma)$ the limit set of Γ. The convex core $\mathrm{Core}(M)$ of $M = X/\Gamma$ is defined as the Γ-quotient of the closed convex hull of $\Lambda(\Gamma)$ in X. Recall also that a point $\xi \in \partial_\infty X$ is a *conical limit point*[2] of Γ if for every $x \in X$ and every geodesic ray l in X asymptotic to ξ, there exists a positive constant A such that the set $\Gamma x \cap N_A(l)$ accumulates to ξ, where $N_A(l)$ denotes the A-neighborhood of l in X. A parabolic fixed point $\xi \in \partial_\infty X$ (i.e. a fixed point of a parabolic element of Γ) is called *bounded* if

$$(\Lambda(\Gamma) - \{\xi\})/\Gamma_\xi$$

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[1] Bonahon uses this result to prove his famous theorem about tameness of hyperbolic 3-manifolds.

[2] Another way is to describe conical limit points of Γ as points $\xi \in \partial_\infty X$ such that one, equivalently, every, geodesic ray $\mathbb{R}_+ \to X$ asymptotic to ξ projects to a non-proper map $\mathbb{R}_+ \to M$.

is compact. Here Γ_ξ is the stabilizer of ξ in Γ.

Bowditch [**12**], gave four equivalent definitions of geometric finiteness for Γ:

THEOREM 1.1.4. *The followings are equivalent for discrete subgroups* $\Gamma < \mathrm{Isom}(X)$*:*

1. *The quotient space* $\bar{M}(\Gamma) = (\bar{X} - \Lambda)/\Gamma$ *has finitely many topological ends each of which is a "cusp".*
2. *The limit set* $\Lambda(\Gamma)$ *of* Γ *consists entirely of conical limit points and bounded parabolic fixed points.*
3. *The noncuspidal part of the convex core* $\mathrm{Core}(M)$ *of* $M = X/\Gamma$ *is compact.*
4. *For some* $\delta > 0$*, the uniform* δ*-neighbourhood of the convex core,* $N_\delta(\mathrm{Core}(M))$*, has finite volume and there is a bound on the orders of finite subgroups of* Γ*.*

If one of these equivalent conditions holds, the subgroup $\Gamma < \mathrm{Isom}(X)$ is said to be geometrically finite; otherwise, Γ is said to be geometrically infinite.

We prove the main result in [**29**]:

THEOREM 1.1.5. *Suppose that* $\Gamma < \mathrm{Isom}(X)$ *is a discrete subgroup. Then the followings are equivalent:*

1. Γ *is geometrically infinite.*
2. *There exists a sequence of closed geodesics* $\lambda_i \subset M = X/\Gamma$ *which escapes every compact subset of* M*.*
3. *The set of nonconical limit points of* Γ *has the cardinality of the continuum.*

COROLLARY 1.1.6. *If* $\Gamma < \mathrm{Isom}(X)$ *is a discrete subgroup then* Γ *is geometrically finite if and only if every limit point of* Γ *is either a conical limit point or a parabolic fixed point.*

These results can be sharpened as follows. We refer the reader to Section 5.4 for the precise definitions of ends e of the orbifolds $Y = \mathrm{Core}(M)$ and $\mathrm{noncusp}_\varepsilon(Y)$, of their *neighborhoods* $C \subset Y$ and of their *end-limit sets* $\Lambda(C)$, $\Lambda(e)$, which are certain subsets of the set of non-conical limit points of Γ.

In [**22**, Section 4], Falk, Matsuzaki and Stratmann conjectured the end-limit set of an end e of Y is countable if and only if $\Lambda(e)$ is the Γ-orbit of a (bounded) parabolic fixed point. A slight modification of the proof of Theorem 1.1.5 proves this conjecture [**29**]:

Corollary 1.1.7. *$\Lambda(e)$ is countable if and only if the end e of Y is a cusp.*

Furthermore:

Corollary 1.1.8. *Let $C \subset Y$ be an unbounded complementary component of a compact subset $K \subset Y$. The $\Lambda(C)$ is countable if and only if C is Hausdorff-close to a finite union of cuspidal neighborhoods of cusps in Y.*

By Theorem 1.1.5, the set of nonconical limit points of a discrete isometry subgroup Γ has the cardinality of the continuum if Γ is geometrically infinite. It is natural to ask:

Question 1.1.9. *What is the Hausdorff dimension of the set of nonconical limit points of Γ? Here, the Hausdorff dimension is defined with respect to any of the* visual metrics *on $\partial_\infty X$, see [**35**].*

Partial results have been obtained by Fernández and Melián [**23**] in the case of Fuchsian subgroups of the 1st kind, $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ and by Bishop and Jones [**8**] in the case of finitely generated discrete torsion-free subgroups $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ of the 2nd kind, such that the manifold $\mathbb{H}^3/\Gamma$ has injectivity radius bounded below. In both cases, the Hausdorff dimension of the set of nonconical limit points equals the Hausdorff dimension of the entire limit set.

Below is an outline of the proof of Theorem 1.1.5. Our proof of the implication (1)⇒(2) mostly follows Bonahon's argument with the following exception: At some point of the proof Bonahon has to show that certain elements of Γ are loxodromic. For this he uses a calculation with 2×2 parabolic matrices: If g, h are parabolic elements of $\mathrm{Isom}(\mathbb{H}^3)$ generating a nonelementary subgroup then either gh or hg is non-parabolic. This argument is no longer valid for isometries of higher dimensional hyperbolic spaces, let alone Hadamard manifolds. We replace this computation with a more difficult argument showing that there exists a number $\ell = \ell(n, \kappa)$ such that for every n-dimensional Hadamard manifold X with sectional curvatures pinched between $-\kappa^2$ and -1 and for any pair of parabolic isometries $g, h \in \mathrm{Isom}(X)$ generating a nonelementary discrete subgroup, a certain word $w = w(g, h)$ of length $\leq \ell$ is loxodromic (Theorem 4.2.5). We later found a stronger result by Breuillard and Fujiwara [**14**] that there exists a loxodromic element of uniformly bounded word length in the discrete nonelementary subgroup generated by any isometry subset (Corollary 4.2.11), which can be used to deal with nonelementary groups generated by elliptic isometries.

Our proof of the implication (2)⇒(3) is similar to Bishop's but is more coarse-geometric in nature. Given a sequence of closed geodesics λ_i in M escaping compact subsets, we define a family of proper piecewise geodesic paths γ_τ in M consisting of alternating geodesic arcs μ_i, ν_i, such that μ_i connects λ_i to λ_{i+1} and is orthogonal to both, while the image of ν_i is contained in the loop λ_i. If the lengths of ν_i are sufficiently long, then the path γ_τ lifts to a uniform quasigeodesic $\tilde{\gamma}_\tau$ in X, which, therefore, is uniformly close to a geodesic $\tilde{\gamma}^*_\tau$. Projecting the latter to M, we obtain a geodesic γ^*_τ uniformly close to γ_τ, which implies that the ideal point $\tilde{\gamma}^*_\tau(\infty) \in \partial_\infty X$ is a nonconical limit point of Γ. Different choices of the arcs ν_i yield distinct limit points, which, in turn implies that $\Lambda(\Gamma)$ contains a set of nonconical limit points with the cardinality of the continuum. The direction (3)⇒(1) is a direct corollary of Theorem 1.1.4.

1.2. Quantitative version of the Tits alternative

In geometric group theory, the Tits alternative is an important theorem about the structure of finitely generated linear groups [**38**]. It is an important ingredient in the proof of Gromov's theorem on groups of polynomial growth [**24**]. A group G is said to satisfy the Tits alternative if for every subgroup H of G, either H is virtually solvable or H contains a nonabelian free subgroup. This was originally proved for matrix group in [**38**].

In this dissertation, we discuss quantitative version of the Tits alternative. If G satisfies the Tits alternative, and $H < G$ is generated by $g_1, \cdots, g_k$, then there exist words $u(g_1, \cdots, g_k)$ and $v(g_1, \cdots, g_k)$ which generate a free rank 2 subgroup of H, unless H is virtually solvable. The quantitative version of the Tits alternative estimates the lengths of the words u, v and describes the shapes of u, v.

Let X be an n-dimensional negatively curved Hadamard manifold, with sectional curvature ranging between $-\kappa^2$ and -1, for some $\kappa \geq 1$. Discrete isometry subgroups of $\mathrm{Isom}(X)$ satisfy the Tits alternative [**6**]. Our second main result [**19**] is the following quantitative version of the

Tits alternative for X, which answers a question asked by Filippo Cerocchi during the Oberwolfach Workshop "Differentialgeometrie im Grossen", 2017, see also [**16**]:

THEOREM 1.2.1. *There exists a function* $\mathcal{L} = \mathcal{L}(n, \kappa)$ *such that the following holds: Let* f, g *be non-elliptic isometries of* X *generating a nonelementary torsion-free discrete subgroup* Γ *of* $\mathrm{Isom}(X)$. *Then there exists an element* $h \in \Gamma$ *whose word length (with respect to the generators* f, g*) is* $\leq \mathcal{L}$ *and a number* $N \leq \mathcal{L}$ *such that the subgroup of* Γ *generated by* f^N, h *is free of rank two.*

For other forms of the quantitative Tits alternative for discrete isometry groups of X, we refer to [**6**, **13**, **14**, **15**].

After replacing g with the element $g' := gfg^{-1}$, and noticing that the subgroup generated by f, g' is still discrete, torsion-free and nonelementary, we reduce the problem to the case when the isometries f and g are conjugate in $\mathrm{Isom}(X)$ which we will assume from now on.

The proof of Theorem 1.2.1 breaks into two cases which are handled by different arguments:

Case 1. f (and, hence, g) has translation length bounded below by some positive number λ. We discuss this case in Section 4.4.1.

Case 2. f has translation length bounded above by some positive number λ. We discuss this case in Section 4.4.2.

REMARK 1.2.2. 1. For the constant λ we will take $\varepsilon(n, \kappa)/10$, where $\varepsilon(n, \kappa)$ is a positive lower bound for the Margulis constant of X.

2. We need to use a power of f only in Case 1, while in Case 2 we can take $N = 1$.

We also note that if f is loxodromic, the free group $\langle f^N, h\rangle$ constructed in our proof is convex-cocompact. See Proposition 4.3.3 and Corollary 4.4.5. One can also show that this subgroup is geometrically finite if f is parabolic but we will not prove it.

1.3. Notation

In a metric space (Y, d), we will use the notation $B(a, r)$ to denote the *open* r*-ball* centered at a in Y. For a subset $A \subset Y$ and a point $y \in Y$, we will denote by $d(y, A)$ the *minimal distance* from y to A, i.e.

$$d(y,A) := \inf\{d(y,a) \mid a \in A\}.$$

Similarly, for two subsets $A, B \subset Y$ define their minimal distance as

$$d(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}.$$

We will use the notation $l(p)$ or *length*(p) for the length of a rectifiable path p in a metric space.

We use the notation $\bar{N}_r(A)$ for the *closed* r-neighborhood of A in Y:

$$\bar{N}_r(A) = \{y \in Y : d(y,A) \leq r\}.$$

The *Hausdorff distance* $\mathrm{hd}(Q_1, Q_2)$ between two closed subsets Q_1 and Q_2 of (Y,d) is the infimum of $r \in [0,\infty)$ such that $Q_1 \subseteq \bar{N}_r(Q_2)$ and $Q_2 \subseteq \bar{N}_r(Q_1)$.

A metric space Y is called a proper metric space if closed balls are compact subsets. Moreover, Y is geodesic if for any pair of points x, y in Y, there is a geodesic segment xy connecting them. Let $x, y, z \in Y$. A geodesic triangle with vertices x, y, z is the union of three geodesic segments xy, yz, zx. If for any point $m \in xy$ and, there is a point in $yz \cup zx$ at distance less than δ of m, and similarly for other points on the other edges, and $\delta > 0$, then the triangle is δ-slim. A geodesic metric space Y is a δ-hyperbolic space if all geodesic triangles are δ-slim.

For a triangle $[xyz]$ in a geodesic metric space (Y,d), let x', y', z' be 3 points in the 2-dimensional Euclidean space $(\mathbb{E}^2, d')$ such that $d(x,y) = d'(x',y'), d(y,z) = d'(y',z')$ and $d(z,x) = d'(z',x')$. A geodesic space (Y,d) is a $\mathrm{CAT}(0)$ space if for any $x, y, z \in Y$ and any p belonging to some geodesic segment between x and y, then

$$d(z,p) \leq d'(z',p')$$

where p' is the unique point in $\mathbb{E}^2$ such that $d(x,p) = d'(x',p')$ and $d(y,p) = d'(y',p')$.

Let (Y,d) be a geodesic δ-hyperbolic metric space or a $\mathrm{CAT}(0)$ space, and let $\gamma, \gamma' : [0,\infty) \to Y$ be two geodesic rays. We say that γ and γ' are asymptotic if there exists $K > 0$ such that $d(\gamma(t), \gamma'(t)) \leq K$ for any $t \in [0,\infty)$. In this case, we write $\gamma \sim \gamma'$ and we define

$$\partial_\infty Y := \{\gamma \text{ geodesic ray in } Y\}/\sim.$$

We call $\partial_\infty Y$ the visual boundary of Y, and we write $\bar{Y} := Y \cup \partial_\infty Y$. If Y is proper then $\bar{Y}$ is a compactification of Y.

For general δ-hyperbolic spaces a geodesic segment xy connecting x and y is *not* unique, but, since any two such segments are within distance δ from each other, this abuse of notation is harmless. We let $|xy| = d(x, y)$ denote the length of xy. Similarly, if $y \in Y, \xi \in \partial_\infty Y$, then $y\xi$ will denote a geodesic ray emanating from y and asymptotic to ξ.

Throughout the thesis, X will denote an n-dimensional negatively pinched Hadamard manifold, unless otherwise stated; we assume that all sectional curvatures of X lie between $-\kappa^2$ and -1, where $\kappa > 0$. We let d denote the Riemannian distance function on X and let $\mathrm{Isom}(X)$ denote the isometry group of X.

For a Hadamard manifold X, the exponential map is a diffeomorphism, in particular, X is diffeomorphic to $\mathbb{R}^n$. Then X can be compactified by adjoining the ideal boundary sphere $\partial_\infty X$. The space $\bar{X}$ is homeomorphic to the closed n-dimensional ball [**12**].

In this dissertation, geodesics will be always parameterized by their arc-length; we will conflate geodesics in X with their images.

Given a closed subset $A \subseteq X$ and $x \in X$, we write

$$\mathrm{Proj}_A(x) = \{y \in A \mid d(x, y) = d(x, A)\}$$

for the nearest-point projection of x to A. It consists of all points in A which are closest to x. If A is convex, then $\mathrm{Proj}_A(x)$ is a singleton.

Given points $P_1, P_2, \cdots, P_m \in X$ we let $[P_1 P_2 \cdots P_m]$ denote the geodesic polygon in X which is the union of geodesic segments $P_i P_{i+1}$, i taken modulo m. Given a sequence of paths $\gamma_1, \cdots, \gamma_k$, assume that the end point of γ_i is the same as the the start point of γ_{i+1} where $1 \le i \le k-1$. We let $\gamma_1 * \cdots * \gamma_k$ denote the concatenation of paths $\gamma_1, \cdots, \gamma_k$.

Given two distinct points $x, y \in X$, and a point $q \in xy$, we define the *normal hypersurface* $\mathscr{N}_q(x, y)$, i.e. the image of the normal exponential map to the segment xy at the point q:

$$\mathscr{N}_q(x, y) = \exp_q(T_q^\perp(xy)),$$

where $T_q^{\perp}(xy) \subset T_qX$ is the orthogonal complement in the tangent space at q to the segment xy. In the special case when q is the midpoint of xy, $\mathscr{N}_q(x,y)$ is the *perpendicular bisector* of the segment xy, and we will denote it $\mathrm{Bis}(x,y)$. Similarly, we define the normal hypersurface $\mathscr{N}_q(\xi,\eta)$ for any point q in the biinfinite geodesic $\xi\eta$.

Given a pair of points p,q in X we let $H(p,q)$ denote the closed half-space in X given by

$$H(p,q) = \{x \in X : d(x,p) \le d(x,q)\}.$$

Then $\mathrm{Bis}(p,q) = \mathrm{Bis}(q,p) = H(p,q) \cap H(q,p)$.

Note that if X is a real-hyperbolic space, then $\mathrm{Bis}(x,y)$ is totally geodesic and equals the set of points equidistant from x and y. For general Hadamard spaces, this is not the case. However, if X is δ-hyperbolic, then each $\mathscr{N}_p(x,y)$ is δ-quasiconvex, see Definition 2.3.1.

We let δ denote the *hyperbolicity constant* of X; hence, $\delta \le \cosh^{-1}(\sqrt{2})$. We will use the notation $\mathrm{Hull}(A)$ for the *closed convex hull* of a subset $A \subset X$, i.e. the intersection of all closed convex subsets of X containing A. The notion of the closed convex hull extends to the closed subsets of $\partial_\infty X$ as follows. Given a closed subset $A \subset \partial_\infty X$, we denote by $\mathrm{Hull}(A)$ the smallest closed convex subset of X whose accumulation set in $\bar{X}$ equals A. (Note that $\mathrm{Hull}(A)$ is nonempty as long as A contains more than one point.)

For a subset $A \subset X$ the *quasiconvex hull* $\mathrm{QHull}(A)$ of A in X is defined as the union of all geodesics connecting points of A. Similarly, for a closed subset $A \subset \partial_\infty X$, the quasiconvex hull $\mathrm{QHull}(A)$ is the union of all biinfinite geodesics asymptotic to points of A. Then $\mathrm{QHull}(A) \subset \mathrm{Hull}(A)$.

We will use the notation Γ for a discrete subgroup of isometries of X. We let $\Lambda = \Lambda(\Gamma) \subset \partial_\infty X$ denote the *limit set* of Γ, i.e. the accumulation set in $\partial_\infty X$ of one (equivalently, any) Γ-orbit in X. The group Γ acts properly discontinuously on $\bar{X} \setminus \Lambda$, [**12**, Proposition 3.2.6]. We obtain an orbifold with boundary

$$\bar{M} = \left(\bar{X} \setminus \Lambda\right)/\Gamma.$$

If Γ is torsion-free, then $\bar{M}$ is a partial compactification of the quotient manifold $M = X/\Gamma$. We let $\pi : X \to M$ denote the covering projection.

CHAPTER 2

Review of negatively pinched Hadamard manifolds

Much of theory of Hadamard manifold can be simplified when the sectional curvature is bounded away from 0. In this case, we assume that all sectional curvatures at most -1 by scaling the metric. This is the assumption in our dissertation.

2.1. Some CAT(-1) computations

In this section, we use X to denote an n-dimensional Hadamard manifold with curvature at most -1. Then X is a δ-hyperbolic space with $\delta = \cosh^{-1}(\sqrt{2})$ [**12**, Proposition 1.1.6].

For any triangle $[ABC]$ in (X, d), we define a comparison triangle $[A'B'C']$ for $[ABC]$ in $(\mathbb{H}^2, d')$ as follows.

Definition 2.1.1. For a triangle $[ABC]$ in (X, d), let A', B', C' be 3 points in the hyperbolic plane $(\mathbb{H}^2, d')$ satisfying that $d'(A', B') = d(A, B), d'(B', C') = d(B, C)$ and $d'(C', A') = d(C, A)$. Then $[A'B'C']$ is called a *comparison triangle* for $[ABC]$.

In general, for any geodesic polygon $[P_1 P_2 \cdots P_m]$ in (X, d), we define a comparison polygon $[P_1' P_2' \cdots P_m']$ for $[P_1 \cdots P_m]$ in $(\mathbb{H}^2, d')$.

Definition 2.1.2. For any geodesic polygon $[P_1 P_2 \cdots P_m]$ in X, we pick points $P_1', \cdots, P_m'$ in $\mathbb{H}^2$ such that $[P_1' P_i' P_{i+1}']$ is a comparison triangle for $[P_1 P_i P_{i+1}]$ and the triangles $[P_1' P_{i-1}' P_i']$ and $[P_1' P_i' P_{i+1}']$ lie on different sides of $P_1' P_i'$ for each $2 \leq i \leq m-1$. The geodesic polygon $[P_1' P_2' \cdots P_m']$ is called a *comparison polygon* for $[P_1 P_2 \cdots P_m]$.

Remark 2.1.3. Such a comparison polygon $[P_1' P_2' \cdots P_m']$ is not necessarily convex and embedded[1]. In the rest of the section, we have additional assumptions for the polygons $[P_1 P_2 \cdots P_m]$.

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[1] I.e. the natural map $S^1 \to \mathbb{H}^2$ defined by tracing the oriented edges of the polygon in the cyclic order need not be injective.

Under these assumptions, their comparison polygons in $\mathbb{H}^2$ are embedded and convex, see Corollary 2.1.8.

One important property of Hadamard manifolds X with curvature bounded away from 0 is the following *angle comparison* theorem; cf. [**17**].

Proposition 2.1.4. *[**12**, Proposition 1.1.2] For a triangle $[ABC]$ in (X, d), let $[A'B'C']$ denote a comparison triangle for $[ABC]$. Then $\angle ABC \le \angle A'B'C', \angle BCA \le \angle B'C'A'$ and $\angle CAB \le \angle C'A'B'$.*

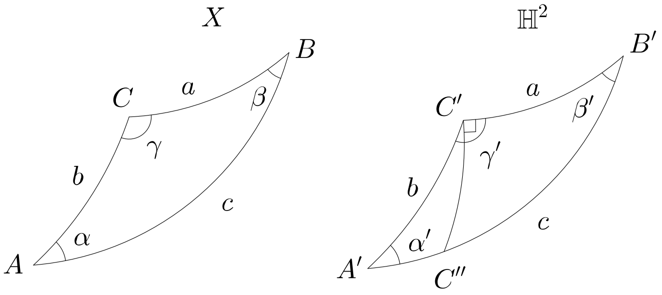

Figure 2.1

Proposition 2.1.4 implies some useful geometric inequalities in X:

Corollary 2.1.5. *[**29**] Consider a triangle in X with vertices ABC such that the angles at A, B, C are α, β, γ and the sides opposite to A, B, C have lengths a, b, c, respectively. If $\gamma \ge \pi/2$, then*

$$\cosh a \sin \beta \le 1.$$

Proof. Let $[A'B'C']$ be a comparison triangle for $[ABC]$ in $(\mathbb{H}^2, d')$. Let α', β', γ' denote the angles at A', B', C' respectively as in Figure 2.1. By Proposition 2.1.4, $d'(A', B') = c, d'(A', C') = b, d'(B', C') = a$ and $\beta' \ge \beta, \gamma' \ge \gamma \ge \pi/2$. Take the point $C'' \in A'B'$ such that $\angle B'C'C'' = \pi/2$. In the right triangle $[B'C'C'']$ in $\mathbb{H}^2$, we have $\cosh a \sin \beta' = \cos(\angle C'C''B')$, see [**4**, Theorem 7.11.3]. So we obtain the inequality:

$$\cosh a \sin \beta \le \cosh a \sin \beta' \le 1. \quad \square$$

REMARK 2.1.6. If $A \in \partial_\infty X$, we use a sequence of triangles in X to approximate the triangle $[ABC]$ and prove that $\cosh a \sin \beta \leq 1$ still holds by continuity.

COROLLARY 2.1.7. *[**19**] Let $[A_1A_2C]$ be a triangle in X such that $\angle A_1CA_2 \geq \pi/2$. Then*

$$d(A_1, A_2) \geq d(A_1, C) + d(A_2, C) - 2\delta.$$

PROOF. Let $D \in A_1A_2$ be the point closest to C. Then at least one of the angles $\angle A_iCD, i = 1, 2$ is $\geq \pi/4$. By Corollary 2.1.5

$$\cosh(d(C, D)) \sin(\frac{\pi}{4}) \leq 1,$$

i.e.

$$d(C, D) \leq \cosh^{-1}(\sqrt{2}) = \delta.$$

The rest follows from the triangle inequalities. □

COROLLARY 2.1. *[**19**] Suppose that $x, x_+, \hat{x}_+, x'_+$ are points in X which lie on a common geodesic and appear on this geodesic in the given order. Assume that*

$$d(\hat{x}_+, x'_+) \geq d(x, x_+) + 2\cosh^{-1}(\sqrt{2}).$$

Then $H(x_+, \hat{x}_+) \subset H(x, x'_+)$.

