Positive Characteristic Phenomena in Linear Series

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DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2019

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June 2019
Mathematics

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Abstract

This dissertation addresses several questions regarding the behavior of linear series on projective curves over a field of positive characteristic—in particular, how the phenomena of wild ramification and inseparability can give different results from those established by Eisenbud and Harris for characteristic 0. We begin by classifying the conditions under which a linear series can be wildly ramified and when that wild ramification can be considered minimal. We then use these results to establish criteria for when a given space of linear series with prescribed ramification has its expected dimension and contains only separable linear series.

Next, we show that a formula given by Osserman for the number of separable linear series on the projective line with given ramification also counts the number of separable limit linear series on a genus 0 curve of compact type. We then use this result to give an example of a limit linear series which cannot be obtained as the limit of any family of linear series with given ramification. And we finish by showing that, unlike in characteristic 0, it is possible for a family of linear series specializing to a refined limit linear series on a curve of compact type to have ramification specializing to a node if the ramification at that node is wild.

Acknowledgments

It is merely a statement of the blatantly obvious to say that this dissertation would not exist without the patient guidance of my thesis advisor, Prof. Brian Osserman. There is no questioning the importance of his contribution to the "thesis" part—in particular, his posing of some interesting questions arising from his own work, his suggestions on approaches to answering them (including significant help with many of the technical details) and his ample provision of references to—and occasional translation of—technical background results (our mutual understanding that everything is more interesting in positive characteristic[1] certainly helped as well).

But possibly even more important was the "advisor" part—it was becoming evident as I was finishing my second year of graduate school that there was a good chance this dissertation would constitute my only serious piece of mathematical research, and I will always be extremely grateful that, even with this acknowledged up front, he was not only willing to take me on as a student but also to indulge my love for discussing the finer points of how to teach calculus. He was without question my favorite professor to TA for, and I think my teaching will forever be improved as a result of my association with him.

I would also like to thank the other members of my thesis committee, Profs. Eric Babson and Greg Kuperberg, as well as Prof. Dmitry Fuchs, both for the graduate-level courses in algebra, complex analysis, and algebraic and differential topology, and also for the thorough grilling they gave me at my qualifying exam—still possibly the highlight of my time in graduate school. And finally, I would like to thank Sarah Driver and Tina Denena for all their help in making every other aspect of my graduate education run smoothly.

[1]And not, say, over an algebraically closed field of characteristic 0 notated as $\mathbb{C}$...

CHAPTER 0

Introduction and Preliminary Material

Let C be a nonsingular projective curve over an algebraically closed field K. This introduction will briefly outline the definitions and basic theory regarding linear series on C (adapted from the exposition in [**10**, Chapter 2]).

0.1. Line Bundles, Divisors, and Linear Series

Given a line bundle $\mathscr{L}$ on C (i.e. a locally free $\mathscr{O}_C$-module of rank 1), we have by [**6**, Theorem II.5.19] that the space of global sections $\Gamma(C, \mathscr{L})$ is finite dimensional. In particular, by [**6**, Theorem II.8.15] the space of differential forms Ω^1_C is a line bundle on C, and we define the *genus* of C to be $g \coloneqq \dim \Gamma(C, \Omega^1_C)$.

A *divisor* on C is a finite formal sum $D = \sum_i a_i P_i$, where the a_i are integers and the P_i are points of C. The *degree* of D is $\deg D \coloneqq \sum_i a_i$ and we say that D is *effective* if $a_i \geq 0$ for each i. Given a nonzero rational function f on C, we can define a divisor $D(f) \coloneqq \sum_{P \in C} a_P P$, where a_P is the order of f at P. A divisor D is *principal* if there is a rational function f with $D = D(f)$, and we call two divisors D and D' *linearly equivalent* if $D - D'$ is principal.

Given a divisor D on C, we can define a line bundle $\mathscr{O}_C(D)$ whose global sections are 0 together with all nonzero rational functions f on C such that $D(f) + D$ is effective (i.e., if D is positive at some $P \in C$, then $\mathscr{O}_C(D)$ allows poles up to that order at P, while if D is negative at P, then $\mathscr{O}_C(D)$ imposes zeros of at least that order at P). By [**10**, Theorem 2.1.2], the map $D \mapsto \mathscr{O}_C(D)$ from divisors on C to line bundles on C is surjective, with two divisors mapping to isomorphic line bundles if and only if they are linearly equivalent. In particular, if $\mathscr{L}$ is a line bundle on C, then $\mathscr{L} \cong \mathscr{O}_C(D)$ for some divisor D on C, and we can define the *degree* of $\mathscr{L}$ to be $\deg \mathscr{L} \coloneqq \deg D$ (since by [**6**, Corollary II.6.10], a principal divisor has degree 0, linearly equivalent divisors have the same degree and therefore the degree of a line bundle is well-defined).

We can now make the following definition:

DEFINITION 0.1.1. A *linear series* of degree d and rank r on C (notated as a $\mathfrak{g}^r_d$) is a pair $(\mathscr{L}, V)$, where $\mathscr{L}$ is a line bundle on C of degree d and $V \subseteq \Gamma(C, \mathscr{L})$ is an $(r+1)$-dimensional subspace of global sections of $\mathscr{L}$.

REMARK 0.1.1.1. We can say the following about how d and r are related:

- It is always the case that $d \geq r$. In particular:
- The Riemann-Roch theorem [**10**, Theorem 2.2.1] gives that if $\mathscr{L}$ is a line bundle of degree $d > 2g-2$, then $r \leq \Gamma(C, \mathscr{L}) - 1 \leq d - g$.
- On the other hand, Clifford's inequality [**10**, Corollary 2.2.13] gives that if $\mathscr{L}$ has degree $0 \leq d \leq 2g-2$, then $(d+1-g)^+ \leq \dim \Gamma(C, \mathscr{L}) \leq \frac{d}{2}+1$. In particular, this implies that $r \leq \frac{d}{2}$.

NOTATION. We will use the notation $x^+ := \max(x, 0)$ throughout.

0.2. Vanishing and Ramification

0.2.1. The Vanishing Sequence. If $(\mathscr{L}, V)$ is a $\mathfrak{g}^r_d$ on C, we have that $\mathscr{L} \cong \mathscr{O}_C(D)$ for some divisor D of degree d. Each nonzero section $s \in V$ then corresponds to a section of $\mathscr{O}_C(D)$, a rational function f such that $D(f) + D$ is an effective divisor, and s vanishes according to the "shifted" divisor $D(s) = D(f) + D$. Since $D(s)$ is an effective divisor of degree d, the vanishing of s at any point of C does not exceed d.

Given a point $P \in C$, let $V(-aP)$ be the subspace of V consisting of sections which vanish to order at least a at P. If we then consider the sequence

$$V \supseteq V(-P) \supseteq V(-2P) \supseteq \cdots \supseteq V(-dP) \supseteq V(-(d+1)P) = 0$$

we have by [**10**, Proposition 2.2.7] that the dimension drops by at most 1 at each step.

Definition 0.2.1. The *vanishing sequence* $a_\bullet(P) = a_0(P), a_1(P), \ldots, a_r(P)$ of $(\mathscr{L}, V)$ at P is the sequence of orders of vanishing at P of sections of V.

That is, the $a_i(P)$ correspond to the points in the sequence of subspaces above where the dimension drops. If we define the corresponding *ramification sequence* by $\alpha_i(P) \coloneqq a_i(P) - i$ for $0 \le i \le r$, we observe that the vanishing sequence is strictly increasing and bounded above by d, while the ramification sequence is nondecreasing and bounded above by $d - r$. If $\alpha_i(P) = 0$ for each i, then $(\mathscr{L}, V)$ is *unramified* at P. Otherwise, it is *ramified* at P and P is a *ramification point*. The *ramification weight* of $(\mathscr{L}, V)$ at P is $\sum_{i=0}^{r} \alpha_i(P)$.

0.2.2. Basepoints. If P is a point such that every section in V vanishes at P (this is equivalent to $a_0(P)$ being nonzero), then P is a *basepoint* of $(\mathscr{L}, V)$. Following [**10**, Remark 2.1.8], we can canonically replace any linear series with a linear series of smaller degree such that P is no longer a basepoint (we will call this "subtracting off" of the basepoint P). The significance of this process comes from the fact that by [**6**, Theorem II.7.1 and Remark II.7.8.1], there is a bijection between basepoint-free $\mathfrak{g}^r_d$s on C and nondegenerate morphisms $C \to \mathbb{P}^r$ of degree d (determined up to automorphism of $\mathbb{P}^r$).

0.2.3. Wild Ramification and Inseparability.

Definition 0.2.2. Given nonnegative integers r and $d \ge r$, an *(r, d)-sequence* is a sequence of nonnegative integers $a_\bullet = 0 \le a_0 < a_1 < \cdots < a_r \le d$. If p is a fixed prime, an (r, d)-sequence $a_\bullet$ is *wild* if $\prod_{i<j} \frac{a_j - a_i}{j-i} \equiv 0$ modulo p. Otherwise, the sequence is *tame*.

Remark 0.2.2.1. We have that $\prod_{i<j} \frac{a_j - a_i}{j-i}$ is the determinant of the matrix $a_{ij} = \binom{a_j}{i}$, and therefore an integer.

We can first observe that the vanishing sequence $a_\bullet(P)$ of a $\mathfrak{g}^r_d$ on C at a point P is an (r, d)-sequence. If the characteristic of K is $p > 0$, we say that the $\mathfrak{g}^r_d$ is *wildly ramified* at P if the vanishing sequence is a wild sequence (otherwise, it is *tamely ramified*).

DEFINITION 0.2.3. A linear series on C is *inseparable* if every point is a ramification point. Otherwise, it is *separable.*

The total amount of ramification possible for a separable linear series is determined by the following formula:

THEOREM 0.2.4 (Plücker formula). *Let $(\mathscr{L}, V)$ be a separable $\mathfrak{g}^r_d$ on C. Then*

$$\sum_{P \in C} \sum_{i=0}^{r} \alpha_i(P) \leq (r+1)(d-r) + r(r+1)g$$

with equality if and only if $(\mathscr{L}, V)$ is everywhere tamely ramified. In particular, $(\mathscr{L}, V)$ can only have finitely many ramification points.

This result was first proved in characteristic 0 by Eisenbud and Harris [**2**, Proposition 1.1] and extended to arbitrary characteristic by Osserman [**8**, Proposition 2.4]. The key point in the proof is that a linear series is inseparable if and only if the determinant of the matrix $A_{ij}(P) = \binom{a_j(P)}{i}$ is zero (modulo p) for every $P \in C$. We can immediately conclude:

COROLLARY 0.2.4.1. *An inseparable $\mathfrak{g}^r_d$ is wildly ramified at every point. As such, inseparability is not possible if K has characteristic 0.*

We will discuss further consequences of the Plücker formula in Section 3.1.

CHAPTER 1

Tame and Wild Sequences

1.1. A Classification of Wild Sequences

This section will be devoted to the proof of the following result, which classifies the circumstances under which it is possible for a linear series to be wildly ramified:

THEOREM 1.1.1. *Given a prime p, there exists a wild (r, d)-sequence for every triple (r, d, p) except in the following cases:*

- $r = 0$, $d = r$, *or* $d < p$
- $r = np^k - 2$ *and* $d = r + 1 = np^k - 1$ *for* $k \geq 1$ *and* $1 \leq n < p$

Before we detail the proof, we can immediately state the following:

COROLLARY 1.1.2. *If K is an algebraically closed field of characteristic $p > 0$, there exists an inseparable $\mathfrak{g}^r_d$ on $\mathbb{P}^1_K$ for every r and $d \geq r$ unless the triple (r, d, p) satisfies an exceptional case from Theorem 1.1.1.*

PROOF. If the triple (r, d, p) satisfies one of the exceptional cases from Theorem 1.1.1, then a $\mathfrak{g}^r_d$ on any nonsingular projective curve over K cannot be wildly ramified at any point, and so must be separable. Conversely, if (r, d, p) is not exceptional, then there exists an (r, d)-sequence $a_\bullet$ such that the matrix $a_{ij} = \binom{a_j}{i}$ has determinant zero. As such, we have that the determinant of the matrix $A_{ij}(t) = \binom{a_j}{i} t^{a_j - i}$ is identically zero and, following the proof of [**2**, Proposition 1.1], we see that the morphism $\mathbb{P}^1_K \to \mathbb{P}^r_K$ given in affine coordinates by $t \mapsto (t^{a_1 - a_0}, \ldots, t^{a_r - a_0})$ defines an inseparable $\mathfrak{g}^r_d$ on $\mathbb{P}^1_K$. $\square$

REMARK 1.1.3. If a curve C has genus $g \geq 2$, then by Remark 0.1.1.1, a $\mathfrak{g}^r_d$ on C (with $r > 0$) can only satisfy the exceptional case $d < p$: If $d > 2g - 2$, then $r \leq d - g \leq d - 2$, so $d = r$ and $d = r + 1$ are not possible. Conversely, if $d \leq 2g - 2$, then $r \leq \frac{d}{2}$. As such, $d = r$ is not possible and $d = r + 1$ is only possible if $d = 2$ and $r = 1$. If $p = 2$, then the $(1, 2)$-sequence $0, 2$ is wild, while if $p > 2$, then we have the exceptional case $d < p$.