PROOF. We observe that the CAT(-1) condition implies that for each z equidistant from $x_+, \hat{x}_+$ we have

$$\angle z x_+ \hat{x}_+ \leq \pi/2, \quad \angle z \hat{x}_+ x_+ \leq \pi/2.$$

Hence,

$$\angle x x_+ z \geq \pi/2, \quad \angle x'_+ \hat{x}_+ z \geq \pi/2.$$

Then the lemma and the triangle inequality implies that

$$d(z, x) \leq d(z, x'_+).$$

and, thus,

$$\text{Bis}(x_+, \hat{x}_+) \subset H(x, x'_+).$$

Since every geodesic connecting $w \in H(x_+, \hat{x}_+)$ to x'_+ passes through some point $z \in \mathrm{Bis}(x_+, \hat{x}_+)$, it follows that

$$d(x, w) \leq d(w, x'_+). \qquad \square$$

Corollary 2.1.8. *[**29**] Let $[ABCD]$ denote a quadrilateral in X such that $\angle ABC \geq \pi/2, \angle BCD \geq \pi/2$ and $\angle CDA \geq \pi/2$ as in Figure 2.2. Then:*

(1) $\sinh(d(B, C)) \sinh(d(C, D)) \leq 1$.

(2) *Suppose that $\angle BAD \geq \alpha > 0$. If $\cosh(d(A, B)) \sin \alpha > 1$, then*

$$\cosh(d(C, D)) \geq \cosh(d(A, B)) \sin \alpha > 1.$$

Proof. Let $[A'B'C'D']$ be a comparison quadrilateral for $[ABCD]$ in $(\mathbb{H}^2, d')$ such that $[A'B'C']$ is a comparison triangle for $[ABC]$ and $[A'C'D']$ is a comparison triangle for $[ACD]$. By Proposition 2.1.4, $\angle A'B'C' \geq \pi/2$, $\angle A'D'C' \geq \pi/2$ and

$$\angle B'C'D' = \angle B'C'A' + \angle A'C'D' \geq \angle BCD \geq \pi/2.$$

Thus, $0 < \angle B'A'D' \leq \pi/2$ and $[A'B'C'D']$ is an embedded convex quadrilateral.

We first prove that $\sinh d(B, C) \sinh(d(C, D)) \leq 1$. In Figure 2.2, take the point $H \in A'B'$ such that $\angle HC'D' = \pi/2$ and take the point $G \in A'H$ such that $\angle GD'C' = \pi/2$. We claim that $\angle C'HA' \geq \pi/2$. Observe that

$$\angle C'HB' + \angle HB'C' + \angle B'C'H \leq \pi$$

$$\angle C'HA' + \angle C'HB' = \pi.$$

Thus $\angle C'HA' \geq \angle C'B'H \geq \pi/2$. We also have $d'(C', H) \geq d'(C', B')$ since

$$\frac{\sinh(d'(C', H))}{\sin(\angle C'B'H)} = \frac{\sinh(d'(C', B'))}{\sin(\angle C'HB')}.$$

Take the point $H' \in GD'$ such that $\angle C'HH' = \pi/2$. In the quadrilateral $[C'HH'D']$, $\cos(\angle HH'D') = \sinh(d'(H, C')) \sinh(d'(C', D'))$, [**4**, Theorem 7.17.1]. Thus, we have

$$\begin{aligned}\sinh(d(C,D))\sinh(d(B,C)) &= \sinh(d'(C',D')\sinh(d'(B',C'))\\ &\le \sinh(d'(C',D'))\sinh(d'(C',H))\\ &\le 1.\end{aligned}$$

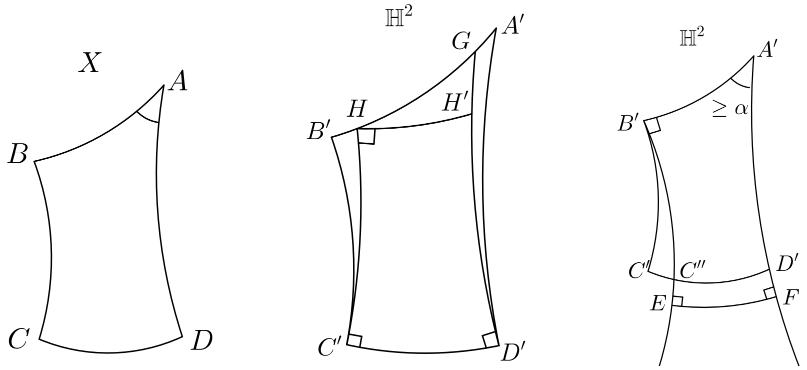

FIGURE 2.2

Next, we prove that if $\cosh(d(A,B))\sin\alpha > 1$, then $\cosh(d(C,D)) \geq \cosh(d(A,B))\sin\alpha$. In Figure 2.2, take the $C'' \in C'D'$ such that $\angle A'B'C'' = \pi/2$. Observe that C'' cannot be on $A'D'$. Otherwise in the right triangle $[A'B'C'']$, we have

$$\cosh(d(A,B))\sin\alpha \le \cosh(d'(A',B'))\sin(\angle B'A'D') \le 1,$$

which is a contradiction. Let EF denote the geodesic segment which is orthogonal to $B'E$ and $A'F$. In the quadrilateral $[A'B'EF]$, $\cosh(d'(E,F)) = \cosh(d'(A',B'))\sin(\angle B'A'F)$ by hyperbolic trigonometry [**4**, Theorem 7.17.1]. Thus,

$$\cosh(d(C,D)) \geq \cosh(d'(C'',D')) \geq \cosh(d'(E,F)) \geq \cosh(d(A,B))\sin\alpha. \quad \square$$

REMARK 2.1.9. If $A \in \partial_\infty X$ and $\angle BAD = 0$, we use quadrilaterals in X to approximate the quadrilateral $[ABCD]$ and prove that $\sinh(d(B,C))\sinh(d(C,D)) \le 1$ by continuity.

For each pair of points $A, B \in \mathbb{H}^2$ and each circle $S \subset \mathbb{H}^2$ passing through these points, we let $\widehat{AB}^S$ denote the (hyperbolic) length of the shorter arc into which A, B divide the circle S.

LEMMA 2.1.10. *[**19**] If $d(A,B) \leq D$, then, for every circle S as above, the length ℓ of $\widehat{AB}^S$ satisfies the inequality:*

$$d(A,B) \leq \ell \leq \frac{2\pi \tanh(D/4)}{1 - \tanh^2(D/4)}.$$

PROOF. The first inequality is clear, so we verify the second. We want to maximize the length of $\widehat{AB}^S$ among all circles S passing through A, B. We claim that the maximum is achieved on the circle S_o whose center o is the midpoint of AB. This follows from the fact that given any other circle S, we have the radial projection from $\widehat{AB}^{S_o}$ to $\widehat{AB}^S$ (with the center of the projection at o). Since this radial projection is distance-decreasing (by convexity), the claim follows. The rest of the proof amounts to a computation of the length of the hyperbolic half-circle with the given diameter. $\square$

LEMMA 2.1.11. *[**19**] There exists a function $c(D)$ so that the following holds. Consider an isosceles triangle ABC in X with $d(A,C) = d(B,C)$, $d(A,B) \leq D$, and an isosceles subtriangle $A'B'C$ with $A' \in AC, B' \in BC$, $d(A,A') = d(B,B') = \tau$. Then*

$$d(A',B') \leq c(D)e^{-\tau}.$$

PROOF. In view of the CAT(-1) assumption, it suffices to consider the case when $X = \mathbb{H}^2$. We will work with the unit disk model of the hyperbolic plane where C is the center of the disk as in Figure 2.3. Let α denote the angle $\angle ACB$. Set $T := d(C,A) = d(C,B)$. For points $A_t \in CA, B_t \in CB$ such that $d(C,A_t) = d(C,B_t) = t$ we let l_t denote the hyperbolic length of

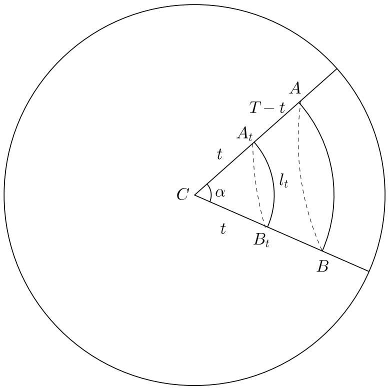

FIGURE 2.3

the (shorter) circular arc $\widehat{A_tB_t} = \widehat{A_tB_t}^{S_t}$ of the angular measure α, centered at C and connecting A_t to B_t. (Here S_t is the circle centered at C and of the hyperbolic radius t.) Let R_t denote the Euclidean distance between C and A_t (same for B_t). Then

$$l_t = \frac{2\alpha R_t}{1 - R_t^2},$$

$$R_t = \tanh(t/2).$$

Thus, for $\tau = T - t$,

$$\begin{aligned}\frac{l_t}{l_T} = \frac{R_t}{R_T}\frac{1-R_T^2}{1-R_t^2} \le \frac{1-R_T^2}{1-R_t^2} \le 2\frac{1-R_T}{1-R_t}\\ = 2\frac{1-\tanh(T/2)}{1-\tanh(t/2)} = 2\frac{e^t+1}{e^T+1} = 2\frac{e^{-T}+e^{-\tau}}{e^{-T}+1} \le 4e^{-\tau}.\end{aligned}$$

In other words,

$$d(A_t, B_t) \le l_t \le 4e^{-\tau} l_T.$$

Combining this inequality with Lemma 2.1.10, we obtain

$$l_t \le 4e^{-\tau}\frac{2\pi\tanh(d(A,B)/4)}{1-\tanh^2(d(A,B)/4)} \le 4e^{-\tau}\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}.$$

Lastly, setting $A' = A_t, B' = B_t, A = A_T, B = B_T$, we get:

$$d(A', B') \le 4\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}e^{-\tau} = c(D)e^{-\tau}.$$ □

Given a point $\xi \in \partial_\infty X$, for any point $y \in X$, we use a map $\rho_y : \mathbb{R}^+ \to X$ to parametrize the geodesic $y\xi$ by its arc-length. The following lemma is deduced from the $CAT(-1)$ inequality, see [**12**]:

LEMMA 2.1.12. *[**12**, Proposition 1.1.11]*

1. *Given any $y, z \in X$, the function $d(\rho_y(t), \rho_z(t))$ is monotonically decreasing in t.*
2. *For each r, there exists a constant $R = R(r)$, such that if $y, z \in X$ lie in the same horosphere about ξ and $d(y,z) \le r$, then $d(\rho_y(t), \rho_z(t)) \le Re^{-t}$ for all t.*

2.2. Volume inequalities

In this section, we assume that all sectional curvatures of X are at least $-\kappa^2$ for some $\kappa \geq 1$. This lower bound of the sectional curvatures gives an upper bound on the volume of uniforms balls.

Let $V(r,n)$ denote the volume of the r-ball in $\mathbb{H}^n$. Then, for a positive constant c_n depending only on n,

$$V(r,n) = c_n \int_0^r \sinh^{n-1}(t)dt \leq \frac{c_n}{2^{n-1}(n-1)} e^{(n-1)r} = C_n e^{(n-1)r},$$

see e.g. [**34**, Sect. 1.5].

Volumes of metric balls $B(x,r) \subset X$ satisfy the inequalities

$$V(r,n) \leq VolB(x,r) \leq V(\kappa r, n)/\kappa^n, \tag{2.2}$$

see e.g. Proposition 1.1.12 and Proposition 1.2.4 in [**12**], or [**9**, Sect. 11.10]. As a corollary of these volume inequalities we obtain the following *packing inequality*:

Lemma 2.2.1. *Suppose that $Z \subset X$ is a subset such that the minimal distance between distinct points of Z is at least $2r$. Then for every $x \in X$, $R \geq 0$, we have*

$$\mathrm{card}\,(B(x,R) \cap Z) \leq \frac{V(\kappa(R+r),n)}{\kappa^n V(r,n)} \leq \frac{C_n}{\kappa^n V(r,n)} e^{\kappa(n-1)(R+r)}.$$

In particular, if

$$\mathrm{card}\,(Z) > \frac{C_n}{V(r,n)} e^{\kappa(n-1)(R+r)}$$

then for any $z \in Z$ there exists $z' \in Z$ such that $d(z,z') > R$.

2.3. Convexity and quasi-convexity

In this section, we assume that X is a negatively pinched Hadamard manifold such that all sectional curvatures lie between -1 and $-\kappa^2$ for some $\kappa \geq 1$.

Definition 2.3.1. A subset $A \subseteq X$ is *convex* if $xy \subseteq A$ for all $x, y \in A$. A closed subset $A \subseteq X$ is λ-*quasiconvex* if $xy \subseteq \bar{N}_\lambda(A)$ for all $x, y \in A$. Convex closed subsets are 0-quasiconvex.

Remark 2.3.2. If A is a λ-quasiconvex set, then $\mathrm{QHull}(A) \subseteq \bar{N}_\lambda(A)$.

Definition 2.3.3. A closed subset $Q \subset \bar{X}$ is *starlike* about $x \in \bar{X}$ if $xy \subset Q$ for all $y \in Q$.

Lemma 2.3.4. *[**12, 19**] Starlike subsets in a δ-hyperbolic space X are δ-quasiconvex.*

Proof. We prove this for subsets $A \subset X$ starlike with respect to $a \in A$; the proof in the case of starlike subsets with respect to $\xi \in \partial_\infty X$ is similar and is left to the reader. Take $z_1, z_2 \in A$. Then, by the δ-hyperbolicity,

$$z_1 z_2 \subset \bar{N}_\delta \left(a z_1 \cup a z_2 \right) \subset \bar{N}_\delta \left(A \right). \qquad \square$$

Proposition 2.3.5. *[**12**, Lemma 2.2.1] For any $m+1$ points $x_0, x_1, \cdots, x_m \in \bar{X}$ we have*

$$x_0 x_m \subseteq \bar{N}_\lambda (x_0 x_1 \cup x_1 x_2 \cup \cdots \cup x_{m-1} x_m)$$

where $\lambda = \lambda_0 \lceil \log_2 m \rceil, \lambda_0 = \cosh^{-1}(\sqrt{2})$.

Proof. Without loss of generality, we assume that $n = 2^r$ for some $r \in \mathbb{N}$. Let $m = n/2 = 2^{r-1}$. By the δ-hyperbolicity of X, we have

$$x_0 x_n \subset N_{\cosh^{-1}(\sqrt{2})} (x_0 x_m \cup x_m x_n).$$

The result follows by induction on r.

$\square$

Remark 2.3.6. For any set $Q \subset \bar{X}$ where any two points can be joined by a piecewise geodesic path with a uniformly bounded number of segments if quasi-convex.

Proposition 2.3.7. *[**12**, Proposition 2.5.4] There is a function $\mathfrak{r}_\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ (depending also on κ) such that for every λ-quasiconvex subset $A \subseteq X$, we have*

$$\mathrm{Hull}(A) \subseteq \bar{N}_{\mathfrak{r}_\kappa(\lambda)}(A).$$

Remark 2.3.8. Note that, by the definition of the hyperbolicity constant δ of X, the quasiconvex hull $\mathrm{QHull}(A)$ is 2δ-quasiconvex for every closed subset $A \subseteq \bar{X}$. Thus, $\mathrm{Hull}(A) \subseteq \bar{N}_r(\mathrm{QHull}(A))$ for some absolute constant $r \in [0, \infty)$.

Remark 2.3.9. For any closed subset $A \subseteq \partial_\infty X$ with more than one point, $\partial_\infty \mathrm{Hull}(A) = A$.

Corollary 2.3.10. *[**12, 19**] For every starlike subset A in a Hadamard manifold X of negatively pinched curvature, the closed convex hull* $\mathrm{Hull}(A)$ *is contained in the* $\mathfrak{q} = \mathfrak{q}(\kappa, \delta)$*-neighborhood of A.*

In what follows, we will suppress the dependence of $\mathfrak{q}$ on κ and δ since these are fixed for our space X.

Lemma 2.3.11. *[**29**] Assume that ξ, η are distinct points in $\partial_\infty X$ and (x_i), (y_i) are sequences in X converging to ξ and to η respectively. Then for every point $p \in \xi\eta \subseteq X$, $p \in \bar{N}_{2\delta}(x_i y_i)$ for all sufficiently large i.*

Proof. Since (x_i) converges to ξ and (y_i) converges to η, we have $d(p, x_i\xi) \to \infty$ and $d(p, y_i\eta) \to \infty$ as $i \to \infty$. By the δ-hyperbolicity of X,

$$p \in \bar{N}_{2\delta}(x_i y_i \cup x_i\xi \cup y_i\eta).$$

Since $d(p, x_i\xi) \to \infty$ and $d(p, y_i\eta) \to \infty$, we obtain

$$p \in \bar{N}_{2\delta}(x_i y_i)$$

for sufficiently large i. □

Remark 2.3.12. This lemma holds for any δ-hyperbolic geodesic metric space.

CHAPTER 3

Groups of isometries

In this section, X denotes the negatively pinched Hadamard manifold with sectional curvature lying between -1 and $-\kappa^2$. We discuss the action of discrete isometry subgroup of $\mathrm{Isom}(X)$ on X.

3.1. Classification of isometries

Every isometry g of X extends to a homeomorphism (still denoted by g) of $\bar{X}$. We let $\mathrm{Fix}(g)$ denote the fixed point set of $g : \bar{X} \to \bar{X}$. For a subgroup $\Gamma < \mathrm{Isom}(X)$, we use the notation

$$\mathrm{Fix}(\Gamma) := \bigcap_{g\in\Gamma} \mathrm{Fix}(g),$$

to denote the fixed point set of Γ in $\bar{X}$. Typically, this set is empty.

Isometries of X are classified as follows:

(1) g is *parabolic* if $\mathrm{Fix}(g)$ is a singleton $\{p\} \subset \partial_\infty X$. In this case, g preserves (setwise) every horosphere centered at p.

(2) g is *loxodromic* if $\mathrm{Fix}(g)$ consists of two distinct points $p, q \in \partial_\infty X$. The loxodromic isometry g preserves the geodesic $pq \subset X$ and acts on it as a nontrivial translation. The geodesic pq is called the *axis* A_g of g.

(3) g is *elliptic* if it fixes a point in X. The fixed point set of an elliptic isometry is a totally-geodesic subspace of X invariant under g. In particular, the identity map is an elliptic isometry of X.

If $g \in \mathrm{Isom}(X)$ is such that $\mathrm{Fix}(g)$ contains three distinct points $\xi, \eta, \zeta \in \partial_\infty X$, then g also fixes pointwise the convex hull $\mathrm{Hull}(\{\xi, \eta, \zeta\})$ and, hence, g is an elliptic isometry of X.

For each isometry $g \in \mathrm{Isom}(X)$ we define the *displacement function* of g as follows:

$$d_g(x) = d(x, gx).$$

The *translation length* $l(g)$ of g is defined as :

$$l(g) = \inf_{x \in X} d(x, g(x)).$$

Isometries $g \in \mathrm{Isom}(X)$ of X are also classified into these three types according to their translation lengths $l(g)$, see [**3**, **6**].

1. An isometry g of X is *loxodromic* if $l(g) > 0$. Equivalently, the infimum is attained and is positive. In this case, the infimum is attained on a g-invariant geodesic, called the *axis* of g, and denoted by A_g.

2. An isometry g of X is *elliptic* of $l(g) = 0$ and the infimum is attained; the set where the infimum is attained is a totally geodesic submanifold of X fixed pointwise by g.

3. An isometry g of X is *parabolic* if the infimum is not attained. In this case, the infimum is necessarily equal to zero.

Thus, only parabolic and elliptic isometries have zero translation lengths. For any $g \in \mathrm{Isom}(X)$ and $m \in \mathbb{Z}$ we have

$$l(g^m) = |m| l(g). \tag{3.1}$$

In terms of geometric groups theory, the following theorem provides an alternative characterization of types of isometries of X, see [**18**].

Theorem 3.2. *Suppose that g is an isometry of X. Then:*

(1) *g is loxodromic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x$ is a quasiisometric embedding $\mathbb{Z} \to X$.*

(2) *g is elliptic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x, N \in \mathbb{Z}$ has bounded image.*

(3) *g is parabolic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x, N \in \mathbb{Z}$ is proper and*

$$\lim_{N \to \infty} \frac{d(x, g^N(x))}{N} = 0.$$

LEMMA 3.1.1. *If f, g are loxodromic isometries of X generating a discrete subgroup of* $\mathrm{Isom}(X)$*, then either the ideal boundaries of the axes A_f, A_g are disjoint or $A_f = A_g$.*

PROOF. Suppose that the ideal boundaries of A_f and A_g have exactly one common point $p \in \partial_\infty X$. Without loss of generality, we assume that p is the attracting fixed point of f and g. Then the sequence $(f^i g^i f^{-i})$ of elements of $\langle f, g\rangle$ tends to the identity, which contradicts to our discreteness assumption. $\square$

For an isometry $g \in \mathrm{Isom}(X)$, we define the *rotation* of g at $x \in X$ as:

$$r_g(x) = \max_{v \in T_x X} \angle(v, P_{g(x),x} \circ g_{*_x} v).$$

Here $g_{*_x} : T_x X \to T_{g(x)} X$ is the differential and $P_{g(x),x} : T_{g(x)} X \to T_x X$ is the parallel transport along the unique geodesic from $g(x)$ to x. Following [**3**], given $a \geq 8$ we define the *norm of* g at x as $n_g(x) = \max(r_g(x), a \cdot d_g(x))$ where $d_g(x) = d(x, g(x))$.

PROPOSITION 3.1.2. *[**19**] There exists a function $\mathfrak{r}(\varepsilon)$ such that for any loxodromic isometry $h \in \mathrm{Isom}(X)$ with translation length*

$$l(h) = l \leq \varepsilon/10,$$

if $A \in X$ satisfies $d(A, h(A)) = \varepsilon$, then there exists $B \in X$ such that $d(B, h(B)) = \varepsilon/3$, $d(A, B) \leq \mathfrak{r} = \mathfrak{r}(\varepsilon)$ and B lies on the shortest geodesic segment connecting A to the axis A_h of h.

PROOF. Let $C \in A_h$ be the closest point to A in A_h. By the convexity of the distance function, there exists a point $B \in AC$ such that $d(B, h(B)) = \varepsilon/3$. Suppose that $d(A, B) = d(h(A), h(B)) = t$ and $d(A, C) = d(h(A), h(C))) = T$ as shown in Figure 3.1. Then $d(C, h(A)) \leq T + l \leq T + \varepsilon/10$. There exist points D, E in the segment $h(A)C$ such that $d(C, D) = d(C, B) = T - t$, $d(h(A), E) = t$ and $d(A', C) = d(A, C) = T$.