1.1.1. The Proof of Theorem 1.1.1. For the following statements, assume that r and $d \geq r$ are nonnegative integers and p is a prime.

Given $k \geq 1$, we say that an (r, d)-sequence $a_\bullet$ has a *collision of order* k (or k-*collision*) whenever $a_j - a_i$ is a multiple of p^k for $i < j$. A sequence is then wild exactly when it has more total collisions (of all orders) than the *unramified sequence* $0, 1, \ldots, r$ (that is, when the a_i are not *maximally distributed* modulo powers of p).

We can define a partial order on (r, d)-sequences by $a_\bullet \leq b_\bullet$ if $a_i \leq b_i$ for every $0 \leq i \leq r$. A wild sequence $a_\bullet$ is *strongly minimal* if $a_\bullet \leq b_\bullet$ for every wild sequence $b_\bullet$ and *weakly minimal* if every sequence $b_\bullet$ with $b_\bullet \leq a_\bullet$ is tame.

PROPOSITION 1.1.4. *If* $r = 0$, $d = r$, *or* $d < p$, *then every* (r, d)-*sequence is tame, while if* $0 < r < p \leq d$, *then the sequence* $0, 1, \ldots, r - 1, p$ *is a strongly minimal wild sequence.*

PROOF. A $(0, d)$-sequence $a_\bullet = a_0$ is vacuously tame because $\prod_{i<j} \frac{a_j - a_i}{j - i} = 1$ is the empty product. If $d = r$, then the only possible (r, d)-sequence is the unramified (and so tame) sequence $0, 1, \ldots, d$. If $d < p$, then $a_j - a_i < p$ for any (r, d)-sequence $a_\bullet$, so the sequence has no collisions and must be tame. Finally, if $0 < r < p \leq d$, then the unramified sequence has no collisions, but $0, 1, \ldots, r - 1, p$ has the collision of p with 0 and so is wild. Furthermore, every (r, d)-sequence $a_\bullet$ with $a_r < p$ must be tame, so $0, 1, \ldots, r - 1, p$ is strongly minimal. □

Now assume that $p \leq r < d$ and let $k \geq 1$ be the largest with $p^k \leq r$.

PROPOSITION 1.1.5. *If* $0 \leq \ell \leq k$ *is the largest such that* $r \equiv 0$ *modulo* p^ℓ, *let* $0 < s < p^{\ell+1}$ *such that* $r \equiv -s$ *modulo* $p^{\ell+1}$. *Then* $0, 1, \ldots, r - 1, r + s$ *is a weakly minimal wild sequence for any* $d \geq r + s$. *In particular, if* $r \equiv -1$ *modulo* p, *then* $0, 1, \ldots, r - 1, r + 1$ *is a strongly minimal wild sequence for any* $d \geq r + 1$.

Proof. Assume first that $\ell = 0$. Then $r \equiv -s$ modulo p, and in particular, r does not collide with 0. Given $t \leq s$, replacing r with $r+t$ in the unramified sequence shifts every collision involving r to the right t spaces (in particular, the collision between r and $r-p$ becomes a collision between $r+t$ and $r-p+t$, where we note that $r-p+t \leq r-p+s \leq r-1$). As such, the sequence $0, 1, \ldots, r-1, r+t$ is wild if and only if the entry $r+t$ collides with 0. By assumption, $r+s$ collides with 0, while $r+t$ does not collide with 0 for $t < s$, so the sequence $0, \ldots, r-1, r+s$ is weakly minimal. In particular, if $s = 1$, then the sequence $a_\bullet = 0, \ldots, r-1, r+1$ is strongly minimal since $a_\bullet \leq b_\bullet$ for every (r, d)-sequence $b_\bullet$ except the unramified sequence.

Now assume that $\ell > 0$. In this case, r and s are both multiples of p^ℓ, and so collide to order ℓ. Let $r - np^m$ (with $1 \leq n < p$ and $1 \leq m \leq k$) be an entry that collides with r to order m. If $np^m > t$ for a given $t \leq s$ (since $s < p^{\ell+1}$, this is always the case if $m > \ell$), then replacing r with $r+t$ in the unramified sequence shifts this collision to the right t spaces as above. If $np^m \leq t$, then shifting to the right t spaces involves first shifting np^m spaces, at which point the collision is "wrapped around" to become a collision between $r + np^m$ and 0, and then shifting the remaining $t - np^m$ spaces. Since r is a multiple of p^ℓ with $\ell \geq m$, the collision between $r + np^m$ and 0 will have order at least m. Furthermore, the order will be greater than m if and only if $m = \ell$ and $r + np^m$ is a multiple of $p^{\ell+1}$. This says exactly that replacing r with $r+t$ gives a wild sequence if and only if r collides with $r-t$ to order ℓ and this collision increases in order when "wrapped". By assumption, the first collision to increase in order is the collision between r and $r-s$, which says exactly that $0, 1, \ldots, r-1, r+s$ is wild and $0, 1, \ldots, r-1, r+t$ is tame for $t < s$. As above, this establishes that the wild sequence $0, 1, \ldots, r-1, r+s$ is weakly minimal. $\square$

Next, a *complete block* is a subsequence of the unramified sequence of the form

$$mp^k, mp^k + 1, \ldots, (m+1)p^k - 1$$

where $0 \leq m < p$. By assumption, the unramified sequence contains at least one complete block $0, 1, \ldots, p^k - 1$. Given an entry a in a complete block, there are $p^{k-\ell}$ entries in any other complete block that collide with a to order ℓ. Similarly, within the same complete block as a, there are $p^{k-\ell} - 1$ entries that collide with a to order ℓ (a does not collide with itself). As such, in an unramified sequence consisting of n complete blocks, any entry a contributes $n(p^{k-1} + p^{k-2} + \cdots + p + 1) - k$ total collisions. In particular, *the number of collisions removed by deleting an entry does not depend on the choice of entry.*

Proposition 1.1.6. *If $r \equiv -2$ modulo p^k, then every $(r, r+1)$-sequence is tame. Furthermore, if $d \geq r + 2$, then $0, 1, \ldots, r-1, r+2$ is a strongly minimal wild sequence.*

Proof. Let $r = np^k - 2$ for some $2 \leq n \leq p$. Then the unramified sequence consists of $n - 1$ complete blocks plus a block missing its final entry, $np^k - 1$. As such, every $(r, r+1)$-sequence is obtained from the unramified sequence by deleting an entry and adding $np^k - 1$. In particular, every $(r, r+1)$-sequence (including the unramified sequence) is obtained by deleting an entry from a sequence of n complete blocks and therefore has the same number of collisions. This says exactly that a wild sequence is not possible.

Conversely, assume that $d \geq r + 2$. If $p \neq 2$, then we have that $r \equiv -2$ modulo p. If $p = 2$, then $r = 2 \cdot 2^k - 2 = 2^{k+1} - 2$. Since $k \geq 1$, this says exactly that $r \equiv 0$ modulo 2 but that $r \equiv -2$ modulo 4. In either case, we have from Proposition 1.1.5 that the sequence $0, 1, \ldots, r-1, r+2$ is wild. $\square$

Proposition 1.1.6 establishes the last of the exceptional cases. The following result proves the existence of a wild sequence in all other situations:

Proposition 1.1.7. *If $r = np^k + s$ for some $1 \leq n < p$ and $0 \leq s \leq p^k - 3$, then the sequence $0, 1, \ldots, np^k - 2, np^k, \ldots, r, r+1$ is wild for any $d \geq r + 1$.*

Proof. In this case, the unramified sequence consists of n complete blocks followed by a block of length $s+1$. Consider first the subsequence $0, 1, \ldots, np^k - 1$ formed by the complete blocks. We have that deleting the entry np^k-1 removes $n(p^{k-1}+p^{k-2}+\cdots+p+1)-k$ collisions with this subsequence, while separately adding $r+1$ creates $n(p^{k-1}+p^{k-2}+\cdots+p+1)$ collisions with this subsequence. Since $s \leq p^k-3$, np^k-1 and $r+1$ can collide to order at most $k-1$, so replacing np^k-1 with $r+1$ creates at least one collision with the complete blocks. Furthermore, any collision between np^k-1 and the final block of length $s+1$ corresponds to a "mirror-image" collision between $r+1$ and the final block: If np^k-1 collides with np^k+t for some $0 \leq t \leq s$, then $r+1$ collides to the same order with np^k+s-t. As such, the number of collisions with the final block is unchanged. This says exactly that $0, 1, \ldots, np^k-2, np^k, \ldots, r, r+1$ is a wild sequence. $\square$

Proposition 1.1.7 completes the proof of Theorem 1.1.1.

1.2. Minimal Wild Sequences

As above, let r and $d \geq r$ be nonnegative integers, p a prime, and, in the case that $p \leq r$, $k \geq 1$ the largest with $p^k \leq r$.

Definition 1.2.1. An (r,d)-*index* is a sequence of nonnegative integers $\alpha_\bullet = 0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq d-r$. Given an (r,d)-index $\alpha_\bullet$, we can associate a unique (r,d)-sequence $a_\bullet$ defined by $a_i = \alpha_i + i$ for $0 \leq i \leq r$, and we say that $\alpha_\bullet$ is *wild* if $a_\bullet$ is wild (and similarly with the other properties of (r,d)-sequences).

Remark 1.2.1.1. We observe that the ramification sequence $\alpha_\bullet(P)$ of a $\mathfrak{g}^r_d$ on a curve C at a point $P \in C$ is an (r,d)-index.

Proposition 1.2.2. *Assume that* $r = np^k + s$ *with* $0 \leq s \leq p^k - 1$ *and* $d < p^{k+1}$*. Then given an* $(s, d-np^k)$*-index* $\alpha_\bullet$*, the* (r,d)*-index* $\overline{\alpha}_\bullet = 0, \ldots, 0, \alpha_0, \ldots, \alpha_s$ *is wild if and only if* $\alpha_\bullet$ *is wild. In particular,* $\overline{\alpha}_\bullet$ *is weakly minimal if and only if* $\alpha_\bullet$ *is weakly minimal.*

Proof. Let $a_\bullet$ be the (r,d)-sequence associated to $\overline{\alpha}_\bullet$. Then $a_\bullet$ consists of n complete blocks followed by $s+1$ entries not exceeding $p^{k+1}-1$. Any of these final entries contributes exactly $n(p^{k-1}+p^{k-2}+\cdots+p+1)$ collisions with the entries in the complete blocks, and this is true for any choice of $\alpha_\bullet$, including the *unramified index* $0,\ldots,0$. As such, we can determine if $a_\bullet$ is wild simply by counting the collisions within the final $s+1$ entries. Since shifting every entry by np^k does not change the number of collisions, this says exactly that $\overline{\alpha}_\bullet$ is wild if and only if $\alpha_\bullet$ is wild. Furthermore, given a wild $(s,d-np^k)$-index $\beta_\bullet$ and the corresponding wild (r,d)-index $\overline{\beta}_\bullet$, we have that $\beta_\bullet \leq \alpha_\bullet$ if and only if $\overline{\beta}_\bullet \leq \overline{\alpha}_\bullet$, which says exacly that $\overline{\alpha}_\bullet$ is weakly minimal if and only if $\alpha_\bullet$ is weakly minimal. □

COROLLARY 1.2.3. *If $r = np^k + s$ with $0 \leq s \leq p^k - 3$ and the triple $(s, s+1, p)$ satisfies one of the exceptional cases from Theorem 1.1.1, then the wild sequence found in Proposition 1.1.7 is weakly minimal. If we also have that $d = r+1$, then it is strongly minimal.*

PROOF. The wild sequence in Proposition 1.1.7 corresponds to the index $0,\ldots,0,\underbrace{1,\ldots,1}_{s+2}$. If $(s,s+1,p)$ satisfies an exceptional case from Theorem 1.1.1, then every $(s,s+1)$-index is tame, so every $(r,r+1)$-index that ends with fewer than $s+2$ 1's must be tame by Proposition 1.2.2. □

REMARK 1.2.3.1. If there exists a wild $(s,s+1)$-index, then by Proposition 1.2.2, a weakly minimal wild $(s,s+1)$-index determines a weakly minimal wild $(r,r+1)$-index. As such, we can determine a weakly minimal wild $(r,r+1)$-index by reducing to the case that $r = s$ and iterating this process.

We conclude this chapter by collecting the previous results into the following classification of minimal wild indices:

THEOREM 1.2.4. *For any r and $d \geq r$ such that the triple (r,d,p) does not satisfy an exceptional case from Theorem 1.1.1, there exists a weakly minimal wild index of one (or both) of the following forms: $0,\ldots,0,1,\ldots,1$ or $0,\ldots,0,t$ for some $t \leq d-r$. In particular:*

(1) *If $r < p \leq d$, then $0,\ldots,0,p-r$ is a strongly minimal wild index.*

(2) *If* $r \equiv -1$ *modulo* p *and* $d \geq r+1$, *then* $0, \dots, 0, 1$ *is a strongly minimal wild index.*

(3) *If* $r \equiv -2$ *modulo* p^k *and* $d \geq r+2$, *then* $0, \dots, 0, 2$ *is a strongly minimal wild index.*

(4) *If* $p = 2$ *and* $d \geq r+2$, *then every weakly minimal wild index is of the form* $0, \dots, 0, 1$, $0, \dots, 0, 1, 1$, *or* $0, \dots, 0, 2$.