Then $d(A, A') \leq \varepsilon + l \leq 11\varepsilon/10$. By Lemma 2.1.11, $c(11\varepsilon/10)$ (defined in that lemma) satisfies

$$d(B, D) \leq c(d(A, A'))e^{-t} \leq c(11\varepsilon/10)e^{-t}.$$

Similarly, by taking the point $A'' \in h(A)C$ satisfying $d(A'', h(A)) = T$, $d(h(C), A'') \leq 2l$, considering the isosceles triangle $\triangle h(C)A''h(A)$ and its subtriangle $\triangle h(B)Eh(A)$, we obtain:

$$d(h(B), E) \leq c(2l)e^{t-T}.$$

Since $l \leq \varepsilon/10$ and $d(B, h(B)) = \varepsilon/3$, convexity of the distance function implies that $T - t > t$. Thus,

$$\begin{aligned} \varepsilon/3 = d(B, h(B)) &\leq d(B, D) + d(D, E) + d(E, h(B)) \\ &\leq c(11\varepsilon/10)e^{-t} + l + c(2l)e^{t-T} \leq c(11\varepsilon/10)e^{-t} + \frac{\varepsilon}{10} + c(\varepsilon/5)e^{-t}, \end{aligned}$$

which simplifies to

$$\frac{7}{30}\varepsilon \leq \left(c(11\varepsilon/10) + c(\varepsilon/5)\right) e^{-t},$$

and consequently

$$d(A, B) = t \leq \mathfrak{r}(\varepsilon) := \log\left([c(11\varepsilon/10) + c(\varepsilon/5)]\frac{30}{7}\varepsilon^{-1}\right). \qquad \square$$

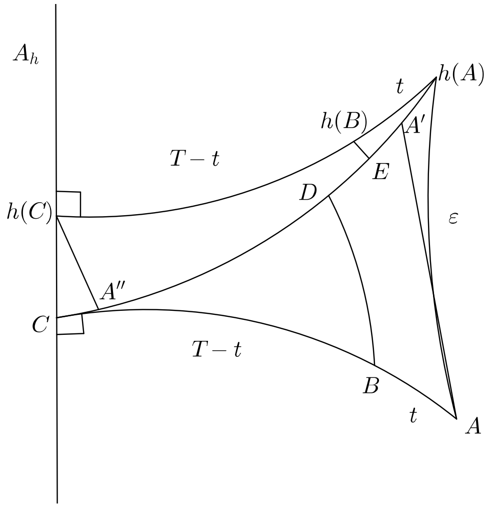

FIGURE 3.1

3.2. Elementary groups of isometries

A discrete subgroup G of isometries of X is called *elementary* if either $\mathrm{Fix}(G) \neq \emptyset$ or if G preserves set-wise some bi-infinite geodesic in X. (In the latter case, G contains an index 2 subgroup G' such that $\mathrm{Fix}(G') \neq \emptyset$.) Based on the fixed point set, elementary groups are divided into the following three classes [**12**]:

(1) $F(G)$ is a nonempty subspace of $\bar{X}$.

(2) $F(G)$ consists of a single point of $\partial_\infty X$.

(3) G has no fixed point in X, and G preserves setwise a unique bi-infinite geodesic in X.

Remark 3.2.1. If $G < \mathrm{Isom}(X)$ is discrete and in the first class, then G is finite by discreteness and consists of elliptic isometries. If G is discrete and in the second class, it is called parabolic, and it contains a parabolic isometry [**12**, Proposition 4.2]. Discrete groups G in the third class will be called elementary loxodromic groups.

Lemma 3.2.2. *[**29**] If $G < \mathrm{Isom}(X)$ is a discrete elementary subgroup consisting entirely of elliptic elements, then G is finite.*

Proof. By Remark 3.2.1, G is either finite or loxodromic. Suppose that G is loxodromic and preserves a geodesic $l \subset X$ setwise. Let $\rho : G \to \mathrm{Isom}(l)$ denote the restriction homomorphism. Since G is loxodromic, the subgroup $\rho(G)$ has no fixed point in l. Hence, there exist two elements $g, h \in G$ such that $\rho(g), \rho(h)$ are distinct involutions. Their product $\rho(g)\rho(h)$ is a nontrivial translation of l. Hence, gh is a loxodromic isometry of X, contradicting our assumption. Hence, G is finite. $\square$

Corollary 3.2.3. *[**29**] Every discrete elementary loxodromic group contains a loxodromic isometry.*

Consider a subgroup Γ of isometries of X. Given any subset $Q \subseteq \bar{X}$, let

$$\mathrm{stab}_\Gamma(Q) = \{\gamma \in \Gamma \mid \gamma(Q) = Q\}$$

denote the setwise stabilizer of Q in Γ.

DEFINITION 3.2.4. A point $p \in \partial_\infty X$ is called a *parabolic fixed point* of a subgroup $\Gamma < \mathrm{Isom}(X)$ if $\mathrm{stab}_\Gamma(p)$ is parabolic.

REMARK 3.2.5. If $p \in \partial_\infty X$ is a parabolic fixed point of a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, then $\mathrm{stab}_\Gamma(p)$ is a maximal parabolic subgroup of Γ, see [**12**, Proposition 3.2.1]. Thus, we have a bijective correspondence between the Γ-orbits of parabolic fixed points of Γ and the Γ-conjugacy classes of maximal parabolic subgroups of Γ.

Consider an elementary loxodromic subgroup $G < \Gamma$ with the axis β. Then $\mathrm{stab}_\Gamma(\beta)$ is a maximal loxodromic subgroup of Γ, see [**12**, Proposition 3.2.1].

3.3. The Thick-Thin decomposition

For an isometry $g \in \mathrm{Isom}(X)$, define the *Margulis region* $Mar(g, \varepsilon)$ of g as:

$$Mar(g, \varepsilon) = \{x \in X \mid d(x, g(x)) \leq \varepsilon\}.$$

Of primary importance are subset $Mar(g, \varepsilon)$ for $\varepsilon < \varepsilon(n, \kappa)$. For any two isometries g, h of X, we have

$$Mar(hgh^{-1}, \varepsilon) = h(Mar(g, \varepsilon)).$$

In particular, if g, h commute, then h preserves $Mar(g, \varepsilon)$. By the convexity of the distance function, $Mar(g, \varepsilon)$ is convex.

For parabolic isometries g of X define the set

$$\mathcal{T}_\varepsilon(g) := \cup_{i \in \mathbb{Z} \setminus \{0\}} Mar(g^i, \varepsilon).$$

This subset is $\langle g \rangle$-invariant.

Suppose that g is a loxodromic isometry of X. Define m_g to be the (unique) positive integer such that

$$l(g^{m_g}) \leq \varepsilon/10, \quad l(g^{m_g+1}) > \varepsilon/10, \tag{3.3}$$

and the set

$$\mathcal{T}_\varepsilon(g) := \bigcup_{1 \le i \le m_g} Mar(g^i, \varepsilon) \subset X.$$

If $l(g) > \varepsilon/10$, then $\mathcal{T}_\varepsilon(g) = \emptyset$.

Since the subgroup $\langle g \rangle$ is abelian, we obtain:

LEMMA 3.3.1. *The subgroup $\langle g \rangle$ preserves $\mathcal{T}_\varepsilon(g)$ and, hence, also preserves* $\mathrm{Hull}(\mathcal{T}_\varepsilon(g))$.

By convexity of the distance function, $\mathcal{T}_\varepsilon(g)$ is a starlike region with respect to any fixed point $p \in \bar{X}$ of g for general g, and with respect to any point on the axis of g if g is loxodromic. As a corollary to Lemma 2.3.4, one obtains,

COROLLARY 3.3.2. *For every isometry $g \in \mathrm{Isom}(X)$, the set $\mathcal{T}_\varepsilon(g)$ is δ-quasiconvex.*

Given $x \in X$ and a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, let $\mathcal{F}_\varepsilon(x) = \{\gamma \in \Gamma \mid d(x, \gamma x) \le \varepsilon\}$ denote the set of isometries in Γ which move x a distance at most ε. Let $\Gamma_\varepsilon(x)$ denote the subgroup generated by $\mathcal{F}_\varepsilon(x)$. We use $\varepsilon(n, \kappa)$ to denote the Margulis constant of X. Then, by the Margulis Lemma, $\Gamma_\varepsilon(x)$ is virtually nilpotent whenever $0 < \varepsilon \le \varepsilon(n, \kappa)$. More precisely,

PROPOSITION 3.3.3. *[**3**, Theorem 9.5] Given $0 < \varepsilon \le \varepsilon(n, \kappa)$ and $x \in X$, the group N generated by the set $\{\gamma \in \Gamma_\varepsilon(x) \mid n_\gamma(x) \le 0.49\}$ is a nilpotent subgroup of $\Gamma_\varepsilon(x)$ of a uniformly bounded index (where the bound depends only on κ and n). Moreover, each coset $\gamma N \subset \Gamma_\varepsilon(x)$ can be represented by an element γ of word length $\le m(n, \kappa)$ in the generating set $\mathcal{F}_\varepsilon(x)$ of $\Gamma_\varepsilon(x)$. Here $m(n, \kappa)$ is a constant depending only on κ and n.*

REMARK 3.3.4. $\Gamma_\varepsilon(x)$ is always finitely generated.

We will use the following important property of nilpotent groups in Section 4.2:

THEOREM 3.3.5. *[**21, 30**] Let G be a nilpotent group. The set of all finite order elements of G forms a characteristic subgroup of G. This subgroup is called the* torsion subgroup *of G and denoted by $Tor(G)$.*

Given $0 < \varepsilon \le \varepsilon(n, \kappa)$ and a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, define the set

$$T_\varepsilon(\Gamma) = \{p \in X \mid \Gamma_\varepsilon(p) \text{ is infinite}\}.$$

Below we establish some properties of $T_\varepsilon(\Gamma)$ where $\Gamma < \mathrm{Isom}(X)$ are discrete subgroups.

Applying Lemma 2.2.1 to the subset $Z = G \cdot x \subset X$ we obtain:

LEMMA 3.3.6. *[**29**] Suppose that $G = \langle g \rangle$ is a (discrete) infinite cyclic subgroup and $x \notin \mathrm{int}(T_\varepsilon(G))$, i.e. $d(x, g^i(x)) \geq \varepsilon$ for all $i \neq 0$. Then for every D there exists i,*

$$0 < i \leq N(\varepsilon, n, \kappa, D) := 1 + \frac{C_n e^{\kappa(n-1)\varepsilon/2}}{\kappa^n V(\varepsilon/2, n)} e^{\kappa(n-1)D}$$

such that $d(x, g^i x) \geq D$.

LEMMA 3.3.7. *[**29**] Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup and $\varepsilon > 0$. For any $z \in T_{\varepsilon/3}(G)$, we have $B(z, \varepsilon/3) \subseteq T_\varepsilon(G)$.*

PROOF. The set $\mathcal{F}_{\varepsilon/3}(z) = \{\gamma \in G | d(z, \gamma(z)) \leq \varepsilon/3\}$ generates an infinite subgroup of G since $z \in T_{\varepsilon/3}(G)$. For any element $\gamma \in \mathcal{F}_{\varepsilon/3}(z)$ and $z' \in B(z, \varepsilon/3)$, we have

$$d(z', \gamma(z')) \leq d(z, z') + d(z, \gamma(z)) + d(\gamma(z), \gamma(z')) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, $\mathcal{F}_\varepsilon(z') = \{\gamma \in G | d(z', \gamma(z')) \leq \varepsilon\}$ also generates an infinite subgroup. Thus $z' \in T_\varepsilon(G_i)$ and $B(z, \varepsilon/3) \subseteq T_\varepsilon(G)$.

□

PROPOSITION 3.3.8. *[**12**, Proposition 3.5.2] Suppose $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$, and $\varepsilon > 0$. Then $T_\varepsilon(G) \cup \{p\}$ is starlike about p, i.e. for each $x \in \bar{X} \setminus \{p\}$, the intersection $xp \cap T_\varepsilon(G)$ is a ray asymptotic to p.*

COROLLARY 3.3.9. *Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$. For every $\varepsilon > 0$, $T_\varepsilon(G)$ is a δ-quasiconvex subset of X.*

PROOF. By Proposition 3.3.8, $T_\varepsilon(G) \cup \{p\}$ is starlike about p. Every starlike set is δ-quasiconvex, [**12**, Corollary 1.1.6]. Thus $T_\varepsilon(G)$ is δ-quasiconvex for every discrete parabolic subgroup $G < \mathrm{Isom}(X)$. □

REMARK 3.3.10. According to Proposition 2.3.7, there exists $r = \mathfrak{r}_\kappa(\delta) \in [0, \infty)$ such that $\mathrm{Hull}(T_\varepsilon(G)) \subseteq \bar{N}_r(T_\varepsilon(G))$ for any $\varepsilon > 0$.

LEMMA 3.3.11. *[**29**] If $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$, then $\partial_\infty T_\varepsilon(G) = \{p\}$.*

PROOF. By Lemma 2.1.12(2), for any $p' \in \partial_\infty X \setminus \{p\}$, both $p'p \cap T_\varepsilon(G)$ and $X \cap (p'p \setminus T_\varepsilon(G))$ are nonempty [**12**, Proposition 3.5.2]. If $p' \in \partial_\infty T_\varepsilon(G)$, there exists a sequence of points $(x_i) \subseteq T_\varepsilon(G)$ which converges to p'. By Proposition 3.3.8, $x_i p \subseteq T_\varepsilon(G)$. Since $T_\varepsilon(G)$ is closed in X, then $p'p \subseteq T_\varepsilon(G)$, which is a contradiction.

□

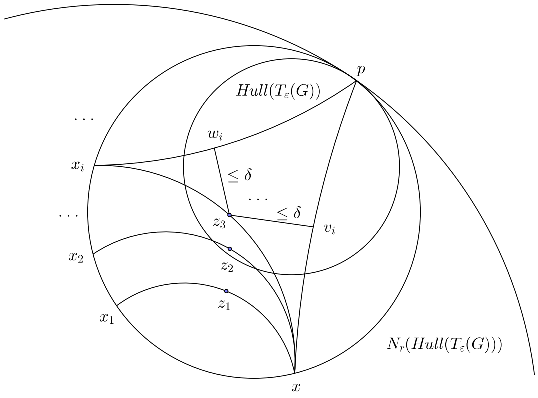

FIGURE 3.2

PROPOSITION 3.3.12. *[**29**] Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$. Given $r > 0$ and $x \in X$ with $d(x, \mathrm{Hull}(T_\varepsilon(G))) = r$, if (x_i) is a sequence of points on the boundary of $\bar{N}_r(\mathrm{Hull}(T_\varepsilon(G)))$ and $d(x, x_i) \to \infty$, then there exists $z_i \in xx_i$ such that the sequence (z_i) converges to p and for every $\varepsilon > 0$, $z_i \in \bar{N}_\delta(T_\varepsilon(G))$ for all sufficiently large i.*

PROOF. By the δ-hyperbolicity of X, there exists a point $z_i \in xx_i$ such that $d(z_i, px) \leq \delta$ and $d(z_i, px_i) \leq \delta$. Let $w_i \in px_i$ and $v_i \in px$ be the points closest to z_i, see Figure 3.2. Then $d(z_i, w_i) \leq \delta$, $d(z_i, v_i) \leq \delta$ and, hence, $d(w_i, v_i) \leq 2\delta$.

According to Lemma 3.3.11, the sequence (x_i) converges to the point p. Hence, any sequence of points on $x_i p$ converges to p as well; in particular, (w_i) converges to p. As $d(w_i, z_i) \leq \delta$, we also obtain

$$\lim_{i\to\infty} z_i = p.$$

Since $d(z_i, v_i) \leq \delta$, it suffices to show that $v_i \in T_\varepsilon(G)$ for all sufficiently large i. This follows from the fact that $d(x, v_i) \to \infty$ and that $xp \cap T_\varepsilon(G)$ is a geodesic ray asymptotic to p.

$\square$

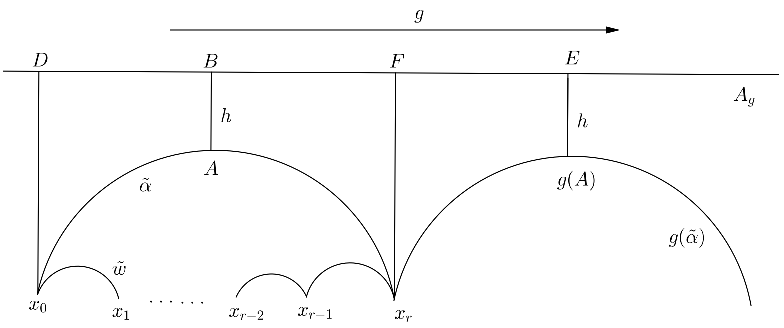

FIGURE 3.3

PROPOSITION 3.3.13. *[**29**] Let $\langle g\rangle < \mathrm{Isom}(X)$ be the cyclic group generated by a loxodromic isometry g. Let γ denote the simple closed geodesic $A_g/\langle g\rangle$ in $M = X/\langle g\rangle$. If $w \subseteq M$ is a piecewise-geodesic loop freely homotopic to γ which consists of r geodesic segments, then*

$$d(w, \gamma \cup (Mar(g,\epsilon)/\langle g\rangle)) \leq \cosh^{-1}(\sqrt{2})\lceil \log_2 r\rceil + \sinh^{-1}(2/\epsilon).$$

PROOF. Let $x \in w$ be one of the vertices. Connect this point to itself by a geodesic segment α in M which is homotopic to w (rel $\{x\}$). The loop $w * \alpha^{-1}$ lifts to a polygonal loop $\beta \subseteq X$ with consecutive vertices $x_0, x_1, \cdots, x_r$ such that the geodesic segment $\tilde{\alpha} := x_0x_r$ covers α. Let $\tilde{w}$ denote the union of edges of β distinct from $\tilde{\alpha}$. By Proposition 2.3.5, $\tilde{\alpha}$ is contained in the λ-neighborhood of the piecewise geodesic path $\tilde{w}$ where $\lambda = \cosh^{-1}(\sqrt{2})\lceil\log_2 r\rceil$. It follows that $\alpha \subseteq \bar{N}_\lambda(w)$.

Suppose that $Mar(g,\epsilon) \neq \emptyset$. It is closed and convex. Let $h = d(\tilde{\alpha}, Mar(g,\varepsilon))$. Choose points $A \in \tilde{\alpha}, B \in Mar(g,\epsilon)$ such that $d(A,B) = h$ realizes the minimal distance between $\tilde{\alpha}$ and $Mar(g,\epsilon)$. Let $F = \mathrm{Proj}_{Mar(g,\epsilon)}(x_r)$. Then we obtain a quadrilateral $[ABFx_r]$ with $\angle ABF =$

$\angle BFx_r = \angle BAx_r \geq \pi/2$. By Corollary 2.1.8,

$$d(B,F) \leq \sinh(d(B,F)) \leq 1/\sinh(h).$$

Take the point $D \in Mar(g,\epsilon)$ which is closest to x_0. By a similar argument, we have $d(B,D) \leq 1/\sinh(h)$. Thus, $d(F,D) \leq 2/\sinh(h)$. The projection $Proj_{A_g}$ is $\langle g\rangle$-equivariant, thus F, D are identified by the isometry g. Hence

$$\epsilon = d(D, g(D)) = d(D,F) \leq 2/\sinh(h)$$

and $h \leq \sinh^{-1}(2/\epsilon)$.

If $Mar(g,\epsilon) = \emptyset$, then the translation length $l(g) \geq \epsilon$. Let $h = d(\tilde{\alpha}, A_g)$. Replacing $Mar(g,\epsilon)$ by A_g, we use a similar argument to obtain that

$$d(A_g, \tilde{\alpha}) \leq \sinh^{-1}(2/\epsilon).$$

Hence,

$$d(w, \gamma \cup (Mar(g,\epsilon)/\langle g\rangle)) \leq \cosh^{-1}(\sqrt{2})\lceil \log_2 r\rceil + \sinh^{-1}(2/\epsilon).$$

□

COROLLARY 3.3.14. *[**29**] Under the conditions in Proposition 3.3.13, if the translation length of g satisfies that $l(g) \geq \epsilon > 0$, then γ is contained in the C-neighborhood of the loop w where $C = \cosh^{-1}(\sqrt{2})\lceil \log_2 r\rceil + \sinh^{-1}(2/\epsilon) + 2\delta$.*

PROOF. We use the same notations as in proof of Proposition 3.3.13. Let $E \in A_g$ be the nearest point to $g(A)$ as in Figure 3.3. Then $\pi(BE)$ in $M = X/\langle g\rangle$ is the geodesic loop γ where π is the covering projection. By δ-hyperbolicity of X, BE is within the $(h+2\delta)$-neighborhood of the lifts of α as in Figure 3.3. Thus γ is within the $(\sinh^{-1}(2/\epsilon) + 2\delta)$-neighborhood of α. Since α is contained in the $(\cosh^{-1}(\sqrt{2})\lceil \log_2 r\rceil)$-neighborhood of w, the loop γ is contained in the $(\cosh^{-1}(\sqrt{2})\lceil \log_2 r\rceil + \sinh^{-1}(2/\epsilon) + 2\delta)$-neighborhood of w.

□

Given $0 < \varepsilon \le \varepsilon(n, \kappa)$ and a discrete subgroup Γ, the set $T_\varepsilon(\Gamma)$ is a disjoint union of the subsets of the form $T_\varepsilon(G)$, where G ranges over all maximal infinite elementary subgroups of Γ, [**12**, Proposition 3.5.5]. If $G < \Gamma$ is a maximal parabolic subgroup, $T_\varepsilon(G)$ is precisely invariant and $\mathrm{Stab}_\Gamma(T_\varepsilon(G)) = G$, [**12**, Corollary 3.5.6]. In this case, by abuse of notation, we regard $T_\varepsilon(G)/G$ as a subset of M, and call it a *Margulis cusp.* Similarly, if $G < \Gamma$ is a maximal loxodromic subgroup, $T_\varepsilon(G)/G$ is called a *Margulis tube.*

For the quotient orbifold $M = X/\Gamma$, set

$$\mathrm{thin}_\varepsilon(M) = T_\varepsilon(\Gamma)/\Gamma.$$

This closed subset is the *thin part*[1] of the quotient orbifold M. It is a disjoint union of its connected components, and each such component has the form $T_\varepsilon(G)/G$, where G ranges over all maximal infinite elementary subgroups of Γ.

The closure of the complement $M \setminus \mathrm{thin}_\varepsilon(M)$ is the *thick part* of M, denoted by $\mathrm{thick}_\varepsilon(M)$. Let $\mathrm{cusp}_\varepsilon(M)$ denote the union of all Margulis cusps of M; it is called the *cuspidal* part of M. The closure of the complement $M \setminus \mathrm{cusp}_\varepsilon(M)$ is denoted by $\mathrm{noncusp}_\varepsilon(M)$; it is called the *noncuspidal* part of M. Observe that $\mathrm{cusp}_\varepsilon(M) \subseteq \mathrm{thin}_\varepsilon(M)$ and $\mathrm{thick}_\varepsilon(M) \subseteq \mathrm{noncusp}_\varepsilon(M)$. If M is a manifold (i.e., Γ is torsion-free), the ε-thin part is also the collection of all points $x \in M$ where the injectivity radius of M at x is no greater than $\varepsilon/2$.

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[1] more precisely, ε-thin part

CHAPTER 4

Tits alternative

4.1. Quasi-geodesics

In this section, X is a Hadamard manifold of sectional curvature ≤ -1. We will prove that certain concatenations of geodesics in X are uniform quasigeodesics, therefore, according to the Morse Lemma, are uniformly close to geodesics.

DEFINITION 4.1.1. A map $q : I \to X$ defined on an interval $I \subset \mathbb{R}$ is called a (λ, α)-quasigeodesic (for $\lambda \geq 1$ and $\alpha \geq 0$) if

$$\lambda^{-1}|s-t| - \alpha \leq d(q(s), q(t)) \leq \lambda|s-t| + \alpha$$

for all $s, t \in I$.

PROPOSITION 4.1.2. [**29**][Piecewise-geodesic paths with long edges] *Define the function*

$$L(\theta) = 2\cosh^{-1}\left(\frac{2}{\sin(\theta/2)}\right) + 1.$$

Suppose that $\gamma = \gamma_1 * \cdots * \gamma_n \subseteq \bar{X}$ *is a piecewise geodesic path*[1] *from* x *to* y *where each* γ_i *is a geodesic of length* $\geq L = L(\theta)$ *and the angles between adjacent arcs* γ_i *and* γ_{i+1} *are* $\geq \theta > 0$. *Then* γ *is a* $(2L, 4L+1)$*-quasigeodesic.*

PROOF. Recall that $\mathrm{Bis}(x_i, x_{i+1})$ denotes the perpendicular bisector of $\gamma_i = x_i x_{i+1}$ where $x_1 = x$ and $x_{n+1} = y$. We claim that the consecutive bisectors are at least unit distance apart:

$$d(\mathrm{Bis}(x_i, x_{i-1}), \mathrm{Bis}(x_i, x_{i+1})) \geq 1. \tag{4.1}$$

If the closures in $\bar{X}$ of the bisectors $\mathrm{Bis}(x_i, x_{i+1})$ and $\mathrm{Bis}(x_{i+1}, x_{i+2})$ intersect each other, then we have a quadrilateral $[ABCD]$ with $\angle DAB = \angle DCB = \pi/2$ as in Figure 4.1, where $B \in \bar{X}$.

---

[1] parameterized by its arc-length

Connecting D, B by a geodesic segment (or a ray), we get two right triangles $[ADB]$ and $[BCD]$, and one of the angles $\angle ADB, \angle CDB$ is $\geq \theta/2$. Without loss of generality, we can assume that $\angle ADB \geq \theta/2$. By Corollary 2.1.5 and Remark 2.1.6, $\cosh(d(A,D)) \sin \angle ADB \leq 1$. However, we know that

$$\cosh(d(A,D)) \sin(\angle ADB) \geq \cosh(L/2) \sin(\theta/2) > 1,$$

which is a contradiction. Thus, the closures of $\mathrm{Bis}(x_i, x_{i+1})$ and $\mathrm{Bis}(x_{i+1}, x_{i+2})$ are disjoint.

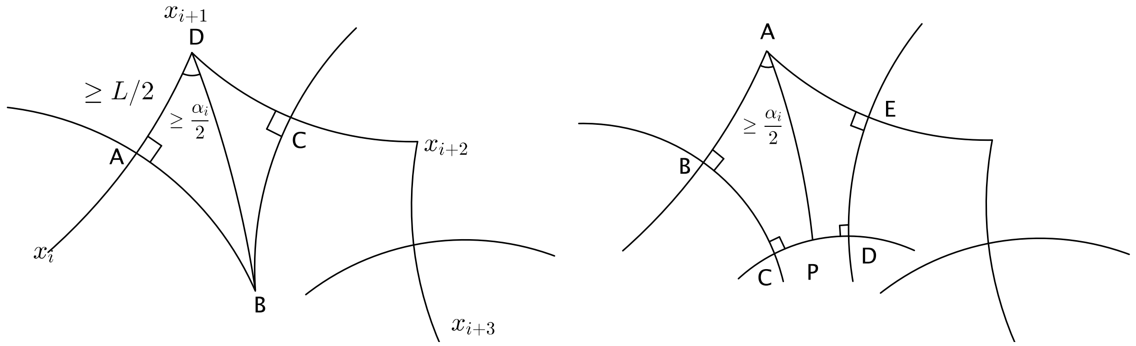

FIGURE 4.1

Let $C \in \mathrm{Bis}(x_i, x_{i+1}), D \in \mathrm{Bis}(x_{i+1}, x_{i+2})$ denote points (not necessarily unique) such that $d(C,D)$ is the minimal distance between these perpendicular bisectors. Since $CB \subset \mathrm{Bis}(x_i, x_{i+1})$, $DE \subset \mathrm{Bis}(x_{i+1}, x_{i+2})$, it follows that the segment CD is orthogonal to both CB and DE. The segment CD lies on a unique (up to reparameterization) bi-infinite geodesic $\xi\eta$. Then $A \in \mathscr{N}_P(\xi, \eta)$ for some point $P \in \xi\eta$. We claim that $P \in CD$. Otherwise, we obtain a triangle in X with two right angles, which is a contradiction. Hence, the geodesic $AP \subseteq \mathscr{N}_P(C, D)$ and AP is orthogonal to CD as in Figure 4.1. We get two quadrilaterals $[ABCP]$ and $[APDE]$. Without loss of generality, assume that $\angle BAP \geq \theta/2$. By Corollary 2.1.8,

$$\cosh(d(C,D)) \geq \cosh(d(C,P)) \geq \cosh(L/2) \sin(\theta/2) > 2 > \cosh(1).$$

Hence, $d(C,D) > 1$. This implies the inequality (4.1).

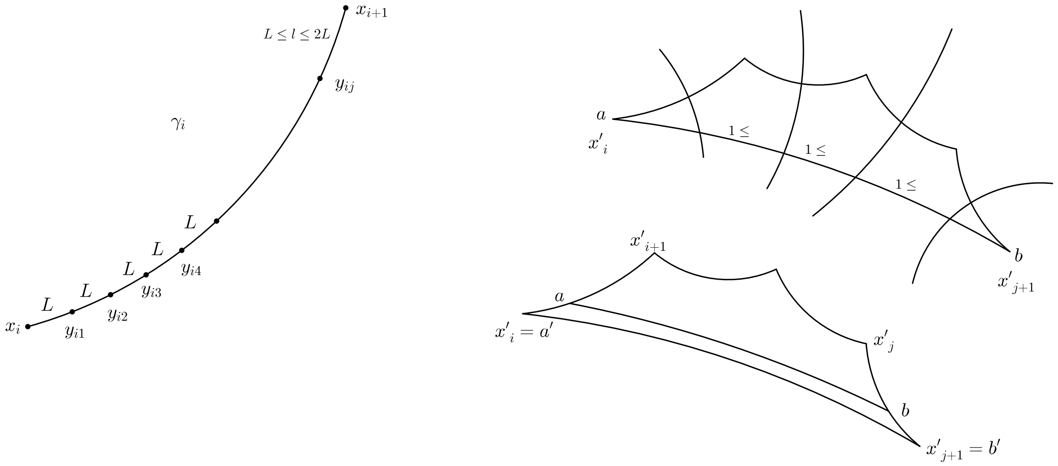

Figure 4.2

We now prove that the path γ is quasigeodesic. For each i, if $d(x_i, x_{i+1}) \geq 2L$, take the point $y_{i1} \in \gamma_i$ such that $d(x_i, y_{i1}) = L$. If $L \leq d(y_{i1}, x_{i+1}) < 2L$, we stop. Otherwise, take the point $y_{i2} \in \gamma_i$ such that $d(y_{i1}, y_{i2}) = L$. If $d(y_{i2}, x_{i+1}) \geq 2L$, we continue the process until we get y_{ij} such that $L \leq d(y_{ij}, x_{i+1}) < 2L$. Thus we get a new partition of the piecewise geodesic path γ:

$$\gamma = \gamma'_1 * \cdots * \gamma'_{n'}$$

such that for each i, $L \leq length(\gamma'_i) < 2L$, and consecutive geodesic arcs γ'_i and γ'_{i+1} meet either at the angle π or, at least, at the angle $\geq \theta$. See Figure 4.2.

In order to prove that γ is (λ, ϵ)-quasigeodesic, with $\lambda \geq 1$ and $\epsilon \geq 0$, we need to verify the inequality

$$\frac{1}{\lambda} length(\gamma|_{[t_a, t_b]}) - \epsilon \leq d(a, b) \leq \lambda \cdot length(\gamma|_{[t_a, t_b]}) + \epsilon$$

for all pairs of points $a, b \in \gamma$, where $\gamma(t_a) = a$ and $\gamma(t_b) = b$. The upper bound (for arbitrary $\lambda \geq 1$ and $\epsilon \geq 0$) follows from the triangle inequality and we only need to establish the lower bound.

The main case to consider is when a, b are both terminal endpoints of some geodesic pieces γ'_i, γ'_j of γ; see Figure 4.2. The bisectors of the geodesic segments of γ divide ab into several pieces, and, by (4.1), each piece has length ≥ 1. At the same time, each arc γ'_k of γ has length $< 2L$.

Thus, $d(a,b) \geq |j-i|$, while

$$2L|j-i| + 2L > \mathit{length}(\gamma|_{[t_a,t_b]}).$$

We obtain:

$$d(a,b) \geq \frac{1}{2L}\mathit{length}(\gamma|_{[t_a,t_b]}) - 1.$$

Lastly, general points $a \in \gamma_i', b \in \gamma_j'$ are within distance $< 2L$ from the terminal endpoints a', b' of these segments. Hence,

$$\begin{aligned} d(a,b) \geq d(a',b') - 4L \geq \frac{1}{2L}\mathit{length}(\gamma|_{[t_{a'},t_{b'}]}) - 1 - 4L &\geq \frac{1}{2L}\mathit{length}(\gamma|_{[t_a,t_b]}) - 1 - 4L \\ &= \frac{1}{2L}\mathit{length}(\gamma|_{[t_a,t_b]}) - (4L+1). \end{aligned}$$

Therefore, γ is a $(2L, 4L+1)$-quasigeodesic. $\square$

Proposition 4.1.3. [**29**][Piecewise-geodesic paths with long and short edges] *Define the function*

$$L(\theta,\varepsilon) = 2\cosh^{-1}\left(\frac{e^2+1}{2\sin(\alpha/2)}\right) + 1$$

where $\alpha = \min\{\theta, \pi/2 - \arcsin(1/\cosh\varepsilon)\}$.

Suppose that $\gamma = \gamma_1 * \cdots * \gamma_n \subseteq \bar{X}$ is a piecewise geodesic path from x to y such that:

1. *Each geodesic arc γ_j has length either at least $\varepsilon > 0$ or at least $L = L(\theta,\varepsilon)$.*
2. *If γ_j has length $< L$, then the adjacent geodesic arcs γ_{j-1} and γ_{j+1} have lengths at least L and γ_j meets γ_{j-1} and γ_{j+1} at angles $\geq \pi/2$.*
3. *Other adjacent geodesic arcs meet at an angle $\geq \theta$.*

Then γ is a (2L, 4L+3)-quasigeodesic.

Proof. We call an arc γ_j *long* if its length is $\geq L$ and *short* otherwise. Notice that γ contains no consecutive short arcs. Unlike the proof of Proposition 4.1.2, we cannot claim that the bisectors of consecutive arcs of γ are unit distance apart (or even disjoint). Observe, however, that by the same proof as in Proposition 4.1.2, the bisectors of every consecutive pair γ_j, γ_{j+1} of *long* arcs are at least unit distance apart.

Consider, therefore, short arcs. Suppose that $\gamma_j = x_j x_{j+1}$ is a short arc. Then $\gamma_{j-1} = x_{j-1}x_j$ and $\gamma_{j+1} = x_{j+1}x_{j+2}$ are long arcs. Consider the geodesic $x_{j-1}x_{j+1}$ and the triangle $[x_{j-1}x_j x_{j+1}]$. By Proposition 2.1.4, $d(x_{j-1}, x_{j+1}) \geq d(x_{j-1}, x_j) \geq L$ and by Corollary 2.1.5,

$$\cosh\varepsilon \sin\angle x_j x_{j+1} x_{j-1} \leq \cosh(d(x_j, x_{j+1})) \sin\angle x_j x_{j+1} x_{j-1} \leq 1.$$

Hence,

$$\angle x_j x_{j+1} x_{j-1} \leq \arcsin\left(\frac{1}{\cosh\varepsilon}\right),$$

and

$$\angle x_{j-1} x_{j+1} x_{j+2} \geq \frac{\pi}{2} - \arcsin\left(\frac{1}{\cosh\varepsilon}\right).$$

By a similar argument to the one of Proposition 4.1.2, the bisectors of the arcs $x_{j-1}x_{j+1}$ and $x_{j+1}x_{j+2}$ are at least distance 2 apart, see Figure 4.3.

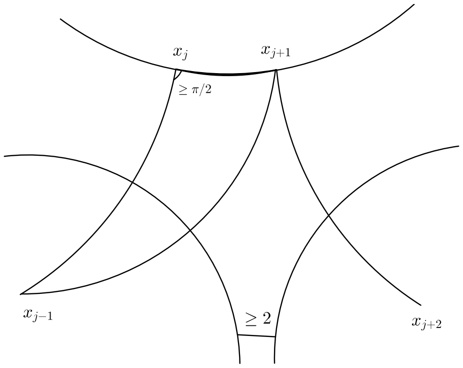

FIGURE 4.3

We now prove that γ is a $(2L, 4L+3)$-quasigeodesic. By the same argument as in Proposition 4.1.2, we can assume that all long arcs of γ are shorter than $2L$ (and short arcs, are, of course, shorter than L).

As in the proof of Proposition 4.1.2, we first suppose that points $a = \gamma(t_a), b = \gamma(t_b)$ in γ are terminal points of arcs γ_i, γ_j, $i < j$. Consider bisectors of $x_{k-1}x_{k+1}$ for short arcs γ_k in $\gamma\mid_{[t_a,t_b]}$ and bisectors of the remaining long arcs except γ_{k-1}. They divide ab into several segments, each of which has length at least 2. By adding these lengths together, we obtain the inequality

$$d(a,b) \geq j - i - 2,$$

while

$$2(j-i+1)L \geq length(\gamma \mid_{[t_a,t_b]}).$$

Putting these inequalities together, we obtain

$$d(a,b) \geq \frac{1}{2L} length(\gamma \mid_{[t_a,t_b]}) - 3.$$

Lastly, for general points a, b in γ, choosing a', b' as in the proof of Proposition 4.1.2, we get:

$$\begin{aligned} d(a,b) \geq d(a',b') - 4L \geq \frac{1}{2L} length(\gamma|_{[t_{a'},t_{b'}]}) - 4L - 3 \geq \frac{1}{2L} length(\gamma|_{[t_a,t_b]}) - 4L - 3 \\ = \frac{1}{2L} length(\gamma|_{[t_a,t_b]}) - (4L+3). \end{aligned}$$

Thus, γ is a $(2L, 4L+3)$-quasigeodesic. ☐

REMARK 4.1.4. By the Morse Lemma, the Hausdorff distance between the quasigeodesic path γ and xy is at most $C = C(L)$, [**21**, Lemma 9.38, Lemma 9.80].

4.2. Loxodromic products

In this section, we construct a loxodromic element with uniformly bounded word length in $\langle f, g\rangle$ where f, g are two parabolic isometries generating a discrete nonelementary subgroup of $\mathrm{Isom}(X)$. This is used in the proof of the generalized Bonahon's theorem, see Section 5.2. To deal with the case of general discrete subgroups, possibly containing elliptic elements, we also need to extend this result to pairs of elliptic isometries g_1, g_2.

We first consider discrete subgroups generated by parabolic isometries. Our goal is to prove Theorem 4.2.5. For the proof of this theorem we will need several technical results.

LEMMA 4.2.1. *[**33**, Theorem Σ_m] Let $F = \{A_1, A_2, \cdots, A_m\}$ be a family of open subsets of an n-dimensional topological space X. If for every subfamily F' of size j where $1 \leq j \leq n+2$, the intersection $\cap F'$ is nonempty and contractible, then the intersection $\cap F$ is nonempty.*

Proof. This lemma is a special case of the topological Helly theorem [**33**]. Here we give another proof of the lemma. Suppose k is the smallest integer such that there exists a subfamily $F' = \{A_{i(1)}, A_{i(2)}, \cdots, A_{i(k)}\}$ of size k with empty intersection $\cap F' = \emptyset$. By the assumption, $k \geq n+3$. Then

$$U := \bigcup_{1\leq j\leq k} A_{i(j)}$$

is homotopy equivalent to the nerve $N(F')$ [**26**, Corollary 4G.3], which, in turn, is homotopy equivalent to S^{k-2}. Then $H_{k-2}(S^{k-2}) \cong H_{k-2}(U) \cong \mathbb{Z}$, which is a contradiction since $k-2 \geq n+1$ and X has dimension n.

□

Proposition 4.2.2. *[**29**] Let X be a δ-hyperbolic n-dimensional Hadamard space. Suppose that $B_1, \cdots, B_k$ are convex subsets of X such that $B_i \cap B_j \neq \emptyset$ for all i and j. Then there is a point $x \in X$ such that $d(x, B_i) \leq n\delta$ for all $i = 1, ..., k$.*

Proof. For $k = 1, 2$, the lemma is clearly true.

We first claim that for each $3 \leq k \leq n+2$, there exists a point $x \in X$ such that $d(x, B_i) \leq (k-2)\delta$. We prove the claim by induction on k. When $k = 3$, pick points $x_{ij} \in B_i \cap B_j$, $i \neq j$. Then $x_{ij}x_{il} \subset B_i$ for all i, j, l. Since X is δ-hyperbolic, there exists a point $x \in X$ within distance $\leq \delta$ from all three sides of the geodesic triangle $[x_{12}x_{23}x_{31}]$. Hence,

$$d(x, B_i) \leq \delta, i = 1, 2, 3$$

as well.

Assume that the claim holds for $k-1$. Set $B_i' = \bar{N}_\delta(B_i)$ and $C_i = B_i' \cap B_1$ where $i \in \{2, 3, \cdots, k\}$. By the convexity of the distance function on X, each B_i' is still convex in X and, hence, is a Hadamard space. Furthermore, each B_i' is again δ-hyperbolic.

We claim that $C_i \cap C_j \neq \emptyset$ for all $i, j \in \{2, 3, \cdots, k\}$. By the nonemptyness assumption, there exist points $x_{1i} \in B_1 \cap B_i \neq \emptyset, x_{1j} \in B_1 \cap B_j \neq \emptyset$ and $x_{ij} \in B_i \cap B_j \neq \emptyset$. By δ-hyperbolicity of X, there exists a point $y \in x_{1i}x_{1j}$ such that $d(y, x_{1i}x_{ij}) \leq \delta$, $d(y, x_{2j}x_{ij}) \leq \delta$.

Therefore, $y \in B_1 \cap \bar{N}_\delta(B_i) \cap \bar{N}_\delta(B_j) = C_i \cap C_j$. By the induction hypothesis, there exists a point $x' \in X$ such that $d(x', C_i) \leq (k-3)\delta$ for each $i \in \{2, 3, \cdots, k\}$. Thus,

$$d(x', B_i) \leq (k-2)\delta, i \in \{1, 2, \cdots, k\}$$

as required.

For $k > n+2$, set $U_i = \bar{N}_{n\delta}(B_i)$. Then by the claim, we know that for any subfamily of $\{U_i\}$ of size j where $1 \leq j \leq n+2$, its intersection is nonempty and the intersection is contractible since it is convex. By Lemma 4.2.1, the intersection of the family $\{U_i\}$ is also nonempty. Let x be a point in this intersection. Then $d(x, B_i) \leq n\delta$ for all $i \in \{1, 2, \cdots, k\}$.

$\square$

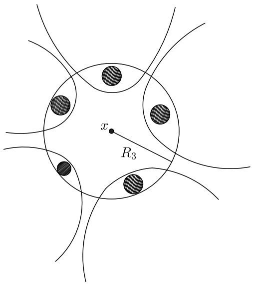

FIGURE 4.4

PROPOSITION 4.2.3. *[**29**] There exists a function $\mathfrak{k} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{N}$ with the following property. Let $g_1, g_2, \cdots, g_k$ be parabolic elements in a discrete subgroup $\Gamma < \mathrm{Isom}(X)$. For each g_i let $G_i < \Gamma$ be the unique maximal parabolic subgroup containing g_i, i.e. $G_i = \mathrm{stab}_\Gamma(p_i)$, where $p_i \in \partial_\infty X$ is the fixed point of g_i. Suppose that*

$$T_\varepsilon(G_i) \cap T_\varepsilon(G_j) = \emptyset$$

for all $i \neq j$. Then, whenever $k \geq \mathfrak{k}(D, \varepsilon)$, there exists a pair of indices i, j with

$$d(T_\varepsilon(G_i), T_\varepsilon(G_j)) > D.$$

PROOF. For each i, $\mathrm{Hull}(T_\varepsilon(G_i))$ is convex and by Remark 3.3.10, $\mathrm{Hull}(T_\varepsilon(G_i)) \subseteq \bar{N}_r(T_\varepsilon(G_i))$, for some uniform constant $r = r_\kappa(\delta)$. Suppose that $g_1, g_2, \cdots, g_k$ and D are such that for all i and

j,

$$d(T_\varepsilon(G_i), T_\varepsilon(G_j)) \leq D.$$

Then $d(\mathrm{Hull}(T_\varepsilon(G_i)), \mathrm{Hull}(T_\varepsilon(G_j))) \leq D$. Our goal is to get a uniform upper bound on k. Consider the $D/2$-neighborhoods $\bar{N}_{D/2}(\mathrm{Hull}(T_\varepsilon(G_i)))$. They are convex in X and have nonempty pairwise intersections. Thus, by Proposition 4.2.2, there is a point $x \in X$ such that

$$d(x, T_\varepsilon(G_i)) \leq R_1 := n\delta + \frac{D}{2} + r, i = 1, ..., k.$$

Then

$$T_\varepsilon(G_i) \cap B(x, R_1) \neq \emptyset, i = 1, ..., k.$$

Next, we claim that there exists $R_2 \geq 0$, depending only on ε, such that

$$T_\varepsilon(G_i) \subseteq \bar{N}_{R_2}(T_{\varepsilon/3}(G_i)).$$

Choose any point $y \in T_\varepsilon(G_i)$ and let $\rho_i : [0, \infty) \to X$ be the ray yp_i. By Lemma 2.1.12, there exists $R = R(\varepsilon)$ such that

$$d(\rho_i(t), g(\rho_i(t))) \leq Re^{-t}$$

whenever $g \in G_i$ is a parabolic (or elliptic) isometry such that

$$d(y, g(y)) \leq \varepsilon.$$

Let $t = \max\{\ln(3R/\varepsilon), 0\}$. Then $d(\rho_i(t), g(\rho_i(t))) \leq \varepsilon/3$ and, therefore,

$$T_\varepsilon(G_i) \subseteq \bar{N}_t(T_{\varepsilon/3}(G_i))$$

for all i. Let $R_2 = t$. By the argument above, $B(x, R_1 + R_2) \cap T_{\varepsilon/3}(G_i) \neq \emptyset$ for all i. Assume that $z_i \in B(x, R_1 + R_2) \cap T_{\varepsilon/3}(G_i)$. Then $B(z_i, \varepsilon/3) \subseteq B(x, R_3)$ where $R_3 = R_1 + R_2 + \varepsilon/3$. By Lemma 3.3.7, $B(z_i, \varepsilon/3) \subseteq B(x, R_3) \cap T_\varepsilon(G_i)$. Since $T_\varepsilon(G_i)$ and $T_\varepsilon(G_j)$ are disjoint for all $i \neq j$, the metric balls $B(z_i, \varepsilon/3)$ and $B(z_j, \varepsilon/3)$ are also disjoint. Recall that $V(r, n)$ denotes the volume of the r-ball in $\mathbb{H}^n$. Then Lemma 2.2.1 implies that for every

$$k \geq \mathfrak{k}(D, \varepsilon) := \frac{C_n e^{\kappa(n-1)R_3}}{V(\varepsilon/3, n)} + 1,$$

there exist i, j, $1 \leq i, j \leq k$, such that $d(T_\varepsilon(G_i), T_\varepsilon(G_j)) > D$. $\square$

PROPOSITION 4.2.4. *[**29**] Suppose that g_1, g_2 are parabolic isometries of X. There exists a constant L which only depends on ε such that if $d(Mar(g_1, \varepsilon), Mar(g_2, \varepsilon)) > L$, then $h = g_2 g_1$ is loxodromic.*

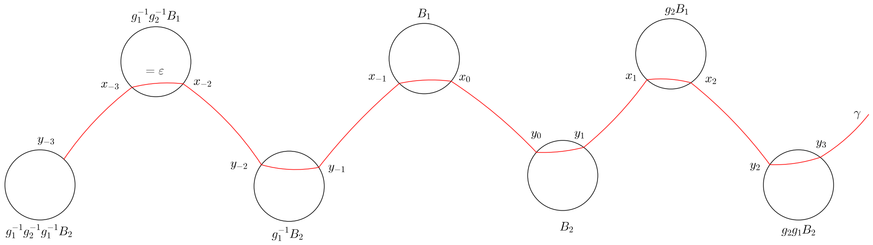

FIGURE 4.5

PROOF. Let $B_i = Mar(g_i, \varepsilon)$, so $d(B_1, B_2) > L$. Consider the orbits of B_1 and B_2 under the action of the cyclic group generated by $g_2 g_1$ as in Figure 4.5. Let $x_0 \in B_1, y_0 \in B_2$ denote points such that $d(x_0, y_0)$ minimizes the distance function between points of B_1 and B_2. For positive integers $m > 0$, we let

$$x_{2m-1} = (g_2g_1)^{m-1} g_2(x_0), \quad x_{2m} = (g_2g_1)^m (x_0)$$

and

$$y_{2m-1} = (g_2g_1)^{m-1} g_2(y_0), \quad y_{2m} = (g_2g_1)^m (y_0).$$

Similarly, for negative integers $m < 0$, we let

$$x_{2m+1} = (g_2g_1)^{m+1} g_1^{-1}(x_0), \quad x_{2m} = (g_2g_1)^m (x_0)$$

and

$$y_{2m+1} = (g_2g_1)^{m+1} g_1^{-1}(y_0), \quad y_{2m} = (g_2g_1)^m (y_0).$$

We construct a sequence of piecewise geodesic paths $\{\gamma_m\}$ where

$$\gamma_m = x_{-2m}y_{-2m} * y_{-2m}y_{-2m+1} \cdots * x_0y_0 * y_0y_1 * y_1x_1 \cdots * x_{2m}y_{2m}$$

for positive integers m. Observe that $d(x_i, y_i) = d(B_1, B_2) > L$ and $d(x_{2i-1}, x_{2i}) = \varepsilon$, $d(y_{2i}, y_{2i+1}) = \varepsilon$ for any integer i. By convexity of B_1, B_2, the angle between any adjacent geodesic arcs in γ_m is at least $\pi/2$. Let γ denote the limit of the sequence (γ_m). By Proposition 4.1.3, there exists a constant $L > 0$ such that the piecewise geodesic path $\gamma : \mathbb{R} \to X$ is unbounded and is a uniform quasigeodesic invariant under the action of h. By the Morse Lemma [**21**, Lemma 9.38, Lemma 9.80], the Hausdorff distance between γ and the complete geodesic which connects the endpoints of γ is bounded by a uniformly constant C. Thus, $g_2 g_1$ fixes the endpoints of γ and acts on the complete geodesic as a translation. We conclude that $g_2 g_1$ is loxodromic. □

THEOREM 4.2.5. *[**29**] Suppose that g_1, g_2 are two parabolic elements with different fixed points. Then there exists a word $w \in \langle g_1, g_2 \rangle$ such that $|w| \leq 4\mathfrak{k}(L, \varepsilon) + 2$ and w is loxodromic where $|w|$ denotes the length of the word and $\mathfrak{k}(L, \varepsilon)$ is the function in Proposition 4.2.3, $0 < \varepsilon \leq \varepsilon(n, \kappa)$ and L is the constant in Proposition 4.2.4.*

PROOF. Let $p_i \in \partial_\infty X$ denote the fixed point of the parabolic isometry g_i, $i = 1, 2$.