(5) *If* $2 < p \leq r$ *and* $d \geq r+2$, *then either* $0, \dots, 0, \underbrace{1, \dots, 1}_{<p}$ *or* $0, \dots, 0, 2$ *is a weakly minimal wild index.*

(6) *If* $d \geq r+p-1$ *and* $p \neq 2$, *then either* $0, \dots, 0, t$ *(for some* $t < p$*) or* $0, \dots, 0, 1, 1$ *is a weakly minimal wild index.*

Furthermore, a wild index not of one of these forms is not weakly minimal.

Proof. The existence result follows from Propositions 1.1.4 and 1.1.5 and Corollary 1.2.3, and statements (1), (2), and (3) follow from Propositions 1.1.4, 1.1.5, and 1.1.6, respectively.

Next, assume that $r = np^k + s$ with $0 \leq s \leq np^k - 3$. We have from Corollary 1.2.3 that the wild index from Proposition 1.1.7 is weakly minimal if the triple $(s, s+1, p)$ satisfies one of the exceptional cases in Theorem 1.1.1. There are two possible ways that this can happen:

- $s + 1 < p$ (this includes the case $s = 0$): Then $s + 2 \leq p$, so the weakly minimal wild index $0, \dots, 0, \underbrace{1, \dots, 1}_{s+2}$ finishes with at most p 1's.
- $s = mp^\ell - 2$ for some $1 \leq m < p$ and $1 \leq \ell < k$: Then if $d \geq r+2$, we have that $0, \dots, 0, 2$ is a weakly minimal wild index by Proposition 1.1.5 (this is true even if $p = 2$ by the argument in the proof of Proposition 1.1.6).

As in Remark 1.2.3.1, if neither of these conditions holds, then we can reduce to the case that $r = s$, and iterate until one of these conditions is satisfied. In the case $p = 2$, this establishes statement (4). If $p \neq 2$, then $s + 2 = p$ implies that $r \equiv -2$ modulo p, so $0, \dots, 0, 2$ is a weakly minimal wild index by Proposition 1.1.5. As such, if this is not the case, then we must have that $s + 2 < p$, which establishes statement (5).

To establish statement (6), assume that $d \geq r + p - 1$ and $r \geq p > 2$ (statement (1) covers the case $0 < r < p$). If r is not a multiple of p, then we can write $r \equiv -t$ modulo p for some $0 < t < p$, and then $0, \ldots, 0, t$ is a weakly minimal wild index by Proposition 1.1.5. Otherwise, assume that r is a multiple of p. If $r = np^k$, then we can apply the above result with $s = 0$ to see that $0, \ldots, 0, 1, 1$ is a weakly minimal wild index. Otherwise, we can write $r = np^k + s$, where s is also a multiple of p. Since $p \neq 2$, we can again reduce to the case that $r = s$ and iterate this process, which must terminate with the weakly minimal wild index $0, \ldots, 0, 1, 1$.

Lastly, assume that the final statement fails for some triple (r, d, p). In particular, let r be the smallest such that there exists $d \geq r$, a prime p, and a weakly minimal wild (r, d)-index $\alpha_\bullet$ which is not of one of the given forms. Then:

- Since every $(r, r+1)$-index is of the form $0, \ldots, 0, 1, \ldots, 1$, we have that $d \geq r + 2$.
- Since each strongly minimal wild index in statements (1), (2), and (3) is of the form $0, \ldots, 0, t$, we have that $r \geq p$ with $r \not\equiv -1$ or -2 modulo p^k.
- By Proposition 1.1.5, the largest t needed to achieve a wild index of the form $0, \ldots, 0, t$ is $t = p^{k+1} - r$, so $\alpha_\bullet$ is not weakly minimal if $\alpha_r \geq p^{k+1} - r$.

As such, we can write $r = np^k + s$ with $0 \leq s \leq p^k - 3$ and we can assume without loss of generality that $d < p^{k+1}$. By Proposition 1.1.7, the (r, d)-index $0, \ldots, 0, \underbrace{1, \ldots, 1}_{s+2}$ is wild. Since $\alpha_\bullet$ is not of this form, only the final $s + 1$ entries can be nonzero, so we have that $\alpha_\bullet = 0, \ldots, 0, \alpha_{np^k}, \ldots, \alpha_r$. By Proposition 1.2.2, this says exactly that the $(s, d - np^k)$-index $\alpha_{np^k}, \ldots, \alpha_r$ is a weakly minimal wild index not of one of the given forms. However, since $s < r$, this contradicts the assumed minimality of r and the result follows. $\square$

CHAPTER 2

Brill-Noether Theory and Inseparability

This chapter will use results from the previous chapter to establish criteria for when spaces of linear series with prescribed ramification have their expected dimension and contain only separable linear series. We begin with the following preliminary material:

Let C_g be a nonsingular projective curve of genus g over an algebraically closed field K, r and $d \geq r$ nonnegative integers, and $a_\bullet$ and $b_\bullet$ (r,d)-sequences. Then if $\alpha_\bullet$ and $\beta_\bullet$ are the corresponding (r,d)-indices, we can define:

- $\rho_0 \coloneqq g-(r+1)(g+r-d) = (r+1)(d-r)-rg$ (the *Brill-Noether number*);
- $\rho_1 \coloneqq \rho_0 - \sum_{i=0}^{r} \alpha_i$;
- $\rho_2 \coloneqq \rho_0 - \sum_{i=0}^{r} (\alpha_i+\beta_i)$;
- $\widehat{\rho}_1 \coloneqq g - \sum_{i=0}^{r} (a_i+(r-i)-(d-g))^+ = g - \sum_{i=0}^{r} (\alpha_i-(d-(g+r)))^+$
- $\widehat{\rho}_2 \coloneqq g - \sum_{i=0}^{r} (a_i+b_{r-i}-(d-g))^+ = g - \sum_{i=0}^{r} (\alpha_i+\beta_{r-i}-(d-(g+r)))^+$.

By [**10**, Proposition 2.6.1], there exists a space $G^r_d(C_g)$ which contains $\mathfrak{g}^r_d$s on C_g. Then, given $Q_1,\ldots,Q_n$ distinct points of C_g and $a^j_\bullet$ (r,d)-sequences for $1 \leq j \leq n$ (with associated (r,d)-indices $\alpha^j_\bullet$), we can set $G^r_d(C_g,\{(Q_j,a^j_\bullet)\}_j)$ to be the space of $\mathfrak{g}^r_d$s on C_g with vanishing sequence at least $a^j_\bullet$ at Q_j. Following [**10**, Proposition 2.7.3], we will refer to

$$\rho \coloneqq \rho_0 - \sum_{j=0}^{n}\sum_{i=0}^{r} \alpha^j_i$$

as the *expected dimension* of the space $G^r_d(C_g,\{(Q_j,a^j_\bullet)\}_j)$.