Assume that every element in $\langle g_1, g_2 \rangle$ of word length at most $2k(L, \varepsilon) + 1$ is parabolic (otherwise, there exists a loxodromic element $w \in \langle g_1, g_2 \rangle$ of word-length $\leq 4\mathfrak{k}(L, \varepsilon) + 2$).

Consider the parabolic elements $g_2^i g_1 g_2^{-i} \in \langle g_1, g_2 \rangle$, $0 \leq i \leq \mathfrak{k}(L, \varepsilon)$. The fixed point (in $\partial_\infty X$) of each $g_2^i g_1 g_2^{-i}$ is $g_2^i(p_1)$. We claim that the points $g_2^i(p_1)$ and $g_2^j(p_1)$ are distinct for $i \neq j$. If not, $g_2^i(p_1) = g_2^j(p_1)$ for some $i > j$. Then $g_2^{i-j}(p_1) = p_1$, and, thus, g_2^{i-j} has two distinct fixed points p_1 and p_2. This is a contradiction since any parabolic element has only one fixed point. Thus, $g_2^i g_1 g_2^{-i}$ are parabolic elements with distinct fixed points for all $0 \leq i \leq k(L, \varepsilon)$. Since $0 < \varepsilon \leq \varepsilon(n, \kappa)$, $T_\varepsilon(\langle g_2^i g_1 g_2^{-i} \rangle), T_\varepsilon(\langle g_2^j g_1 g_2^{-j} \rangle)$ are disjoint for any pair of indices i, j [**12**]. By Proposition 4.2.3, there exist $0 \leq i, j \leq \mathfrak{k}(L, \varepsilon)$ such that

$$d(Mar(g_2^i g_1 g_2^{-i}, \varepsilon), Mar(g_2^j g_1 g_2^{-j}, \varepsilon)) > L.$$

By Proposition 4.2.4, the element $g_2^j g_1 g_2^{i-j} g_1 g_2^{-i} \in \langle g_1, g_2 \rangle$ is loxodromic, and its word length is $\leq 4\mathfrak{k}(L, \varepsilon) + 2$. Thus we can find a word $w \in \langle g_1, g_2 \rangle$ such that $|w| \leq 4k(L, \varepsilon) + 2$ and w is loxodromic. □

Remark 4.2.6. According to Lemma 3.3.6, for every parabolic isometry $g \in \mathrm{Isom}(X)$ and $x \notin T_\varepsilon(\langle g\rangle)$, there exists $i \in (0, N(\varepsilon, n, \kappa, L)]$ such that $d(x, g^i(x)) > L$. Therefore, using an argument similar to the one in the proof of Proposition 4.2.4, we conclude that one of the products $g_1^{k_1} g_2^{k_2}$ is loxodromic, where $k_1, k_2 > 0$ are uniformly bounded from above. This provides an alternative proof of the existence of loxodromic elements of uniformly bounded word length. We are grateful to the referee for suggesting this alternative argument.

We now consider discrete subgroups generated by elliptic elements. In this setting, we will prove that every infinite discrete elementary subgroup $\Gamma < \mathrm{Isom}(X)$ contains an infinite order element of uniformly bounded word-length (Lemma 4.2.7 and Proposition 4.2.8).

Lemma 4.2.7. *Suppose that the set* $T = \{g_1, g_2, \cdots, g_m\} \subset \mathrm{Isom}(X)$ *consists of elliptic elements, and the group* $\langle T\rangle$ *is an elementary loxodromic group. Then there is a pair of indices* $1 \le i, j \le m$ *such that* $g_i g_j$ *is loxodromic.*

Proof. Let l denote the geodesic preserved setwise by $\langle T\rangle$. We claim that there exists g_i which swaps the endpoints of l. Otherwise, l is fixed pointwise by $\langle T\rangle$, and $\langle T\rangle$ is a finite elementary subgroup of $\mathrm{Isom}(X)$ which is a contradiction. Since $g_i(l) = l$, there exists $x \in l$ such that $g_i(x) = x$. By the same argument as in Lemma 3.2.2, there exists g_j such that $g_j(x) \ne x$, and $g_i g_j$ is loxodromic. □

For discrete parabolic elementary subgroups generated by elliptic isometries, we have the following result.

Proposition 4.2.8. *[**29**] Given* $x \in X, 0 < \varepsilon \le \varepsilon(n, \kappa)$ *and a discrete subgroup* $\Gamma < \mathrm{Isom}(X)$, *suppose that the set* $\mathcal{F}_\varepsilon(x) \subset \Gamma$ *consists of elliptic elements and the group* $\Gamma_\varepsilon(x) < \Gamma$ *generated by this set is a parabolic elementary subgroup. Then there is a parabolic element* $g \in \Gamma_\varepsilon(x)$ *of word length in* $\mathcal{F}_\varepsilon(x)$ *uniformly bounded by a constant* $C(n, \kappa)$.

Proof. Let N be the subgroup of $\Gamma_\varepsilon(x)$ generated by the set $\{\gamma \in \Gamma_\varepsilon(x) \mid n_\gamma(x) \le 0.49\}$. By Proposition 3.3.3, N is a nilpotent subgroup of $\Gamma_\varepsilon(x) = s_1 N \cup s_2 N \cdots \cup s_I N$ where the index I is uniformly bounded and each s_i has uniformly bounded word length $\le m(n, \kappa)$ with respect to the generating set $\mathcal{F}_\varepsilon(x)$ of $\Gamma_\varepsilon(x)$.

Let $F = F_S$ denote the free group on $S = \mathcal{F}_\varepsilon(x)$. Consider the projection map $\pi : F \to \Gamma_\varepsilon(x)$, and the preimage $\pi^{-1}(N) < F$. Let T denote a left Schreier transversal for $\pi^{-1}(N)$ in F (i.e a transverse for $\pi^{-1}(N)$ in F so that every initial segment of an element of T itself belongs to T). By the construction, every element $t \in T$ in the Schreier transversal has the minimal word length among all the elements in $t\pi^{-1}(N)$. Then the word length of t is also bounded by $m(n, \kappa)$ since $t\pi^{-1}(N) = s_i\pi^{-1}(N)$ for some i. By the Reidemeister-Schreier Theorem, $\pi^{-1}(N)$ is generated by the set

$$\mathrm{Y} = \{t\gamma_i s \mid t, s \in T, \gamma_i \in \mathcal{F}_\varepsilon(x), \text{ and } s\pi^{-1}(N) = t\gamma_i\pi^{-1}(N)\}.$$

Since the word length of elements in a Schreier transversal is not greater than $m(n, \kappa)$, then the word length of elements in the generating set Y is not greater than $2m(n, \kappa) + 1$.

Next, we claim that there exists a parabolic element in $\pi(\mathrm{Y})$. If not, then all the elements in $\pi(\mathrm{Y})$ are elliptic. By Theorem 3.3.5, all the torsion elements in N form a subgroup of N. Hence all elements in $N = \langle \pi(\mathrm{Y}) \rangle$ are elliptic. By Lemma 3.2.2, N is finite, which contradicts our assumption that $\Gamma_\varepsilon(x)$ is infinite. Therefore, there exists a parabolic element in $\pi(Y)$ whose word length is $\leq 2m(n, \kappa) + 1$. We let $C(n, \kappa) = 2m(n, \kappa) + 1$.

□

Remark 4.2.9. The virtually nilpotent group $\Gamma_\varepsilon(x)$ is uniformly finitely generated by at most $S(n, \kappa)$ isometries α satisfying $d(x, \alpha(x)) \leq \varepsilon$, [**3**, Lemma 9.4]. Let F be the free group on the set A consisting of such elements α. Since the number of subgroups of F with a given finite index is uniformly bounded, and each subgroup has a finite free generating set it follows that $\pi^{-1}(N)$ has a generating set where each element has word length (with respect to A) uniformly bounded by some constant $C(n, \kappa)$. Hence there is a generating set of N where the word length of each element is uniformly bounded by $C(n, \kappa)$. Similarly, there exists a parabolic element g in this generating set of word length bounded by $C(n, \kappa)$ in elements α. This argument provides an alternative proof of the existence of a parabolic isometry of uniformly bounded word length in $\Gamma_\varepsilon(x)$.

The methods of the proof of the above results are insufficient for treating nonelementary discrete subgroups generated by elliptic elements. After proving our results we learned about the recent paper by Breuillard and Fujiwara which can handle this case. Their theorem also implies Theorem

4.2.5. We decided to keep the proof of our theorem since it presents independent interest and is used in the proof of quantitative version of Tits alternative, see Section 4.3 and Section 4.4.

Given a finite subset A of isometries of a metric space X, we let A^m denote the subset of $\mathrm{Isom}(X)$ consisting of products of $\leq m$ elements of A. Furthermore, define

$$L(A) = \inf_{x \in X} \max_{g \in A} d(x, gx).$$

If X is a Hadamard space then $L(A)$ satisfies the inequality

$$L(A^m) \geq \frac{\sqrt{m}}{2} L(AA^{-1}),$$

see [**14**, Proposition 3.6]. If, in addition, X is an n-dimensional Riemannian manifold of sectional curvature bounded below by $-\kappa^2$, and the subgroup $\langle A \rangle < \mathrm{Isom}(X)$ is discrete and nonelementary, then $L(A) > \varepsilon(n, \kappa)$, the Margulis constant of X. We will need the following result proven in [**14**, Theorem 13.1]:

THEOREM 4.2.10 (Breuillard and Fujiwara). *There exists an absolute constant $C > 0$ such that for every δ-hyperbolic space X and every subset $A \subset \mathrm{Isom}(X)$ generating a nonelementary subgroup Γ one of the following holds:*

(i) $L(A) \leq C\delta$.

(ii) If $m > C$ then Γ contains a loxodromic element of word-length $\leq m$.

This theorem implies:

COROLLARY 4.2.11 (Breuillard and Fujiwara). *There exists a function $N = N(n, \kappa)$ satisfying the following. Suppose that X is a negatively curved Hadamard manifold whose sectional curvature belongs to the interval $[-\kappa^2, -1]$. Then for any subset $A = A^{-1} \subset \mathrm{Isom}(X)$ generating a discrete nonelementary subgroup $\Gamma < \mathrm{Isom}(X)$, there exists a loxodromic element of word-length $\leq N$.*

PROOF. By the Margulis lemma, $L(A) > \varepsilon(n, \kappa) = \mu$. Moreover, $\delta = \cosh^{-1}(\sqrt{2})$ and, as noted above,

$$L(A^k) \geq \frac{\sqrt{k}}{2} L(A) \geq \frac{\sqrt{k}}{2} \mu.$$

Therefore, by Theorem 4.2.10, for

$$m = N(n,\kappa) := \left\lceil (C+1)\left(\frac{2C\delta}{\mu}\right)^2 \right\rceil$$

the set A^m contains a loxodromic element. ☐

4.3. Ping-pong

This section is a generalization of Proposition 4.2.4, and it is used to prove the quantitative version of the Tits alternative, see Section 4.4.

PROPOSITION 4.3.1. *[**19**] Suppose that $g, h \in \mathrm{Isom}(X)$ are parabolic/hyperbolic elements with equal translation lengths $\leq \varepsilon/10$, and*

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g)), \mathrm{Hull}(\mathcal{T}_\varepsilon(f))) \geq L, \tag{4.2}$$

where $L = L(\varepsilon/10)$ is as in Proposition 4.1.3. Then $\Phi := \langle g, h\rangle < \mathrm{Isom}(X)$ is a free subgroup of rank 2.

PROOF. To simplify notation, for a non-elliptic element $f \in \mathrm{Isom}(X)$, we denote $\mathrm{Hull}(\mathcal{T}_\varepsilon(f))$ by $\widehat{\mathcal{T}}_\varepsilon(f)$.

Using Lemma 3.3.1, we obtain

$$d(\widehat{\mathcal{T}}_\varepsilon(g), g^k\widehat{\mathcal{T}}_\varepsilon(h)) = d(\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)) \geq L, k \in \mathbb{Z}.$$

Our goal is to show that every nonempty word $w(g,h)$ represents a nontrivial element of $\mathrm{Isom}(X)$. It suffices to consider cyclically reduced words w which are not powers of g, h.

We will consider a cyclically reduced word

$$w = w(g,h) = g^{m_k}h^{m_{k-1}}g^{m_{k-2}}h^{m_{k-3}}\dots g^{m_2}h^{m_1}, \tag{4.3}$$

words with the last letter g are treated by relabeling. Since w is cyclically reduced and is not a power of g, h, the number k is ≥ 2 and all of the m_i's in this equation are nonzero.

For each $N \geq 1$ we define the *r-suffix* of w^N as the following subword of w^N:

$$w_r = \begin{cases} g^{m_r} h^{m_{r-1}} g^{m_{r-2}} h^{m_{r-3}} \dots g^{m_2} h^{m_1}, & r \text{ even} \\ h^{m_r} g^{m_{r-1}} h^{m_{r-2}} \dots g^{m_2} h^{m_1}, & r \text{ odd} \end{cases} \tag{4.4}$$

where, of course, $m_i \equiv m_j$ modulo N. Since w is reduced, each w_r is reduced as well.

We will prove that the map

$$\mathbb{Z} \to X, \quad N \mapsto w^N x$$

is a quasiisometric embedding. This will imply that $w(g, h)$ is nontrivial. In fact, this will also show that $w(g, h)$ is hyperbolic, see Theorem 3.2.

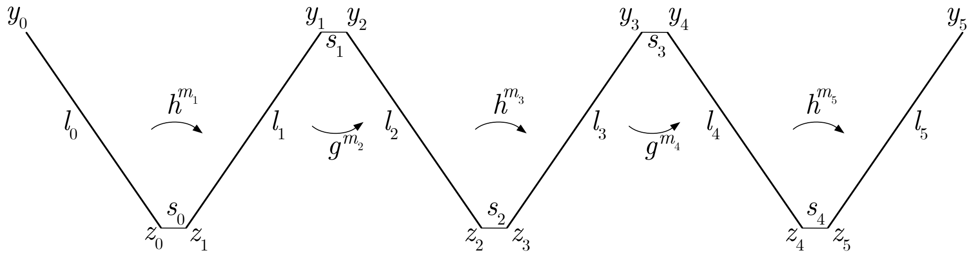

FIGURE 4.6

Let $l = yz$ be the unique shortest geodesic segment connecting points in $\widehat{\mathcal{T}}_\varepsilon(g)$ and $\widehat{\mathcal{T}}_\varepsilon(h)$, where $y \in \widehat{\mathcal{T}}_\varepsilon(g)$ and $z \in \widehat{\mathcal{T}}_\varepsilon(h)$. For $r \geq 0$, we denote $w_r l$, $w_r y$ and $w_r z$ by l_r, y_r and z_r, respectively. In particular, $y_0 = y, z_0 = z$ and $l_0 = l$.

Since l is the shortest segment between $\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)$ and these are convex subsets of X, for every $y' \in \widehat{\mathcal{T}}_\varepsilon(g)$ (resp. $z' \in \widehat{\mathcal{T}}_\varepsilon(h)$),

$$\angle y'yz \geq \pi/2, \quad (\text{resp. } \angle yzz' \geq \pi/2). \tag{4.5}$$

Since g and h have equal translation lengths, h is parabolic (resp. hyperbolic) if and only if g is parabolic (resp. hyperbolic). When both of them are hyperbolic, since y and z are not in the interior of $\mathcal{T}_\varepsilon(g)$ and $\mathcal{T}_\varepsilon(h)$, respectively, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$, for all $1 \leq i \leq m_g$, $1 \leq j \leq m_h$. Also, when $i > m_g$, $j > m_h$, it follows from (3.1) and (3.3) that

$$\min\left(d(y, g^i y), d(z, h^j z)\right) \geq \frac{\varepsilon}{10}.$$

Moreover, when both g and h are parabolic, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$, for all $1 \leq i$, $1 \leq j$. Therefore, in the general case,

$$\min\big(d(y, g^i y), d(z, h^j z)\big) \geq \frac{\varepsilon}{10}, \quad \forall i \geq 1, \forall j \geq 1. \tag{4.6}$$

Let s_r be the segment

$$s_r = \begin{cases} y_r y_{r+1}, & \text{when } r \text{ is odd} \\ z_r z_{r+1}, & \text{when } r \text{ is even} \end{cases}.$$

See the arrangement of the points and segments in Figure 4.6.

Let $\tilde{l}_N$ be the concatenation of the segments l_r's and s_r's as shown in Figure 4.6, $0 \leq r \leq kN$. According to (4.6), the length of each segment s_r is at least $\varepsilon/10$, while by the assumption, the length of each l_r is $\geq L = L(\varepsilon/10)$. Moreover, according to (4.5), the angle between any two consecutive segments in $\tilde{l}_N$ is at least $\pi/2$. Using Proposition 4.1.3, we conclude that $\tilde{l}_N$ is a (λ, α)-quasigeodesic.

Consequently,

$$d(w^N x, x) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{kN-1} |s_i| + NkL\right) - \alpha \geq \frac{kL}{\lambda} N - \alpha. \tag{4.7}$$

From this inequality it follows that the map $\mathbb{Z} \to X, N \mapsto w^N x$ is a quasiisometric embedding. □

Remark 4.3.2. In fact, this proof also shows that every nontrivial element of the subgroup $\Phi < \operatorname{Isom}(X)$ is either conjugate to one of the generators or is hyperbolic.

For the next proposition and the subsequent remark, one needs the notion of *convex-cocompact* subgroups of $\operatorname{Isom}(X)$. Convex-cocompact subgroups are geometric finite. Several equivalent definitions of geometric finiteness have been give in the Theorem 1.1.4. More precisely, a discrete isometry subgroup Γ is convex-cocompact if it acts cocompactly on the convex hull of the limit set $\Lambda(\Gamma)$, i.e $\operatorname{Core}(M)$ is compact. Given any finitely generated, discrete subgroup $\Gamma < \operatorname{Isom}(X)$, Γ is convex-cocompact if and and only if any orbit of Γ is a quasiconvex subset of X. For our purpose, it suffices to say that a subgroup Γ in $\operatorname{Isom}(X)$ is *convex-cocompact* if it is finitely generated and for some (equivalently, every) $x \in X$, the orbit map $\Gamma \to \Gamma x \subset X$ is a quasiisometric embedding, where Γ is equipped with a word metric [**20**].

PROPOSITION 4.3.3. *[**19**] Let $g, h \in \operatorname{Isom}(X)$ be hyperbolic isometries satisfying the hypothesis of Proposition 4.3.1. Then the subgroup $\Phi = \langle g, h\rangle < \operatorname{Isom}(X)$ is convex-cocompact.*

PROOF. We equip the free group $\mathbb{F}_2$ on two generators (denoted g, h) with the word metric corresponding to this free generating set. Since g, h are hyperbolic, by (3.1) the lengths of the segments s_r's in the proof of Proposition 4.3.1 are $\geq \tau|m_{r+1}|$, where

$$\tau = l(g) = l(h).$$

Then, for $N = 1$, $r = k$, and a reduced but not necessarily cyclically reduced word w, the inequality (4.7) becomes

$$d(wy, y) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{k-1} |s_i|\right) - \alpha \geq \frac{\tau}{\lambda}|w| - \alpha, \tag{4.8}$$

where $|w| \geq |m_1| + |m_2| + \cdots + |m_k|$ is the (word) length of w. Therefore, the orbit map $\mathbb{F}_2 \to \Phi y \subset X$ is a quasiisometric embedding. $\square$

REMARK 4.3.4. One can also show that if g, h are parabolic then the subgroup Φ is geometrically finite. We will not prove it in this paper since a proof requires further geometric background material on geometrically finite groups.

4.4. Quantitative Tits alternative

In this section, we prove Theorem 1.2.1.

4.4.1. Case 1: Displacement bounded below. In this section we consider discrete torsion-free nonelementary subgroups of $\operatorname{Isom}(X)$ generated by two hyperbolic elements g, h whose translation lengths are equal to $\tau \geq \lambda$. Our goal is to show that in this case the subgroup $\langle g^N, h^N\rangle$ is free of rank 2 provided that N is greater than some constant depending only on the Margulis constant of X and on λ. The strategy is to bound from above how 'long' the axes A_g, A_h of g and h can stay 'close to each other' in terms of the constant λ. Once we get such an estimate, we find a uniform upper bound on N such that Dirichlet domains for $\langle g^N\rangle$, $\langle h^N\rangle$ (based at some points on A_g, A_h) have disjoint complements. This implies that g^N, h^N generate a free subgroup of rank two by a classical ping-pong argument.

Let α, β be complete geodesics in the Hadamard manifold X. These geodesics eventually will be the axes of g and h, hence, we assume that these geodesics do not share ideal end-points. Let x_-x_+ denote the (nearest point) projection of β to α and let y_-y_+ denote the projection of x_-x_+ to β. Let x, y denote the mid-points of x_-x_+ and y_-y_+ respectively.

Then

$$L_\beta := d(y_-, y_+) \le L_\alpha := d(x_-, x_+).$$

Fix some $T \ge 0$, and let $\hat{x}_-\hat{x}_+$ and $\hat{y}_-\hat{y}_+$ denote the subsegments of α and β containing x_-x_+ and y_-y_+ respectively, such that

$$d(x_\pm, \hat{x}_\pm) = T, \quad d(y_\pm, \hat{y}_\pm) = T. \tag{4.9}$$

We let $U_\pm$ and $V_\pm$ denote the 'half-spaces' in X equal to $H(\hat{x}_\pm, x_\pm)$ and $H(\hat{y}_\pm, y_\pm)$ respectively. See Figure 4.7.