If C_g is assumed to be a general curve (cf. [**10**, Remark 2.5.6]), and P and Q are general points of C_g, then we have the following result due to Osserman [**9**, Theorem 1.1]:

THEOREM 2.0.1. *The space $G^r_d(C_g, (P, a_\bullet), (Q, b_\bullet))$ is nonempty if and only if $\widehat{\rho}_2 \geq 0$, in which case it has the expected dimension ρ_2.*

REMARK 2.0.1.1. If $b_\bullet$ is the unramified sequence, then we have that $\rho_2 = \rho_1$ and $\widehat{\rho}_2 = \widehat{\rho}_1$, so Theorem 2.0.1 gives that the space $G^r_d(C_g, (P, a_\bullet))$ of linear series with vanishing imposed only at P is nonempty if and only if $\widehat{\rho}_1 \geq 0$, in which case it has the expected dimension ρ_1.

REMARK 2.0.2. Note also that since we can write

$$\rho_2 = g - \sum_{i=0}^{r}(a_i + b_{r-i} - (d-g))$$

we have that $\widehat{\rho}_2 \leq \rho_2$ (and similarly $\widehat{\rho}_1 \leq \rho_1$), with equality if $d \leq g + r$.

2.1. A Criterion For Expected Dimension

Assume that there exists a $\mathfrak{g}^r_d$ on C_g with vanishing sequence $a_\bullet$ at P and $b_\bullet$ at Q.

PROPOSITION 2.1.1. *In this situation, we have that $b_r \leq r + \rho_1$. In particular, if $d \leq g + r$, then*

$$b_r \leq d - r(g + r - d) - \sum_{i=0}^{r} \alpha_i \leq d$$

PROOF. Assume that $b_r = r + \rho_1 + s$ for some $s > 0$. Then:

$$\begin{aligned}
\rho_2 &= g - \sum_{i=0}^{r}(a_i + b_{r-i} - (d-g)) \\
&\leq g - \left(\sum_{i=1}^{r}(a_i + (r-i) - (d-g))\right) - (a_0 + (r + \rho_1 + s) - (d-g)) \\
&= \left(g - \sum_{i=0}^{r}(a_i + (r-i) - (d-g))\right) - \rho_1 - s \\
&= -s
\end{aligned}$$

As such, if $b_r > r+\rho_1$, then $\widehat{\rho}_2 \le \rho_2 < 0$, so by Theorem 2.0.1, the space of linear series with this vanishing must be empty. In particular, if $d \le g+r$, we have

$$r+\rho_1 = r+\rho_0 - \sum_{i=0}^{r} \alpha_i = d - r(g+r-d) - \sum_{i=0}^{r} \alpha_i \le d$$

$\square$

Proposition 2.1.2. *If $d \ge g+r$, then*

$$b_r \le (d-g)+\widehat{\rho}_1 = d - \sum_{i=0}^{r} (\alpha_i - (d-(g+r)))^+ \le d$$

In particular, if $d = g+r$, we have that

$$b_r \le d - \sum_{i=0}^{r} \alpha_i$$

Proof. Because there exists a $\mathfrak{g}^r_d$ which vanishes to $a_\bullet$ at P, the space $G^r_d(C_g,(P,a_\bullet))$ is nonempty. This means that the sequence $a_\bullet$ gives $\widehat{\rho}_1 \ge 0$. We can also assume that P is not a basepoint of the $\mathfrak{g}^r_d$, which gives $a_0 = 0$.

Assume that $b_r = (d-g)+\widehat{\rho}_1 + s$ for some $s > 0$. Then:

$$\begin{aligned}
\widehat{\rho}_2 &= g - \sum_{i=0}^{r} (a_i + b_{r-i} - (d-g))^+ \\
&\le g - \left(\sum_{i=1}^{r} (a_i + (r-i) - (d-g))^+ \right) - (a_0 + ((d-g)+\widehat{\rho}_1 + s) - (d-g))^+ \\
&= \left(g - \sum_{i=1}^{r} (a_i + (r-i) - (d-g))^+ \right) - \widehat{\rho}_1 - s
\end{aligned}$$

In the last equality, we are using that $a_0 = 0$ and that $(\widehat{\rho}_1 + s)^+ = \widehat{\rho}_1 + s$. The assumption that $d \ge g+r$ gives that $(a_0 + r - (d-g))^+ = 0$, so

$$\widehat{\rho}_1 = g - \sum_{i=1}^{r} (a_i + (r-i) - (d-g))^+$$

and $\widehat{\rho}_2 \leq -s$. As such, if $b_r > (d-g) + \widehat{\rho}_1$, then $\widehat{\rho}_2 < 0$ and the space of linear series with this vanishing must be empty. In particular:

$$\begin{aligned}(d-g) + \widehat{\rho}_1 &= (d-g) + g - \sum_{i=0}^{r}(a_i + (r-i) - (d-g))^+ \\ &= d - \sum_{i=0}^{r}(a_i - i - (d-(g+r)))^+ \\ &= d - \sum_{i=0}^{r}(\alpha_i - (d-(g+r)))^+ \leq d\end{aligned}$$

Taking $d = g + r$ then gives the final inequality. $\square$

Now let C_0 be the nodal curve obtained by gluing a copy of $\mathbb{P}^1$ to C_g at the point Q.

Definition 2.1.3. Given a nodal curve C_0 defined as the union of two nonsingular projective curves C_1 and C_2 glued at a point Q, a *limit linear series* (or *limit* $\mathfrak{g}^r_d$) on C_0 is a pair of $\mathfrak{g}^r_d$s, $(\mathscr{L}^1, V^1)$ on C_1 and $(\mathscr{L}^2, V^2)$ on C_2, such that $a^1_i(Q) + a^2_{r-i}(Q) \geq d$ for $0 \leq i \leq r$ (where $a^j_\bullet(Q)$ is the vanishing sequence of $(\mathscr{L}^j, V^j)$ at Q). The *vanishing sequence* (respectively, *ramification sequence*) of a limit $\mathfrak{g}^r_d$ at a nonsingular point is the vanishing sequence (respectively, ramification sequence) of the corresponding $\mathfrak{g}^r_d$ at that point.

If $c_\bullet$ is the *complementary sequence* to $b_\bullet$ (i.e. the (r,d)-sequence given by $c_i := d - b_{r-i}$ for $0 \leq i \leq r$), then every limit $\mathfrak{g}^r_d$ on C_0 with vanishing sequence $b_\bullet$ at Q on the C_g component must have vanishing sequence at least $c_\bullet$ at Q on the $\mathbb{P}^1$ component. This gives:

Corollary 2.1.4. *If $d \leq g + r$ then*

$$c_0 \geq r(g+r-d) + \sum_{i=0}^{r} \alpha_i$$

while if $d \geq g + r$, then

$$c_0 \geq \sum_{i=0}^{r}(\alpha_i - (d-(g+r)))^+$$

With Corollary 2.1.4 in mind, we can then define $d' \coloneqq d - \sigma$, where

$$\sigma \coloneqq \begin{cases} r(g+r-d) + \sum_{i=0}^{r} \alpha_i & d \leq g+r \\ \sum_{i=0}^{r} (\alpha_i - (d-(g+r)))^+ & d \geq g+r \end{cases} \tag{2.1}$$

PROPOSITION 2.1.5. *Let $P'_1, \ldots, P'_{n-1}$ be distinct nonsingular points on the $\mathbb{P}^1$ component of C_0 and $a^1_\bullet, \ldots, a^{n-1}_\bullet$ (r,d)-sequences. If K has characteristic $p > 0$, assume that the triple (r, d', p) satisfies one of the exceptional cases from Theorem 1.1.1. Then the space of limit linear series $G^r_d(C_0, \{(P'_j, a^j_\bullet)\}_j, (P, a_\bullet))$ has the expected dimension if it is nonempty.*

PROOF. Since $b_\bullet$ and $c_\bullet$ are complementary sequences, we can establish this result by checking that the spaces $G^r_d(\mathbb{P}^1, \{(P'_j, a^j_\bullet)\}_j, (Q, c_\bullet))$ and $G^r_d(C_g, (P, a_\bullet), (Q, b_\bullet))$ each have the expected dimension if they are nonempty and then applying [**10**, Corollary 3.3.9]. Since C_g, P, and Q were assumed to be general, the statement on C_g follows from Theorem 2.0.1. Furthermore, by Corollary 2.1.4, the sequence $a_\bullet$ forces a basepoint at Q on the $\mathbb{P}^1$ component. Then subtracting off this basepoint (cf. Section 0.2.2) produces an isomorphism between the spaces $G^r_d(\mathbb{P}^1, \{(P'_j, a^j_\bullet)\}_j, (Q, c_\bullet))$ and $G^r_{d'}(\mathbb{P}^1, \{(P'_j, a^j_\bullet)\}_j, (Q, c_\bullet - c_0))$ (i.e. we are lowering the degree from d to d'). By Corollary 1.1.2, it is not possible for any $\mathfrak{g}^r_{d'}$ on $\mathbb{P}^1$ to be inseparable, so we can therefore apply [**10**, Proposition 2.8.2] (the Eisenbud-Harris result for linear series on $\mathbb{P}^1$ in characteristic 0) to see that $G^r_{d'}(\mathbb{P}^1, \{(P'_j, a^j_\bullet)\}_j, (Q, c_\bullet - c_0))$ has the expected dimension if it is nonempty. $\square$

THEOREM 2.1.6. *Let K be an algebraically closed field of characteristic $p > 0$, g, r, and $d \geq r$ nonnegative integers, and $a_\bullet$ an (r,d)-sequence with $a_0 = 0$. If we set $d' \coloneqq d - \sigma$, where σ is as defined in (2.1), above, assume that the triple (r, d', p) satisfies one of the exceptional cases from Theorem 1.1.1. Then there exists a nonsingular projective curve C of genus g and $P_1, \ldots, P_n$ distinct points on C such that given (r,d)-sequences $a^1_\bullet, \ldots, a^{n-1}_\bullet$, the space $G^r_d(C, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ has the expected dimension if it is nonempty.*

Proof. By Winters's theorem on families of curves [**10**, Theorem 3.4.4], there exists a nonsingular curve B, a point $b_0 \in B$, and a genus g family of curves $\pi \colon X \to B$, such that:

- The total space X and the fibers $X|_b$ are nonsingular for $b \neq b_0$;
- The fiber $X_0 = X|_{b_0}$ is isomorphic to the nodal curve C_0;
- There exist disjoint sections $P_1, \ldots, P_n$ of π with each $P_1, \ldots, P_{n-1}$ specializing to $P_1', \ldots, P_{n-1}'$, respectively, and P_n specializing to P.

By Proposition 2.1.5, the space $G^r_d(X_0, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ has exactly the expected dimension if it is nonempty. We can therefore apply "specialization" [**10**, Proposition 3.4.2] to π to see that for a fixed choice of (r,d)-sequences $a^j_\bullet$, the spaces $G^r_d(X|_b, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ can only be nonempty of greater than expected dimension for finitely many points $b \in B$. As such, there is a nonempty open subset $U \subseteq B$ such that $G^r_d(X|_b, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ is either empty or has the expected dimension for every $b \in U$. We obtain such an open subset for each of the finitely many choices of the $a^j_\bullet$, and taking their intersection gives a nonempty open subset $V \subseteq B$ such that for every $b \in V$ and any choice of (r,d)-sequences $a^j_\bullet$, the space $G^r_d(X|_b, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ has the expected dimension if it is nonempty. We can then fix any point $b_1 \in V$ and set $C \coloneqq X|_{b_1}$. □

Our main result is then the following:

Corollary 2.1.7. *Let g, r, d, K, and $a_\bullet$ satisfy the assumptions of Theorem 2.1.6. Then if C is a general nonsingular projective curve of genus g over K with n general marked points $P_1, \ldots, P_n$, and $a^1_\bullet, \ldots, a^{n-1}_\bullet$ are (r,d)-sequences, the space $G^r_d(C, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ has the expected dimension if it is nonempty and contains only separable linear series.*

Proof. By [**10**, Corollary 4.2.4], the property of the space $G^r_d(C, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ having the expected dimension if it is nonempty is open in families of curves with marked points. As such, the existence result from Theorem 2.1.6 is sufficient to establish the result for a general n-marked curve. Furthermore, in the case that the space is nonempty, we can apply Propositions 2.1.1 and 2.1.2 to the point P_n and any other general point Q to see that the vanishing sequence at Q of any $\mathfrak{g}^r_d$ in the space is in fact an (r, d')-sequence and so cannot be a wild sequence. Since by Corollary 0.2.4.1 an inseparable linear series is wildly ramified at every point, $G^r_d(C, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ can only contain separable $\mathfrak{g}^r_d$s. $\square$

REMARK 2.1.7.1. The significance of this result is the following: Without involving the sequence $a_\bullet$ in the assumptions of Theorem 2.1.6, we have by [**10**, Theorem 2.7.7] that the space $G^r_d(C, \{(P_j, a^j_\bullet)\}_j, (P_n, a_\bullet))$ has its expected dimension as long as the triple (r, d, p) satisfies an exceptional case from Theorem 1.1.1. By including $a_\bullet$ in the initial conditions, we only need the weaker assumption that (r, d', p) satisfies an exceptional case.