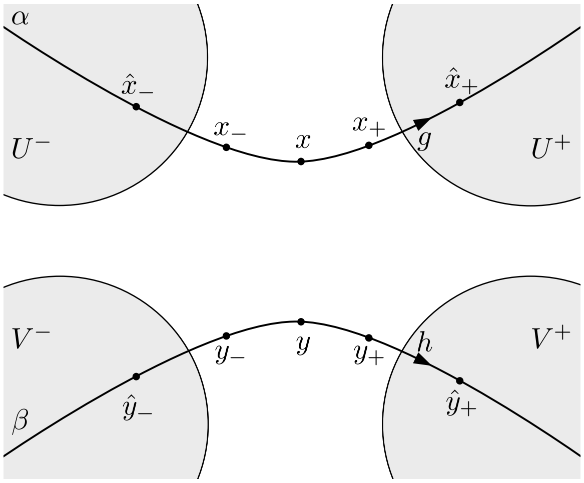

FIGURE 4.7

The following is proven in [**6**, Appendix]:

LEMMA 4.4.1. *If $T \ge 5$ then the sets $U_\pm$, $V_\pm$ are pairwise disjoint.*

Suppose now that g, h are hyperbolic isometries of X with the axes α, β respectively, and equal translation length $\tau(g) = \tau(h) = \tau \ge \lambda > 0$. We let $\Gamma = \langle g, h \rangle < \mathrm{Isom}(X)$ denote the, necessarily nonelementary, torsion-free (but not necessarily discrete), subgroup of isometries of X generated by g and h.

As an application of the above lemma, as in [**6**, Appendix], we obtain:

LEMMA 4.4.2. *If* $N\tau \geq L_\alpha + 5 + 2\delta$ *then the half-spaces* $H(g^{\pm N}x, x)$, $H(h^{\pm N}y, y)$ *are pairwise disjoint.*

PROOF. The inequality

$$N\tau \geq L_\alpha + 5 + 2\delta \geq L_\beta + 5 + 2\delta.$$

implies that the quadruples

$$(x, x_+, \hat{x}_+, g^N(x)), (x, x_-, \hat{x}_-, g^{-N}(x)), (y, y_+, \hat{y}_+, h^N(y)), (y, y_-, \hat{y}_-, h^{-N}(y))$$

satisfy the assumptions of Corollary 2.1 where $\hat{x}_\pm$ and $\hat{y}_\pm$ are given by taking $T = 5$ in (4.9). Therefore, according to this corollary, we have

$$H(g^{\pm N}(x), x) \subset U^\pm, \quad H(h^{\pm N}(y), y) \subset V^\pm.$$

Now, the assertion of the lemma follows from Lemma 4.4.1. □

COROLLARY 4.10. *If*

$$N\tau \geq L_\alpha + 5 + 2\delta \tag{4.11}$$

then the subgroup $\Gamma_N < \Gamma$ *generated by* g^N, h^N *is free with the basis* g^N, h^N.

PROOF. We have

$$g^{\pm N}\left(H(h^{-N}(y), y) \ \cup \ H(h^{+N}(y), y)\right) \subset H(g^{\pm N}x, x)$$

and

$$h^{\pm N}\left(H(g^{-N}(x), x) \ \cup \ H(g^{+N}(x), x)\right) \subset H(h^{\pm N}y, y).$$

Thus, the conditions of the standard ping-pong lemma (see e.g. [**21**, **25**]) are satisfied and, hence, Γ_N is free with the basis g^N, h^N. □

Let $\eta = d(\alpha, \beta)$ denote the minimal distance between α, β and pick some $\eta_0 > 0$ (we will eventually take $\eta_0 = 0.01\varepsilon(n, \kappa)$). Let $\beta_0 = z_-^0 z_+^0 \subset \beta$ be the (possibly empty!) maximal closed subinterval such that the distance from the end-points of β_0 to α is $\le \eta_0$. Thus, $\beta_0 \subset \bar{N}_{\eta_0}(\alpha)$.

REMARK 4.4.3. $\beta_0 = \emptyset$ if and only if $\eta_0 < \eta$.

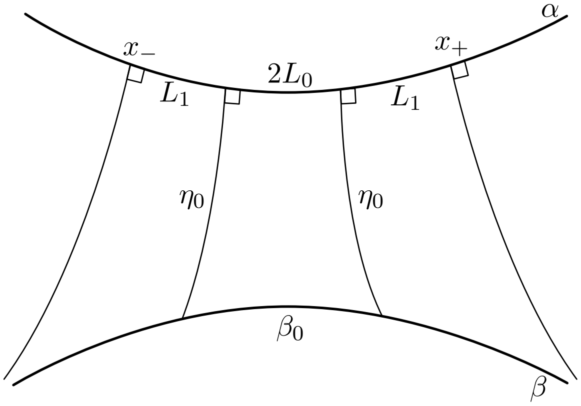

FIGURE 4.8

Let $\alpha_0 = x_-^0 x_+^0$ denote the projection of β_0 to α, let $2L_0$ denote the length of α_0. Hence, the intervals α_0, β_0 are within Hausdorff distance η_0 from each other.

Furthermore, $\angle\beta(-\infty)z_-^0 x_-^0 \ge \pi/2$ and $\angle\beta(-\infty)z_+^0 x_+^0 \ge \pi/2$; see Figure 4.8. Hence, according to [**29**, Corollary 3.7], for

$$L_1 = \sinh^{-1}\left(\frac{1}{\sinh(\eta_0)}\right),$$

we have

$$d(x_-, x_-^0) \le L_1, \quad d(x_+, x_+^0) \le L_1.$$

Thus, the interval $x_- x_+$ breaks into the union of two subintervals of length $\le L_1 = L_1(\eta_0)$ and the interval α_0 of the length $2L_0$. In other words, $L_\alpha = 2(L_0 + L_1)$.

Most of our discussion below deals with the case when the interval β_0 is nonempty.

Our goal is to bound from above L_α in terms of λ, η_0 and the Margulis constant $\varepsilon(n, \kappa)$ of X, provided that $\eta_0 = 0.01\varepsilon(n, \kappa)$ and Γ is discrete.

LEMMA 4.4.4. *Let* $S \subset \Gamma$ *be the subset consisting of elements of word-length* ≤ 4 *with respect to the generating set* g, h. *Let* $P_-P_+ \subset \alpha_0$ *be the middle subinterval of* α_0 *whose length is* $\frac{2}{9}L_0$. *Assume that* $\tau \leq d(P_-, P_+)$. *Then for each* $\gamma \in S$ *the interval* $\gamma(P_-P_+)$ *is contained in the* $3\eta_0$*-neighborhood of* α_0.

PROOF. The proof is a straightforward application of the triangle inequalities taking into account the fact that the Hausdorff distance between α_0 and β_0 is $\leq \eta_0$. ☐

Then, arguing as in the proof of [**27**, Theorem 10.24][2], we obtain that each of the commutators

$$[g^{\pm 1}, h^{\pm 1}], \quad [h^{\pm 1}, g^{\pm 1}]$$

moves each point of P_-P_+ by at most

$$28 \times 3\eta_0 \leq 100\eta_0.$$

Therefore, by applying the Margulis Lemma as in the proof of [**27**, Theorem 10.24], we obtain:

COROLLARY 4.12. *If* Γ *is discrete and* $\eta_0 = 0.01\varepsilon(n, \kappa)$, *then*

$$\tau \geq \frac{2}{9}L_0 = \frac{1}{9}(L_\alpha - 2L_1).$$

COROLLARY 4.13. *If* Γ *is discrete and* $\tau \geq \lambda$, *then the subgroup* $\langle g^N, h^N \rangle = \Gamma_N < \Gamma$ *is free of rank 2 whenever one of the following holds:*

i. Either $L_\alpha \leq 3L_1$ *and*

$$N \geq \frac{5 + 2\delta + 3L_1}{\lambda}.$$

ii. Or $L_\alpha \geq 3L_1$ *and*

$$N \geq 27 + \frac{9(5 + 2\delta)}{L_1}.$$

PROOF. In view of Corollary 4.10, it suffices to ensure that the inequality (4.11) holds.

---

[2] In fact, the argument there is a variation on a proof due to Culler-Shalen-Morgan and Bestvina, Paulin

(i) Suppose first that $L_\alpha \le 3L_1$, hence, $L_\beta \le 3L_1$. Then, in view of the inequality $\tau \ge \lambda > 0$, the inequality (4.11) will follow from

$$N \ge \frac{5 + 2\delta + 3L_1}{\lambda}.$$

(ii) Suppose now that $L_\alpha \ge 3L_1$. The function

$$\frac{9(t + 5 + 2\delta)}{t - 2L_1}$$

attains its maximum on the interval $[3L_1, \infty)$ at $t = 3L_1$. Therefore,

$$\frac{9(L_\alpha + 5 + 2\delta)}{L_\alpha - 2L_1} \le 27 + \frac{9(5 + 2\delta)}{L_1}.$$

Thus, the inequality

$$\tau \ge \frac{L_\alpha - 2L_1}{9}$$

implies that for any

$$N \ge 27 + \frac{9(5 + 2\delta)}{L_1},$$

we have $N\tau \ge L_\alpha + 5 + 2\delta$. $\square$

Consider now the remaining case when for $\eta_0 := \frac{1}{100}\varepsilon(n, \kappa)$, the subinterval β_0 is empty, i.e. $\eta > \eta_0 = \frac{1}{100}\varepsilon(n, \kappa)$. Then, as above, the length L_α of the segment $x_- x_+$ is at most $2L_1$. Therefore, similarly to the case (i) of Corollary 4.13, in order for N to satisfy the inequality (4.11), it suffices to get

$$N \ge \frac{5 + 2\delta + 3L_1}{\lambda}.$$

To conclude:

Theorem 4.14. *[**19**] Suppose that g, h are hyperbolic isometries of X generating a discrete torsion-free nonelementary subgroup, whose translation lengths are equal to some $\tau \ge \lambda > 0$. Let L_1 be such that*

$$\sinh(L_1) \sinh\left(\frac{1}{100}\varepsilon\right) = 1,$$

where $\varepsilon = \varepsilon(n, \kappa)$. Then for every

$$N \geq \max\left(\frac{5 + 2\delta + 3L_1}{\lambda}, 27 + \frac{9(5 + 2\delta)}{L_1}\right) \tag{4.15}$$

the group generated by g^N, h^N is free of rank 2.

We note that proving that (some powers of) g and h generate a free subsemigroup, is easier, see [**6**] and [**14**, section 11].

Corollary 4.4.5. *[**19**] Given g, h as in Theorem 4.14, and any N satisfying (4.15), the free group $\Gamma_N = \langle g^N, h^N \rangle$ is convex-cocompact.*

Proof. Let $\mathcal{U}^{\pm} = H(g^{\pm N}x, x)$ and $\mathcal{V}^{\pm} = H(h^{\pm N}y, y)$. Observe that

$$g^{\pm N}(X \setminus \mathcal{U}^{\mp}) \subset \mathcal{U}^{\pm}$$

and

$$h^{\pm N}(X \setminus \mathcal{V}^{\mp}) \subset \mathcal{V}^{\pm}.$$

We let $\mathfrak{D}_{g^N}, \mathfrak{D}_{h^N}$ denote the closures in $\bar{X}$ of the domains

$$X \setminus (\mathcal{U}^{-} \cup \mathcal{U}^{+}), \quad X \setminus (\mathcal{V}^{-} \cup \mathcal{V}^{+})$$

respectively and set

$$\mathfrak{D} = \mathfrak{D}_{g^N} \cap \mathfrak{D}_{h^N}.$$

It is easy to see (cf. [**32**]) that this intersection is a fundamental domain for the action of Γ_N on the complement $\bar{X} \setminus \Lambda$ to its limit set Λ. Therefore, $(\bar{X} \setminus \Lambda)/\Gamma_N$ is compact. Hence, Γ_N is convex-cocompact (see [**12**]). $\square$

Remark 4.4.6. It is also not hard to see directly that the orbit maps $\Gamma_N \to \Gamma_N x \subset X$ are quasiisometric embeddings by following the proofs in [**29**, section 7] and counting the number of bisectors crossed by geodesics connecting points in Γx.

4.4.2. Case 2: Displacement bounded above. The strategy in this case is to find an element g' conjugate to g (by some uniformly bounded power of f) such that the Margulis regions of g, g' are sufficiently far apart, i.e. are at distance $\geq L$, where L is given by the local-to-global principle for piecewise-geodesic paths in X, see Proposition 4.3.1.

Proposition 4.4.7. *[**19**] There exists a function*

$$\mathfrak{k} : [0, \infty) \times (0, \varepsilon] \to \mathbb{N}$$

for $0 < \varepsilon \leq \varepsilon(n, \kappa)$ with the following property: Let $g_1, \cdots, g_k$ be nonelliptic isometries of the same type (hyperbolic or parabolic) with translation lengths $\leq \varepsilon/10$ and

$$k \geq \mathfrak{k}(L, \varepsilon).$$

Suppose that $\langle g_i, g_j \rangle$ are nonelementary discrete torsion-free subgroup for all $i \neq j$. Then, there exists a pair of indices $i, j \in \{1, \ldots, k\}$, $i \neq j$ such that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))) > L.$$

Proof. If all the isometries g_i are parabolic, then the proposition is established in Proposition 4.2.3. Therefore, we only consider the case when all these isometries are hyperbolic. Our proof follows closely the proof of Proposition 4.2.3.

Since for all $i \neq j$ the subgroup $\langle g_i, g_j \rangle$ is a discrete and nonelementary, and $\varepsilon \leq \varepsilon(n, \kappa)$, we have

$$\mathcal{T}_\varepsilon(g_i) \cap \mathcal{T}_\varepsilon(g_j) = \emptyset.$$

Given $L > 0$, suppose that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))) \leq L, \quad \forall i, j \in \{1, \ldots, k\}$$

Our goal is to get a uniform upper bound of k.

Consider the $L/2$-neighborhoods $\bar{N}_{L/2}(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)))$. They are convex in X, and have nonempty pairwise intersections. Thus, by Proposition 4.2.2, there exists a point $x \in X$ such that

$$d(x, \mathcal{T}_\varepsilon(g_i)) \leq R_1 := n\delta + L/2 + \mathfrak{q}, \quad i = 1, \ldots, k,$$

where δ is the hyperbolicity constant of X and $\mathfrak{q}$ is as in Proposition 2.3.10. Then

$$\mathcal{T}_\varepsilon(g_i) \cap B(x, R_1) \neq \emptyset, \quad i = 1, \ldots, k.$$

For each $i = 1, \ldots, k$ take a point $x_i \in \mathcal{T}_\varepsilon(g_i) \cap B(x, R_1)$ satisfying $d(x_i, g_i^{p_i}(x_i)) = \varepsilon$ for some $0 < p_i \leq m_{g_i}$. Since the translation lengths of the elements $g_i^{p_i}$ are $\leq \varepsilon/10$, by Proposition 3.1.2 there exist points $y_i \in X$ such that

$$d(y_i, g_i^{p_i}(y_i)) = \varepsilon/3, \quad d(x_i, y_i) \leq \mathfrak{r}(\varepsilon).$$

Consider the $\varepsilon/3$-balls $B(y_i, \varepsilon/3)$. Then $B(y_i, \varepsilon/3) \subset \mathcal{T}_\varepsilon(g_i)$ since

$$d(z, g_i^{p_i}(z)) \leq d(z, y_i) + d(y_i, g_i^{p_i}(y_i)) + d(g_i^{p_i}(y_i), g_i^{p_i}(z)) \leq \varepsilon$$

for any point $z \in B(y_i, \varepsilon/3)$. Thus, the balls $B(y_i, \varepsilon/3)$ are pairwise disjoint. Observe that $B(y_i, \varepsilon/3) \subset B(x, R_2)$ where $R_2 = R_1 + \mathfrak{r}(\varepsilon) + \varepsilon/3$.

Let $V(r, n)$ denote the volume of the r-ball in $\mathbb{H}^n$. Then for each i, $\mathrm{Vol}(B(y_i, \varepsilon/3))$ is at least $V(\varepsilon/3, n)$, see [**12**, Proposition 1.1.12]. Moreover, the volume of $B(x, R_2)$ is at most $V(\kappa R_2, n)/\kappa^n$, see [**12**, Proposition 1.2.4]. Let

$$\mathfrak{k}(L, \varepsilon) := \frac{V(\kappa R_2, n)/\kappa^n}{V(\varepsilon/3, n)} + 1.$$

Then $k < \mathfrak{k}(L, \varepsilon)$, because otherwise we would obtain

$$\mathrm{Vol}\left(\bigcup_{i=1}^{k} B(y_i, \varepsilon/3)\right) > \mathrm{Vol}(B(x, R_2)),$$

where the union of the balls on the left side of this inequality is contained in $B(x, R_2)$, which is a contradiction.

Therefore, whenever $k \geq \mathfrak{k}(L, \varepsilon)$, there exist a pair of indices i, j such that

$$d\left(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))\right) > L. \qquad \square$$

Remark 4.4.8. Proposition 4.4.7 also holds for isometries of mixed types (i.e. some g_i's are parabolic and some are hyperbolic). The proof is similar to the one given above.

Theorem 4.4.9. *[**19**] For every nonelementary torsion-free discrete subgroup* $\Gamma = \langle g, h \rangle < \mathrm{Isom}(X)$ *with* g, h *nonelliptic isometries satisfying*

$$\tau(g) \le \varepsilon/10 \le \varepsilon(n, \kappa)/10,$$

there exists i, $1 \le i \le \mathfrak{k}(L(\varepsilon/10), \varepsilon)$, *such that* $\langle g, h^i g h^{-i} \rangle$ *is a free subgroup of rank 2, where* $\mathfrak{k}$ *is the function given by Proposition 4.4.7 and* $L(\varepsilon/10)$ *is the constant in Proposition 4.1.3.*

Proof. Consider isometries $g_i := h^i g h^{-i}$, $i \ge 1$. We first claim that no pair g_i, g_j, $i \ne j$, generates an elementary subgroup of $\mathrm{Isom}(X)$. There are two cases to consider:

(i) Suppose that g is parabolic with the fixed point $p \in \partial_\infty X$. We claim that for all $i \ne j$, $h^i(p) \ne h^j(p)$. Otherwise, $h^{j-i}(p) = p$, and p would be a fixed point of h. But this would imply that Γ is elementary, contradicting our hypothesis.

(ii) The proof in the case when g is hyperbolic is similar. The axis of g_i equals $h^i(A_g)$. If hyperbolic isometries g_i, g_j, $i \ne j$, generate a discrete elementary subgroup of Γ, then they have to share the axis, and we would obtain $h^i(A_g) = h^j(A_g)$. Then $h^{j-i}(A_g) = A_g$. Since h^{j-i} is nonelliptic, it cannot swap the fixed points of g, hence, it fixes both of these points. Therefore, g, h have common axis, contradicting the hypothesis that Γ is nonelementary.

All the isometries g_i have equal translation lengths $\le \varepsilon/10$. Therefore, by Proposition 4.4.7, there exists a pair of natural numbers $i, j \le \mathfrak{k}(L(\varepsilon/10), \varepsilon)$ such that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^i g h^{-i})), \mathrm{Hull}(\mathcal{T}_\varepsilon(h^j g h^{-j}))) > L(\varepsilon/10)$$

where $\mathfrak{k}(L(\varepsilon/10), \varepsilon)$ is the function as in Proposition 4.4.7. It follows that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^{j-i} g h^{i-j})), \mathrm{Hull}(\mathcal{T}_\varepsilon(g))) > L(\varepsilon/10).$$

Setting $f := h^{j-i} g h^{i-j}$, and applying Proposition 4.3.1 to the isometries f, g, we conclude that the subgroup $\langle f, g \rangle < \Gamma$ is free of rank 2. The word length of f is at most

$$2|j - i| + 1 \le 2\mathfrak{k}(L(\varepsilon/10), \varepsilon) + 1. \qquad \square$$

4.4.3. Conclusion. Now we are in a position to complete the proof of Theorem 1.2.1.

PROOF OF THEOREM 1.2.1. We set $\lambda := \varepsilon/10$, where $\varepsilon = \varepsilon(n, \kappa)$ is the Margulis constant. Let g, h be non-elliptic isometries of X generating a discrete torsion-free nonelementary subgroup of $\mathrm{Isom}(X)$, such that $\tau(g) = \tau(h) = \tau$.

If $\tau \geq \lambda$, then by Theorem 4.14, the subgroup $\Gamma_N < \Gamma$ generated by g^N, h^N is free of rank 2, where

$$N := \left\lceil \max\left(\frac{5 + 2\delta + 3L_1}{\lambda}, 27 + \frac{9(5 + 2\delta)}{L_1}\right)\right\rceil.$$

Here $\delta = \cosh^{-1}(\sqrt{2})$, and

$$L_1 = \sinh^{-1}\left(\frac{1}{\sinh(\varepsilon/100)}\right).$$

If $\tau \leq \lambda$, then by Theorem 4.4.9 there exists $i \in [1, \mathfrak{k}(L(\lambda), \varepsilon)]$ such that $\langle g, h^i g h^{-i}\rangle$ is free of rank 2 where $\mathfrak{k}(L(\lambda), \varepsilon)$ is a constant as in Theorem 4.4.9. $\square$

CHAPTER 5

Geometric finiteness

In this section, we prove Theorem 1.1.5, Corollary 1.1.6 and sharpen the result to ends of the convex core of the oribfold $Y = \text{Core}(M)$ and $\text{noncusp}_\varepsilon(Y)$ where $M = X/\Gamma$.

5.1. Escaping sequences of closed geodesics in negatively curved manifolds

In this section, X is a Hadamard manifold of negative curvature ≤ -1 with the hyperbolicity constant δ, $\Gamma < \text{Isom}(X)$ is a discrete isometry subgroup and $M = X/\Gamma$ is the quotient orbifold. A sequence of subsets $A_i \subset M$ is said to *escape every compact subset of* M if for every compact $K \subset M$, the subset

$$\{i \in \mathbb{N} : A_i \cap K \neq \emptyset\}$$

is finite. Equivalently, for every $x \in M$, $d(x, A_i) \to \infty$ as $i \to \infty$.

Lemma 5.1.1. *[**29**] Suppose that (a_i) is a sequence of closed geodesics in $M = X/\Gamma$ which escapes every compact subset of M and $x \in M$. Then, after passing to a subsequence in (a_i), there exist geodesic arcs b_i connecting a_i, a_{i+1} and orthogonal to these geodesics, such that the sequence (b_i) also escapes every compact subset of M.*

Proof. Consider a sequence of compact subsets $K_n := \bar{B}(x, 7\delta n)$ exhausting M. Without loss of generality, we may assume that $a_i \cap K_n = \emptyset$ for all $i \geq n$.

We first prove the following claim:

Claim. *For each compact subset $K \subset M$ and for each infinite subsequence $(a_i)_{i \in I}, I \subset \mathbb{N}$, there exists a further infinite subsequence, $(a_i)_{i \in J}, J \subset I$, such that for each pair of distinct elements $i, j \in J$, there exists a geodesic arc b_{ij} connecting a_i to a_j and orthogonal to both, which is disjoint from K.*

PROOF. Given two closed geodesics a, a' in M, we consider the set $\pi_1(M, a, a')$ of *relative homotopy classes* of paths in M connecting a and a', where the relative homotopy is defined through paths connecting a to a'.

In each class $[b'] \in \pi_1(M, a, a')$, there exists a continuous path b which is the length minimizer in the class. By minimality of its length, b is a geodesic arc orthogonal to a and a' at its end-points.

For each compact subset $K \subset M$, there exists $m \in \mathbb{N}$ such that for all $i \in I_m := I \cap [m, \infty)$, $a_i \cap K' = \emptyset$ where $K' = \bar{N}_{7\delta}(K)$. For $i \in I_m$ let c_i denote a shortest arc between a_i and K'; this geodesic arc terminates a point $x_i \in K'$. By compactness of K', the sequence $(x_i)_{i \in I_m}$ contains a convergent subsequence, $(x_i)_{i \in J}, J \subset I_m$ and, without loss of generality, we may assume that for all $i, j \in J$, $d(x_i, x_j) \leq \delta$. Let $x_i x_j$ denote a (not necessarily unique) geodesic in M of length $\leq \delta$ connecting x_i to x_j. For each pair of indices $i, j \in J$, consider the concatenation

$$b'_{ij} = c_i * x_i x_j * c_j^{-1},$$

which defines a class $[b'_{ij}] \in \pi_1(M, a_i, a_j)$. Let $b_{ij} \in [b'_{ij}]$ be a length-minimizing geodesic arc in this relative homotopy class. Then b_{ij} is orthogonal to a_i and a_j. By the δ-hyperbolicity of X,

$$b_{ij} \subseteq \bar{N}_{7\delta}(a_i \cup c_i \cup c_j \cup a_j).$$

Hence, $b_{ij} \cap K = \emptyset$ for any pair of distinct indices $i, j \in J$. This proves the claim. $\square$

We now prove the lemma. Assume inductively (by induction on N) that we have constructed an infinite subset $S_N \subset \mathbb{N}$ such that:

For the N-th element $i_N \in S_N$, for each $j > i_N, j \in S_N$, there exists a geodesic arc b_j in M connecting a_{i_N} to a_j and orthogonal to both, which is disjoint from K_{N-1}.

Using the claim, we find an infinite subset $S_{N+1} \subset S_N$ which contains the first N elements of S_N, such that for all $s, t > i_N, s, t \in S_{N+1}$, there exists a geodesic $b_{s,t}$ in M connecting a_s to a_t, orthogonal to both and disjoint from K_N.

The intersection

$$S := \bigcap_{N \in \mathbb{N}} S_N$$

equals $\{i_N : N \in \mathbb{N}\}$ and, hence, is infinite. We, therefore, obtain a subsequence $(a_i)_{i\in S}$ such that for all $i, j \in S, i < j$, there exists a geodesic b_{ij} in M connecting a_i to a_j and orthogonal to both, which is disjoint from K_{i-1}. $\square$

REMARK 5.1.2. It is important to pass a subsequence of (a_i), otherwise, the lemma is false. A counter-example is given by a geometrically infinite manifold with two distinct ends E_1 and E_2 where we have a sequence of closed geodesics a_i (escaping every compact subset of M) contained in E_1 for odd i and in E_2 for even i. Then b_i will always intersect a compact subset separating the two ends no matter what b_i we take.

5.2. A generalized Bonahon's theorem

In this section, we use the construction in Section 4.2 to generalize Bonahon's theorem for any discrete subgroup $\Gamma < \mathrm{Isom}(X)$ where X is a negatively pinched Hadamard manifold.

LEMMA 5.2.1. *[**29**] For every* $\tilde{x} \in \mathrm{Hull}(\Lambda(\Gamma))$,

$$\mathrm{hd}(\mathrm{QHull}(\Gamma\tilde{x}), \mathrm{QHull}(\Lambda(\Gamma))) < \infty$$

PROOF. By the assumption that $\tilde{x} \in \mathrm{Hull}(\Lambda(\Gamma))$ and Remark 2.3.8, there exists $r_1 = \mathfrak{r}_\kappa(2\delta) \in [0, \infty)$ such that

$$\mathrm{QHull}(\Gamma\tilde{x}) \subseteq \mathrm{Hull}(\Lambda(\Gamma)) \subseteq \bar{N}_{r_1}(\mathrm{QHull}(\Lambda(\Gamma)))$$

Next, we want to prove that there exists a constant $r_2 \in [0, \infty)$ such that $\mathrm{QHull}(\Lambda(\Gamma)) \subseteq \bar{N}_{r_2}(\mathrm{QHull}(\Gamma\tilde{x}))$.

Pick any point $p \in \mathrm{QHull}(\Lambda(\Gamma))$. Then p lies on some geodesic $\xi\eta$ where $\xi, \eta \in \Lambda(\Gamma)$ are distinct points. Since ξ and η are in the limit set, there exist sequences of elements (f_i) and (g_i) in Γ such that the sequence $(f_i(\tilde{x}))$ converges to ξ and the sequence $(g_i(\tilde{x}))$ converges to η. By Lemma 2.3.11, $p \in \bar{N}_{2\delta}(f_i(\tilde{x})g_i(\tilde{x}))$ for all sufficiently large i. Let $r = \max\{r_1, 2\delta\}$. Thus,

$$\mathrm{hd}(\mathrm{QHull}(\Gamma\tilde{x}), \mathrm{QHull}(\Lambda(\Gamma))) = r < \infty. \quad \square$$

REMARK 5.2.2. Let $\gamma_i = f_i(\tilde{x})g_i(\tilde{x})$. Then there exists a sequence of points $p_i \in \gamma_i$, which converges to p.

If $\Gamma < \mathrm{Isom}(X)$ is geometrically infinite, then

$$\mathrm{Core}(M) \cap \mathrm{noncusp}_{\varepsilon}(M)$$

is noncompact, [**12**]. By Lemma 5.2.1, $(\mathrm{QHull}(\Gamma\tilde{x})/\Gamma) \cap \mathrm{noncusp}_{\varepsilon}(M)$ is unbounded.

We now generalize Bonahon's theorem to geometrically infinite discrete subgroup $\Gamma < \mathrm{Isom}(X)$.

Proof of the implication (1) $\Rightarrow$ (2) **in Theorem 1.1.5:** If there exists a sequence of closed geodesics $\beta_i \subseteq M$ whose lengths tend to 0 as $i \to \infty$, the sequence (β_i) escapes every compact subset of M. From now on, we assume that there exists a constant $\epsilon > 0$ which is a lower bound on the lengths of closed geodesics β in M.

Consider Margulis cusps $T_{\varepsilon}(G)/G$, where $G < \Gamma$ are maximal parabolic subgroups. There exists a constant $r \in [0, \infty), r = \mathfrak{r}_{\kappa}(\delta)$ such that

$$\mathrm{Hull}(T_{\varepsilon}(G)) \subseteq \bar{N}_r(T_{\varepsilon}(G))$$

for every maximal parabolic subgroup G. Let $B(G) = \bar{N}_{2+4\delta}(\mathrm{Hull}(T_{\varepsilon}(G)))$. Let M^o be the union of all subsets $B(G)/\Gamma$ where G ranges over all maximal parabolic subgroups of Γ. Further, we let M^c denote the closure of $\mathrm{Core}(M) \setminus M^o$. Since Γ is geometrically infinite, the noncuspidal part of the convex core,

$$\mathrm{noncusp}_{\varepsilon}(\mathrm{Core}(M) = \mathrm{Core}(M)) \setminus \mathrm{cusp}_{\varepsilon}(M)$$

is unbounded by Theorem 1.1.4. Then M^c is also unbounded since

$$M^o \subseteq \bar{N}_{r+2+4\delta}(\mathrm{cusp}_{\varepsilon}(M)),$$

Fix a point $x \in M^c$ and a point $\tilde{x} \in \pi^{-1}(x) \subset X$. Let

$$C_n = B(x, nR) = \{y \in M^c \mid d(x, y) \leq nR\},$$

where

$$R = r + 2 + 4\delta + m\varepsilon$$

and $m = C(n, \kappa)$ is the constant in Proposition 4.2.8. Let $\boldsymbol{\delta} C_n$ denote the relative boundary

$$\partial C_n \setminus \partial M^c_{cusp}$$

of C_n where

$$M^c_{cusp} = M^o \cap \mathrm{Core}(M).$$

By Lemma 5.2.1 $(\mathrm{QHull}(\Gamma\tilde{x})/\Gamma) \cap M^c$ is unbounded. For every C_n, there exists a sequence of geodesic loops (γ_i) connecting x to itself in $\mathrm{Core}(M)$ such that the Hausdorff distance $\mathrm{hd}(\gamma_i \cap M^c, C_n) \to \infty$ as $i \to \infty$. Let $y_i \in \gamma_i \cap M^c$ be such that $d(y_i, C_n)$ is maximal on $\gamma_i \cap M^c$. We pick a component α_i of $\gamma_i \cap M^c$ in the complement of C_n such that $y_i \in \alpha_i$. Consider the sequence of geodesic arcs (α_i).

After passing to a subsequence in (α_i), one of the following three cases occurs:

Case (a): Each α_i has both endpoints x'_i and x''_i on ∂M^c_{cusp} as in Figure 5.1. By the construction, there exist y'_i and y''_i in the cuspidal part such that $d(x'_i, y'_i) \leq r_1, d(y'_i, y''_i) \leq r_1$ where $r_1 = 2+4\delta+r$. Let $\tilde{y}'_i$ be a lift of y'_i such that $\tilde{y}'_i \in T_\varepsilon(G')$ for some maximal parabolic subgroup $G' < \Gamma$. By the definition, the subgroup $\Gamma_\varepsilon(\tilde{y}'_i)$ generated by the set

$$\mathcal{F}_\varepsilon(\tilde{y}'_i) = \{\gamma \in G' \mid d(\tilde{y}'_i, \gamma(\tilde{y}'_i)) \leq \varepsilon\}$$

is infinite.

We claim that there exists a parabolic element $g' \in \Gamma_\varepsilon(\tilde{y}'_i)$ such that $d(\tilde{y}'_i, g'(\tilde{y}'_i)) \leq m\varepsilon$. Assume that $\mathcal{F}_\varepsilon(\tilde{y}'_i) = \{\gamma_1, \cdots, \gamma_b\}$. If γ_j is parabolic for some $1 \leq j \leq b$, we have $d(\tilde{y}'_i, \gamma_j(\tilde{y}'_i)) \leq \varepsilon$. Now assume that γ_j are elliptic for all $1 \leq j \leq b$. By Proposition 4.2.8, there is a parabolic element $g' \in \Gamma_\varepsilon(\tilde{y}'_i)$ of word length (in the generating set $\mathcal{F}_\varepsilon(\tilde{y}'_i)$) bounded by m. By the triangle inequality, $d(\tilde{y}'_i, g'(\tilde{y}'_i)) \leq m\varepsilon$.

Then we find a nontrivial geodesic loop α'_i contained M^o such that α'_i connects y'_i to itself and has length $l(\alpha'_i) \leq m\varepsilon$. Similarly, there exists a nontrivial geodesic loop α''_i which connects y''_i to itself and has length $l(\alpha''_i) \leq m\varepsilon$. Let

$$w' = x'_i y'_i * \alpha'_i * y'_i x'_i \in \Omega(M, x'_i)$$

and

$$w'' = \alpha_i * x_i''y_i'' * \alpha_i'' * y_i''x_i'' * \alpha_i^{-1} \in \Omega(M, x_i'),$$

where $\Omega(M, x_i')$ denotes the loop space of M. Observe that $w' \cap C_{n-1} = \emptyset$ and $w'' \cap C_{n-1} = \emptyset$.

Let g', g'' denote the elements of $\Gamma = \pi_1(M, x_i')$ represented by w' and w'' respectively. By the construction, g' and g'' are both parabolic. We claim that g' and g'' have different fixed points in $\partial_\infty X$. Otherwise, $g, g'' \in G'$ where $G' < \Gamma$ is some maximal parabolic subgroup. Then $y_i', y_i'' \in T_\varepsilon(G')/\Gamma$ and $x_i', x_i'' \in B(G')/\Gamma$. Since $\mathrm{Hull}(T_\varepsilon(G'))$ is convex, $B(G') = \bar{N}_{2+4\delta}(\mathrm{Hull}(T_\varepsilon(G')))$ is also convex by convexity of the distance function. Thus, $x_i'x_i'' \subseteq B(G')/\Gamma$. However, $x_i'x_i''$ lies outside of $B(G')/\Gamma$ by construction, which is a contradiction.

By Theorem 4.2.5, there exists a loxordomic element $\omega_n \in \langle g', g'' \rangle < \Gamma = \pi_1(M, x_i')$ with the word length uniformly bounded by a constant $K = \mathfrak{k}(\varepsilon, \kappa)$ independent of n. Let w_n be a concatenation of w_i', w_i'' and their reverses which represents ω_n. Then the number of geodesic arcs in w_n is uniformly bounded by $5K$. The piecewise geodesic loop w_n is freely homotopic to a closed geodesic w_n^* in M; hence, by Proposition 3.3.14, w_n^* is contained in some D-neighborhood of the loop w_n where

$$D = \cosh^{-1}(\sqrt{2})\lceil \log_2 5K \rceil + \sinh^{-1}(2/\epsilon) + 2\delta.$$

Thus, $d(x, w_n^*) \geq (n-1)R - D$.

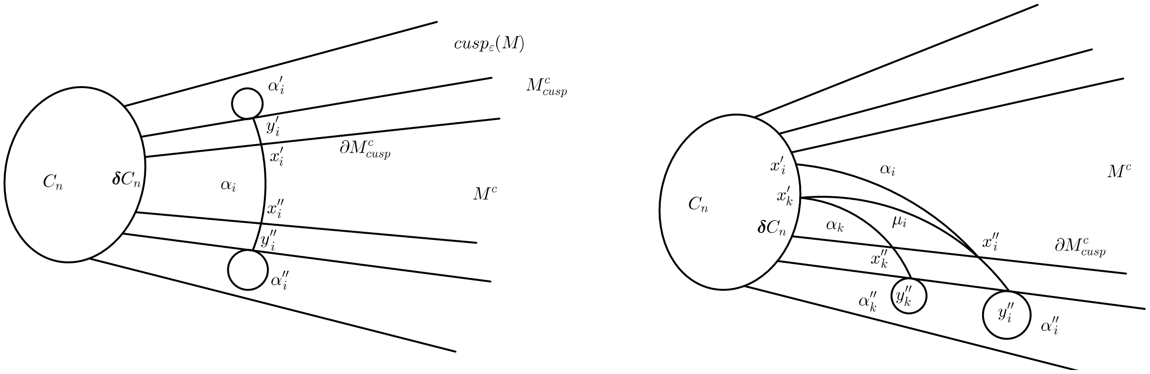

FIGURE 5.1

Case (b): For each i, the geodesic arc α_i connects $x_i' \in \boldsymbol{\delta} C_n$ to $x_i'' \in \partial M^c_{cusp}$, as in Figure 5.1. For each x_i'', there exists a point $y_i'' \in \mathrm{cusp}_\varepsilon(M)$ such that $d(x_i'', y_i'') \leq r_1$ and a short nontrivial geodesic loop α_i'' contained in M^o which connects y_i'' to itself and has length $l(\alpha_i'') \leq m\varepsilon$. Since $\boldsymbol{\delta} C_n$ is compact, after passing to a further subsequence in (α_i), there exists $k \in \mathbb{N}$ such that for all $i \geq k$, $d(x_i', x_k') \leq 1$ and less than the injectivity radius of M at x_k'. Hence, there exists a unique shortest geodesic $x_k'x_i'$ in the manifold M. Let $\mu_i = x_k'x_i''$ denote the geodesic arc homotopic to the concatenation $x_k'x_i' * x_i'x_i''$ rel. $\{x_i', x_i''\}$. Then, by the δ-hyperbolicity of X, the geodesic $\mu_i = x_k'x_i''$ is contained in the $(1+\delta)$-neighborhood of α_i.

Let

$$w_k' = \alpha_k * x_k''y_k'' * \alpha_k'' * y_k''x_k'' * \alpha_k^{-1} \in \Omega(M, x_k')$$

and

$$w_i' = \mu_i * x_i''y_i'' * \alpha_i'' * y_i''x_i'' * (\mu_i)^{-1} \in \Omega(M, x_k')$$

for all $i > k$. By the construction, $w_i' \cap C_{n-1} = \emptyset$ for each $i \geq k$.

Let g_i denote the element of $\Gamma = \pi_1(M, x_k')$ represented by w_i', $i \geq k$. Then each g_i is parabolic. We claim that there exists a pair of indices $i, j \geq k$ such that g_i and g_j have distinct fixed points. Otherwise, assume that all parabolic elements g_i have the same fixed point p. Then $x_i'' \in B(G')/\Gamma$ for any $i \geq k$ where $G' = \mathrm{Stab}_\Gamma(p)$.

Since $\mu_i \cup \alpha_k$ is in the $(1+\delta)$-neighborhood of M^c, by the δ-hyperbolicity of X we have that $x_k''x_i''$ is in $(1+2\delta)$-neighborhood of M^c for every $i > k$. By the definition of M^c, it follows that

$$x_k''x_i'' \cap \bar{N}_\delta(\mathrm{Hull}(T_\varepsilon(G')))/\Gamma = \emptyset.$$

By the construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$. Hence, the length $l(\mu_i) \to \infty$ and the length $l(x_k''x_i'') \to \infty$ as $i \to \infty$. By Lemma 3.3.12, there exists points $z_i \in x_k''x_i''$ such that $z_i \in \bar{N}_\delta(T_\varepsilon(G'))/\Gamma$ for sufficiently large i. Therefore,

$$x_k''x_i'' \cap \bar{N}_\delta(\mathrm{Hull}(T_\varepsilon(G')))/\Gamma \neq \emptyset,$$

which is a contradiction.

We conclude that for some $i, j \geq k$, the parabolic elements g_i, g_j of Γ have distinct fixed points and, hence, generate a nonelementary subgroup of $\mathrm{Isom}(X)$. By Theorem 4.2.5, there exists a loxodromic element $\omega_n \in \langle g_i, g_j \rangle$ with the word length uniformly bounded by a constant K. By the same argument as in Case (a), we obtain a closed geodesic w_n^* (representing the conjugacy class of ω_n) in M such that $d(x, w_n^*) \geq (n-1)R - D$.

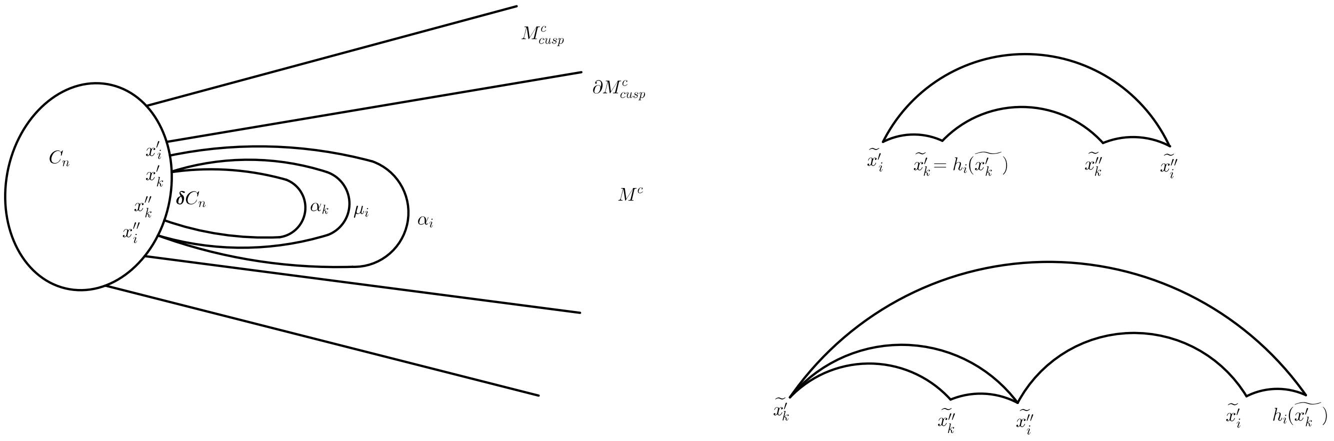

Figure 5.2

Case (c): We assume that for each i, the geodesic arc α_i connects $x'_i \in \boldsymbol{\delta} C_n$ to $x''_i \in \boldsymbol{\delta} C_n$. The argument is similar to the one in Case (b). Since $\boldsymbol{\delta} C_n$ is compact, after passing to a further subsequence in (α_i), there exists $k \in \mathbb{N}$ such that for all $i \geq k$, $d(x'_i, x'_k) \leq 1$, $d(x''_i, x''_k) \leq 1$ and there are unique shortest geodesics $x'_k x'_i$ and $x''_k x''_i$. For each $i > k$ we define a geodesic $\mu_i = x'_k x''_i$ as in Case (b), see Figure 5.2. Then, by the δ-hyperbolicity of X, each μ_i is in the $(\delta+1)$-neighborhood of α_i. Let $v_i = \alpha_k * x''_k x''_i * (\mu_i)^{-1} \in \Omega(M, x'_k)$ for $i > k$. By the construction, $v_i \cap C_{n-1} = \emptyset$.

Let h_i denote the element in $\Gamma = \pi_1(M, x'_k)$ represented by v_i. If h_i is loxodromic for some $i > k$, there exists a closed geodesic w_n^* contained in the D-neighborhood of v_i, cf. Case (a). In this situation, $d(x, w_n^*) \geq (n-1)R - D$.

Assume, therefore, that h_i are not loxodromic for all $i > k$.

We first claim that h_i is not the identity for all sufficiently large i. Let $\widetilde{x'_k}$ be a lift of x'_k in X. Pick points $\widetilde{x''_k}, \widetilde{x''_i}, \widetilde{x'_i}$ and $h_i(\widetilde{x'_k})$ in X such that $\widetilde{x'_k}\widetilde{x''_k}$ is a lift of α_k, $\widetilde{x''_k}\widetilde{x''_i}$ is a lift of $x''_k x''_i$, $\widetilde{x'_i}\widetilde{x''_i}$ is a lift of α_i and $\widetilde{x'_i} h_i(\widetilde{x'_k})$ is a lift of $x'_i x'_k$ as in Figure 5.2. If $h_i = 1$, then $h_i(\widetilde{x'_k}) = \widetilde{x'_k}$ and $d(\widetilde{x'_i}, \widetilde{x''_i}) \leq 2 + d(\widetilde{x'_k}, \widetilde{x''_k})$. By construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$, so $d(\widetilde{x'_i}, \widetilde{x''_i}) \to \infty$. Thus for sufficiently large i, $h_i(\widetilde{x'_k}) \neq \widetilde{x'_k}$.

Assume, therefore, that h_i are not loxodromic and not the identity for all $i > k$. Then h_i could be either parabolic or elliptic for $i > k$.

Claim. *For every k, there exist $i, j > k$ and a loxodromic element in $\langle h_i, h_j \rangle$ whose word length is bounded by a constant independent of k.*

Proof. Suppose there is a subsequence in $(h_i)_{i>k}$ consisting of parabolic elements. For simplicity, we assume that h_i are parabolic for all $i > k'$ where $k' > k$ is a sufficiently large number. We claim that there exists a pair of indices $i, j > k'$ such that h_i and h_j have distinct fixed points in $\partial_\infty X$. Otherwise, all the parabolic elements h_i have the same fixed point p for $i > k'$. By the δ-hyperbolicity of X, $\widetilde{x'_k}h_i(\widetilde{x'_k}) \subseteq \bar{N}_{3\delta+2}(\widetilde{x'_k}\widetilde{x''_k} \cup \widetilde{x''_i}\widetilde{x'_i})$. Since α_k and α_i lie outside of $B(G')/\Gamma$ where $G' = \mathrm{Stab}_\Gamma(p)$, the segment $\widetilde{x'_k}h_i(\widetilde{x'_k})$ lies outside of $\bar{N}_\delta(\mathrm{Hull}(T_\varepsilon(G')))$. Let $r_3 = d(\widetilde{x'_k}, \mathrm{Hull}(T_\varepsilon(G')))$. Then $d(h_i(\widetilde{x'_k}), \mathrm{Hull}(T_\varepsilon(G'))) = r_3$.

By the construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$. Then the length $l(\widetilde{x'_k}h_i(\widetilde{x'_k})) \to \infty$ as well. Observe that the points $\widetilde{x'_k}$ and $h_i(\widetilde{x'_k})$ lie on the boundary of $\bar{N}_{r_3}(\mathrm{Hull}(T_\varepsilon(G)))$ for all $i > k'$. By Lemma 3.3.12, there exist points $\widetilde{z}_i \in \widetilde{x'_k}h_i(\widetilde{x'_k})$ such that $\widetilde{z}_i \in \bar{N}_\delta(T_\varepsilon(G'))$ for sufficiently large i, which is a contradiction. Hence, for some $i > k', j > k'$, parabolic isometries h_i and h_j have distinct fixed points.

By Theorem 4.2.5, there exists a loxodromic element $\omega_n \in \langle h_i, h_j \rangle$ of the word length bounded by a uniform constant K.

Now assume that h_i are elliptic for all $i > k$.

If there exist $i, j > k$ such that $\langle h_i, h_j \rangle$ is nonelementary, by Corollary 4.2.11, there exists a loxodromic element $\omega_n \in \langle h_i, h_j \rangle$ of word length uniformly bounded by a constant K. Now suppose that $\langle h_i, h_j \rangle$ is elementary for any pair of indices $i, j > k$. If one of the elementary subgroups is infinite and preserves a geodesic, by Lemma 4.2.7, $h_i h_j$ is loxodromic.

Assume that all the elementary subgroups $\langle h_i, h_j \rangle$ are either finite or parabolic for all $i, j > k$. Let B_i denote the closure of $Mar(h_i, \varepsilon)$ in $\bar{X}$. If there exist i, j such that B_i and B_j are disjoint, then $\langle h_i, h_j \rangle$ is nonelementary which contradicts our assumption. Thus for any pair of indices $i, j > k$, $B_i \cap B_j \neq \varnothing$. There exists a uniform constant r' such that $N_{r'}(B_i) \cap N_{r'}(B_j) \neq \varnothing$ in X. Hence, by Proposition 4.2.2, there exists $\tilde{z} \in X$ such that for all $i > k$ we have $d(\tilde{z}, N_{r'}(B_i)) \leq n\delta$.

For any $q \in N_{r'}(B_i)$, $d(q, h_i(q)) \leq 2r' + \varepsilon$ by the triangle inequality. Thus,

$$d(\tilde{z}, h_i(\tilde{z})) \leq 2n\delta + 2r' + \varepsilon$$

for all $i > k$. Let $\widetilde{x}'_k$ denote a lift of x'_k in X, and $l = d(\tilde{z}, \widetilde{x}'_k)$. Then

$$d(\widetilde{x}'_k, h_i(\widetilde{x}'_k)) \leq 2l + 2n\delta + 2r' + \varepsilon$$

for all $i > k$. Note that $d(\widetilde{x}'_k, h_i(\widetilde{x}'_k)) \to \infty$ as $i \to \infty$, which is a contradiction. ☐

Thus, for some pair of indices $i, j > k$, there exists a loxodromic element $\omega_n \in \langle h_i, h_j \rangle$ whose word length is uniformly bounded by some constant K. By the same argument as in Case (a), there exists a closed geodesic w_n^* such that $d(x, w_n^*) \geq (n-1)R - D$.

Thus in all cases, for each n, the orbifold M contains a closed geodesic w_n^* such that $d(x, w_n^*) \geq (n-1)R - D$. The sequence of closed geodesics $\{w_n^*\}$, therefore, escapes every compact subset of M. ☐

5.3. Continuum of nonconical limit points

In this section, using the generalized Bonahon theorem in Section 5.2, for each geometrically infinite discrete subgroup $\Gamma < \mathrm{Isom}(X)$ we find a set of nonconical limit points with the cardinality of the continuum. This set of nonconical limit points is used to prove Theorem 1.1.5.

THEOREM 5.3.1. *[**29**] If* $\Gamma < \mathrm{Isom}(X)$ *is a geometrically infinite discrete isometry subgroup, then the set of nonconical limit points of* Γ *has the cardinality of the continuum.*

PROOF. The proof is inspired by Bishop's construction of nonconical limit points of geometrically infinite Kleinian groups in the 3-dimensional hyperbolic space $\mathbb{H}^3$; [**7**, Theorem 1.1]. Let $\pi : X \to M = X/\Gamma$ denote the covering projection. Pick a point $\tilde{x} \in X$ and set $x := \pi(\tilde{x})$. If Γ is geometrically infinite, by the generalized Bonahon theorem in Section 5.2, there exists a sequence of oriented closed geodesics (λ_i) in M which escapes every compact subset of M, i.e.

$$\lim_{i \to \infty} d(x, \lambda_i) = \infty.$$

Let L be the constant as in Proposition 4.1.2 when $\theta = \pi/2$. After passing to a subsequence if necessary, we can assume that $d(x, \lambda_1) \geq L$ and the minimal distance between any consecutive pair of geodesics λ_i, λ_{i+1} is at least L. For each i, let l_i denote the length of the closed geodesic λ_i and let m_i be a positive integer such that $m_i l_i > L$.

We then pass to a subsequence in (λ_i) as in Lemma 5.1.1 (retaining the notation (λ_i) for the subsequence), so that there exists a sequence of geodesic arcs $\mu_i := x_i^+ x_{i+1}^-$ meeting λ_i, λ_{i+1} orthogonally at its end-points, for which

$$\lim_{i \to \infty} d(x, \mu_i) = \infty.$$

Let D_i denote the length of the shortest positively oriented arc of λ_i connecting x_i^- to x_i^+. We let μ_0 denote the shortest geodesic in M connecting x to x_1^-.

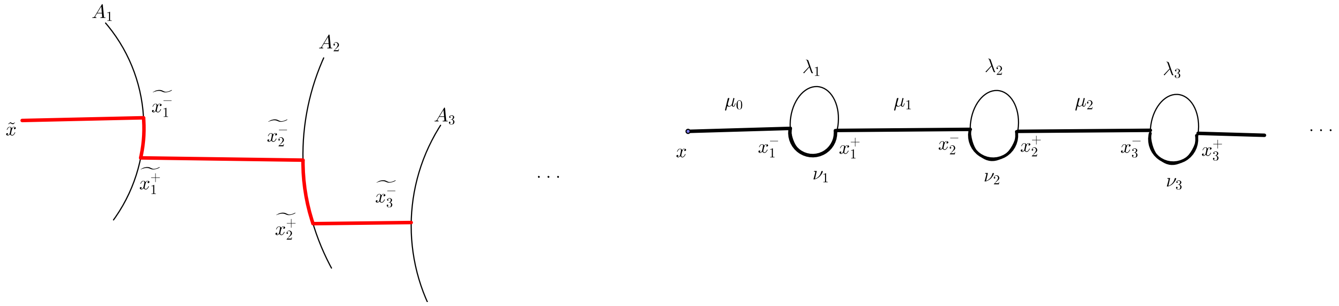

FIGURE 5.3. Here A_i denotes a geodesic in X covering the loop λ_i, $i \in \mathbb{N}$.

We next construct a family of piecewise geodesic paths γ_τ in M starting at x such that the geodesic pieces of γ_τ are the arcs μ_i above and arcs ν_i whose images are contained in λ_i and which have the same orientation as λ_i: Each ν_i wraps around λ_i a certain number of times and connects x_i^- to x_i^+. More formally, we define a map $\mathscr{P} : \mathbb{N}^\infty \to P(M)$ where $\mathbb{N}^\infty$ is the set of sequences of positive integers and $P(M)$ is the space of paths in M as follows:

$$\mathscr{P} : \tau = (t_1, t_2, \cdots, t_i, \cdots) \mapsto \gamma_\tau = \mu_0 * \nu_1 * \mu_1 * \nu_2 * \mu_2 * \cdots * \nu_i * \mu_i * \cdots$$

where the image of the geodesic arc ν_i is contained in λ_i and ν_i has length

$$l(\nu_i) = t_i m_i l_i + D_i.$$

Observe that for $i \geq 1$, the arc μ_i connects λ_i and λ_{i+1} and is orthogonal to both, with length $l(\mu_i) \geq L$ and ν_i starts at x_i^- and ends at x_i^+ with length $l(\nu_i) \geq L$.

For each γ_τ, we have a canonical lift $\tilde{\gamma}_\tau$ in X, which is a path starting at $\tilde{x}$. We will use the notation $\tilde{\mu}_i, \tilde{\nu}_i$ for the lifts of the subarcs μ_i, ν_i respectively, see Figure 5.3. By the construction, each γ_τ has the following properties:

(1) Each geodesic piece of $\tilde{\gamma}_\tau$ has length at least L.

(2) Adjacent geodesic segments of $\tilde{\gamma}_\tau$ make the angle equal to $\pi/2$ at their common endpoint.

(3) The path $\gamma_\tau : [0, \infty) \to M$ is a proper map.

By Proposition 4.1.2, $\tilde{\gamma}_\tau$ is a $(2L, 4L+1)$-quasigeodesic. Hence, there exists a limit

$$\lim_{t\to\infty} \tilde{\gamma}_\tau(t) = \tilde{\gamma}_\tau(\infty) \in \partial_\infty X,$$

and the Hausdorff distance between $\tilde{\gamma}_\tau$ and $x\tilde{\gamma}_\tau(\infty)$ is bounded above by a uniform constant C, depending only on L and κ.

We claim that each $\tilde{\gamma}_\tau(\infty)$ is a nonconical limit point. Observe that $\tilde{\gamma}_\tau(\infty)$ is a limit of loxodromic fixed points, so $\tilde{\gamma}_\tau(\infty) \in \Lambda(\Gamma)$. Let γ_τ^* be the projection of $x\tilde{\gamma}_\tau(\infty)$ under π. Then the image of γ_τ^* is uniformly close to γ_τ. Since γ_τ is a proper path in M, so is γ_τ^*. Hence, $\tilde{\gamma}_\tau(\infty)$ is a nonconical limit point of Γ.

We claim that the set of nonconical limit points $\tilde{\gamma}_\tau(\infty)$, $\tau \in \mathbb{N}^\infty$, has the cardinality of the continuum. It suffices to prove that the map

$$\mathscr{P}_\infty : \tau \mapsto \tilde{\gamma}_\tau(\infty)$$

is injective.

Let $\tau = (t_1, t_2, \cdots, t_i)$ and $\tau' = (t_1', t_2', \cdots, t_i', \cdots)$ be two distinct sequences of positive integers. Let m be the smallest positive integer such that $t_m \neq t_m'$. Then the paths $\tilde{\gamma}_\tau, \tilde{\gamma}_{\tau'}$ can be written as concatenations

$$\tilde{\alpha}_\tau \star \tilde{\nu}_m * \tilde{\beta}_\tau, \quad \tilde{\alpha}_\tau \star \tilde{\nu}_m' * \tilde{\beta}_{\tau'},$$

where $\tilde{\alpha}_\tau$ is the common initial subpath

$$\tilde{\mu}_0 * \tilde{\nu}_1 * \tilde{\mu}_1 * \tilde{\nu}_2 * \tilde{\mu}_2 * \cdots * \tilde{\nu}_{m-1} * \tilde{\mu}_{m-1}.$$

The geodesic segments $\tilde{\nu}_m, \tilde{\nu}'_m$ have the form

$$\tilde{\nu}_m = \tilde{x}_n^- \tilde{x}_m^+,$$

$$\tilde{\nu}'_m = \tilde{x}_n^- \tilde{x}'^+_m.$$

Consider the bi-infinite piecewise geodesic path

$$\sigma := \tilde{\beta}_\tau^{-1} \star \tilde{x}_n^+ \tilde{x}'^+_n \star \tilde{\beta}_{\tau'}$$

in X. Each geodesic piece of the path has length at least L and adjacent geodesic segments of the path are orthogonal to each other. By Proposition 4.1.2, σ is a complete $(2L, 4L+1)$-quasigeodesic and, hence, it is backward/forward asymptotic to distinct points in $\partial_\infty X$. These points in $\partial_\infty X$ are respectively $\tilde{\gamma}_\tau(\infty)$ and $\tilde{\gamma}_{\tau'}(\infty)$. Hence, the map $\mathscr{P}_\infty$ is injective. We conclude that the endpoints of the piecewise geodesic paths $\tilde{\gamma}_\tau$ yield a set of nonconical limit points of Γ which has the cardinality of the continuum. ☐

Remark 5.3.2. This proof is a simplification of Bishop's argument in [**7**], since, unlike [**7**], we have orthogonality of the consecutive segments in each γ_τ.

Proof of Theorem 1.1.5: The implication (1) ⇒ (2) (a generalization of Bonahon's theorem) is the main result of Section 5.2. The implication (2) ⇒ (3) is the content of Theorem 5.3.1. It remains to prove that (3) ⇒ (1). If Γ is geometrically finite, by Theorem 1.4 $\Lambda(\Gamma)$ consists of conical limit points and bounded parabolic fixed points. Since Γ is discrete, it is at most countable; therefore, the set of fixed points of parabolic elements of Γ is again at most countable. If $\Lambda(\Gamma)$ contains a subset of nonconical limit points of the cardinality of the continuum, we can find a point in the limit set which is neither a conical limit point nor a parabolic fixed point. It follows that Γ is geometrically infinite. ☐

Proof of Corollary 1.1.6: If Γ is geometrically finite, by Theorem 1.1.4, $\Lambda(\Gamma)$ consists of conical limit points and bounded parabolic fixed points. Now we prove that if $\Lambda(\Gamma)$ consists of conical limit points and parabolic fixed points, then Γ is geometrically finite. Suppose that Γ is geometrically infinite. By Theorem 1.1.5, there is a set of nonconical limit points with the cardinality of the continuum. Since the set of parabolic fixed points is at most countable, there exists a limit point in $\Lambda(\Gamma)$ which is neither a conical limit point nor a parabolic fixed point. This contradicts to our assumption. Hence, Γ is geometrically finite. $\square$

5.4. Limit set of ends

We start by reviewing the notion of *ends* of locally path-connected, locally compact, Hausdorff topological spaces Z. We refer to [**21**] for a more detailed treatment.

An *end* of Z is the equivalence class of a sequence of connected nonempty open sets

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

of Z, where each $C_i, i \in \mathbb{N}$, is a component of $K_i^c = Z \setminus K_i$, and $\{K_i\}_{i\in\mathbb{N}}$ is an increasing family of compact subsets exhausting Z with

$$K_i \subset K_j, \qquad \text{whenever } i \le j,$$

so that

$$\bigcup_{i\in\mathbb{N}} K_i = Z.$$

Here two sequences (C_i), (C'_i) are equivalent if each C_i contains some C'_j and vice-versa. The sets C_i are called *neighborhoods* of e in Z. A proper continuous map (a *ray*) $\rho : \mathbb{R}_+ \to Z$ is said to be *asymptotic to the end* e if for every neighborhood C_i of e, the subset $\rho^{-1}(C_i) \subset \mathbb{R}_+$ is unbounded.

In this paper we will be considering ends of two classes of topological spaces:

(1) $Z = Y = \mathrm{Core}(M)$, with $M = X/\Gamma$, where Γ is a discrete isometry group of a Hadamard manifold X of pinched negative curvature.

(2) $Z = \mathrm{noncusp}_\varepsilon(Y)$ (with Y as above), where ε is less than the Margulis constant of X.

An end e of $Y = \mathrm{Core}(M)$ is called *cuspidal* or a *cusp* if it can be represented by a sequence C_i consisting of projections of $\mathrm{Hull}(\Lambda) \cap B_i$, where B_i's are nested horoballs in X. (As before, $\Lambda \subset \partial_\infty X$ denotes the limit set of Γ.) Equivalently, e can be represented by a sequence C_i of components of the ε_i-thin part $\mathrm{thin}_{\varepsilon_i}(Y)$ of Y, with $\lim_{i\to\infty} \varepsilon_i = 0$. When ε is less than the *Margulis constant* of X, components of $\mathrm{thin}_\varepsilon(Y)$ which are neighborhoods of e are called *cuspidal neighborhoods* of e.

In view of Theorem 1.1.4, the group Γ is geometrically infinite if and only if Y has at least one non-cuspidal end. Equivalently, Γ is geometrically finite if and only if Z is compact, equivalently, has no ends.

Consider a neighborhood C of an end e of Z, where Z is either $Y = \mathrm{Core}(M)$ or is the noncuspidal part of Y. The preimage $\pi^{-1}(C) \subset \mathrm{Hull}(\Lambda)$ under the quotient map $\pi : X \to M$ is a countable union of components E_j. Then C is naturally isometric to the quotients E_j/Γ_j, where $\Gamma_j = \mathrm{Stab}_\Gamma(E_j)$ is the stabilizer of E_j in Γ. A point

$$\lambda \in \bigcup_j \Lambda(\Gamma_j) \subset \Lambda$$

is an *end-limit point* of C if one (equivalently, every) geodesic ray β in $\mathrm{Hull}(\Lambda)$ asymptotic to λ projects to a proper ray in $Y = \mathrm{Core}(M)$ asymptotic to e. We let $\Lambda(C)$ denote the set of end-limit points of C and let $\Lambda(e)$, the *end-limit set of* e, denote the intersection

$$\bigcap_i \Lambda(C_i)$$

taken over all neighborhoods C_i of e. (It suffices to take the intersection over a sequence (C_i) representing e.) Clearly, for every end e, $\Lambda(e)$ is disjoint from the conical limit set of Γ.

The main result of this section is

THEOREM 5.1. *[**29**] For every end e of $Z = \mathrm{noncusp}_\varepsilon(Y)$, $\Lambda(e)$ has the cardinality of continuum.*

PROOF. The end e is represented by a nested sequence (C_i) of components of

$$K_i^c = \mathrm{noncusp}_\varepsilon(Y) \setminus K_i,$$

where $K_i = B(x, iR)$ with $x \in \mathrm{noncusp}_\varepsilon(Y)$ and R is the same constant as in proof of Theorem 1.1.5.

We first claim that there exists a sequence of closed geodesics (λ_i) exiting e, i.e. $\lambda_i \subset C_i$, $i \in \mathbb{N}$. We follow Bonahon's proof in [**10**]. By Lemma 5.2.1, every intersection

$$(\mathrm{QHull}(\Gamma\tilde{x})/\Gamma) \cap C_i$$

is unbounded, where $\tilde{x}$ is a lift of x to X.

By the argument in the proof of Theorem 1.1.5, for every C_n, there exists a sequence of geodesic arcs $(\alpha_i) \subset C_n$ such that the Hausdorff distance $\mathrm{hd}(\alpha_i, K_n) \to \infty$ as $i \to \infty$, and there exists a sequence of piecewise geodesic loops $w_n \subset C_n$ exiting e. These geodesic loops w_n represent loxodromic isometries $\omega_n \in \mathrm{Isom}(X)$. Up to a subsequence, there are two possible cases:

(1) $l(\omega_n) \geq \epsilon > 0$ for some positive constant ϵ and all n.
(2) $l(\omega_n) \to 0$ as $n \to \infty$.

For case (1), we use the same argument as in the proof of Theorem 1.1.5 to construct a sequence of closed geodesics (λ_i) exiting e.

For case (2), let $\mathrm{T} \subset \Gamma$ be the set consisting of elliptic isometries and the identity. For $\tilde{x} \in X$, we define

$$d_\Gamma(\tilde{x}) = \min_{\gamma \in \Gamma \backslash T} d(\gamma\tilde{x}, \tilde{x}).$$

For $x \in M$, set

$$r(x) = d_\Gamma(\tilde{x})$$

where $\tilde{x} \in X$ is a lift of x. (If Γ is torsion free, then $r(x)$ is twice of the injectivity radius at x.) It is clear that r is a continuous function on Y, hence, it is bounded away from zero on compact subsets of Y.

Thus, $r_k := \min_{x \in K_k} r(x) > 0$. By passing to a subsequence, we assume that $l(\omega_n) < r_1/2$. Then $Mar(\omega_n, r_1/2)$ is nonempty and disjoint from K_1 for all n. By Proposition 3.3.13,

$$d(w_n, Mar(\omega_n, r_1/2)) \leq D,$$

where

$$D = \cosh^{-1}(\sqrt{2})\lceil \log_2 5K \rceil + \sinh^{-1}(4/r_1)$$

and K is the same constant as in the proof of Theorem 1.1.5. Thus, $Mar(\omega_n, r_1/2) \subset C_1$ for all n. Inductively, we find a subsequence (ω_{i_k}) such that $Mar(\omega_{i_k}, r_k/2) \subset C_k$. The closed geodesics $w^*_{i_k} \subset Mar(\omega_{i_k}, r_k/2)$ are also contained in C_k. This is the required sequence of closed geodesics (λ_i) exiting the end e.

We then continue to argue as in the proof of Theorem 5.3.1. Namely, we define a family of proper piecewise-geodesic paths γ_τ in Z. Since these rays are proper and the sequence (λ_i) exits the end e, the paths γ_τ are asymptotic to the end e. Hence, the geodesic rays γ_τ^* are also asymptotic to e.

After choosing a lift of the starting point x of all piecewise geodesic paths γ_τ in Z, there is a canonical choice of the lift $\bar{\gamma}_\tau$ of γ_τ. We claim that all the endpoints $\bar{\gamma}_\tau(\infty)$ belong to $\Lambda(e)$. It suffices to prove that $\bar{\gamma}_\tau(\infty) \in \Lambda(C_i)$ for all $i \geq 1$.

Since the sequence (C_i) is nested, we can find a nested sequence (E_i) of lifts of C_i to X. Recall that $\lambda_i \subset C_i$ for every i. Pick a complete geodesic $A_i \subset E_i$ which is a lift of λ_i. Each loop λ_i represents an element (unique up to conjugation) $\omega_i \in \Gamma$. We choose $\omega_i \in \Gamma$ which preserves the geodesic A_i. Then ω_i preserves E_i as well and, hence, the ideal fixed points of ω_i (the ideal end-points of the geodesic A_i) are in the limit set of $\Gamma_i = \mathrm{Stab}_\Gamma(E_i)$. By the construction, $\tilde{\gamma}_\tau(\infty)$ is the limit of the sequence of geodesics (A_j). Hence, $\tilde{\gamma}_\tau(\infty)$ is a limit point of Γ_i. Since γ_τ^* is a proper geodesic ray asymptotic to e, it follows that $\tilde{\gamma}_\tau(\infty) \in \Lambda(C_i)$, as required. As in the proof of Theorem 1.1.5, the rays γ_τ^* define continuum of distinct limit points of $\Lambda(e)$. Hence, $\Lambda(e)$ has the cardinality of the continuum. $\square$

Since $\Lambda(e)$ is the intersection of the limit sets $\Lambda(C)$ taken over all neighborhoods $C \subset Z = \mathrm{noncusp}_\varepsilon(Y)$, we obtain

COROLLARY 5.2. *For every neighborhood $C \subset Z$ of an end e of Z, the limit set $\Lambda(C)$ has the cardinality of continuum.*

Proof of Corollary 1.1.8: If a complementary component C of a compact subset of Y is Hausdorff-close to a finite union of cuspidal neighborhoods of cusps in Y, then $\Lambda(C)$ is a finite union of orbits of the bounded parabolic fixed points corresponding to the cusps. Suppose, therefore, that C is not Hausdorff-close to a finite union of cuspidal neighborhoods of cusps in Y. Thus, C is also a neighborhood of an end e of Y which is not a cusp. In particular, $C \cap \mathrm{noncusp}_\varepsilon(Y)$ contains an unbounded component C'. Since $\Lambda(C') \subset \Lambda(C)$ and $\Lambda(C')$ has the cardinality of continuum (Corollary 5.2), so does $\Lambda(C)$. $\square$

Proof of Corollary 1.1.7: If e is a cuspidal end of Y, then $\Lambda(e)$ is the orbit of the bounded parabolic fixed point corresponding to e under the group Γ. Hence, $\Lambda(e)$ is countable. Suppose, therefore, that e is a non-cuspidal end. As we noted above, for every neighborhood C of e in Y, the intersection $C \cap Z = \mathrm{noncusp}_{\varepsilon}(Y)$ contains an unbounded component C'. Therefore, every nested sequence (C_i) representing e gives rise to a nested sequence (C_i') in Z representing an end e' of Z. Since $\Lambda(e')$ has the cardinality of continuum (Theorem 5.1) and, by the construction, $\Lambda(e') \subset \Lambda(e)$, it follows that $\Lambda(e)$ also has the cardinality of continuum. $\square$

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