2.1.1. Examples. For the following examples, set $\rho \coloneqq \rho_1$ and $\widehat{\rho} \coloneqq \widehat{\rho}_1$. Because the statement of Theorem 2.1.6 is trivial if the space $G^r_d(C, (P, a_\bullet))$ is empty, we will assume that $\widehat{\rho} \geq 0$.

EXAMPLE 2.1.8. In the the case that $d \geq g + r$, we observe that $\widehat{\rho} = g - \sigma$, and so $\sigma \leq g$. If we have that $\alpha_r = d - r$ and $\alpha_i \leq d - (g + r)$ for $i < r$, then $\sigma = \alpha_r - (d - (g + r)) = g$, so we achieve the maximum value for σ (and therefore the minimum value of $d - g$ for $d' = d - \sigma$). In particular, we can take $\alpha_i = 0$ for $i < r$, in which case we have $\rho = r(d - (g + r))$.

EXAMPLE 2.1.9. Now consider the case that $d \leq g + r$. We note that in this case

$$\rho = (d - r) - r(g + r - d) - \sum_{i=0}^{r} \alpha_i = d - r - \sigma = d' - r$$

If we take $\alpha_i = 0$ for each i, we achieve a minimum value of $\sigma = r(g + r - d)$, while we need $\sigma \leq d - r$ in order to achieve $\widehat{\rho} = \rho \geq 0$. This gives that $r(g + r - d) \leq d - r$ and therefore

$$d - r \leq g \leq \frac{r + 1}{r}(d - r)$$

If we let $0 \leq s \leq \frac{d}{r} - 1$ such that $g = d - r + s$, we have that $\sigma = sr + \sum_{i=0}^{r} \alpha_i$ and $\rho = d - (s + 1)r - \sum_{i=0}^{r} \alpha_i$. Assuming that $\alpha_i = 0$ for $i < r$, we can consider three cases:

- Since $s \leq \frac{d}{r} - 1$, we can take $\alpha_r = d - (s+1)r$. Then $\rho = 0$ and $d' = r$.
- If $s \leq \frac{d-1}{r} - 1$, we can take $\alpha_r = d - (s+1)r - 1$. Then $\rho = 1$ and $d' = r + 1$.
- Assume that $d \geq p > r > 0$ and $s > 0$. If $d - p \geq sr$, taking $\alpha_r = d - sr - p + 1$ gives $\rho = p - r - 1$ and $d' = p - 1 < p$. Conversely, if $d - p < sr$, we can achieve the same result by taking $\alpha_r = 0$: We have that $\rho = \rho_0 = d - (s+1)r$ and $d' = d - sr < p$.

In particular, the $\alpha_\bullet$ in each example produces a d' which might (depending on p in the $d' = r + 1$ case) allow (r, d', p) to satisfy one of the exceptional cases from Theorem 1.1.1.

We finish this section by observing that if we take $a_\bullet$ to be the unramified sequence in Corollary 2.1.7, we achieve the following:

Corollary 2.1.10. *Let g, r, and d be nonnegative integers such that $r \leq d < g + r$, p a prime, and $s \coloneqq g + r - d$ as in Example 2.1.9. Assume that one of the following holds:*

- $d = (s+1)r$ *(or equivalently,* $d = \frac{rg}{r+1} + r$*);*
- $d = (s+1)r + 1$ *and* $r = np^k - 2$ *for* $k \geq 1$ *and* $1 \leq n < p$*;*
- $d < p + sr$.

Then every $\mathfrak{g}^r_d$ on a general nonsingular projective curve of genus g over an algebraically closed field of characteristic p must be separable. In particular, this is always true if $\rho_0 = 0$.

Proof. If we assume that $d < g + r$ and $a_\bullet$ is the unramified sequence, we have from Example 2.1.9 that $\sigma = sr$ and so $d' = d - sr$. For each of the listed conditions, the triple (r, d', p) satisfies an exceptional case from Theorem 1.1.1, and so the result follows from Corollary 2.1.7.

Furthermore, we have that $\rho_0 = g - (r+1)(g + r - d)$ cannot equal 0 unless $d \leq g + r$. If $d = g + r$, then $\rho_0 = 0$ implies $g = 0$, which in turn gives that $d = r$. Otherwise, we have that $d < g + r$ and we can apply the above result: Because $d - (s+1)r = \rho = \rho_0$, $\rho_0 = 0$ implies that $d = (s+1)r$, which gives the exceptional case $d' = r$. $\square$

2.2. The Value of $\widehat{\rho}$ Assuming Minimal Wild Ramification

In this section, we briefly examine the consequences of the classification of weakly minimal wild indices from the previous chapter for the value of $\widehat{\rho}$.

Let C be a general nonsingular projective curve of genus g over an algebraically closed field of characteristic $p > 0$, P and Q general points of C, and r and $d \geq r$ nonnegative integers such that the triple (r, d, p) does not satisfy one of the exceptional cases from Theorem 1.1.1. Then by Theorem 1.2.4, there exists a weakly minimal wild (r, d)-index of either one or both of the following forms: $\alpha_\bullet = 0, \ldots, 0, t$, where $t \leq d - r$, or $\beta_\bullet = 0, \ldots, 0, \underbrace{1, \ldots, 1}_{s}$. Assuming that $r \leq d - g$, we can then calculate $\widehat{\rho}$ for each of the following spaces:

- For the space $G^r_d(C, (P, \alpha_\bullet), (Q, \alpha_\bullet))$, we have:

$$\widehat{\rho} = \begin{cases} 2(d - r - t) - g & r + t \geq d - g \\ g & r + t \leq d - g \end{cases}$$

- For the space $G^r_d(C, (P, \alpha_\bullet), (Q, \beta_\bullet))$, we have:

$$\widehat{\rho} = \begin{cases} g - t - s & r = d - g \\ d - r - t & r + t \geq d - g > r \\ g & r + t \leq d - g \end{cases}$$

- For the space $G^r_d(C, (P, \beta_\bullet), (Q, \beta_\bullet))$, we have:

$$\widehat{\rho} = \begin{cases} g - 2s & r = d - g \\ g - (2s - (r + 1))^+ & r + 1 = d - g \\ g & r + 2 \leq d - g \end{cases}$$

In the particular case that $r < p \leq d$, we have that $t = p - r$ and that there is no weakly minimal wild index of the form $\beta_\bullet$. As such, we only need to calculate $\widehat{\rho}$ in the first of the above cases. Since $2(d-p) < g$ implies that $d-p < g$, and therefore that $r+t = p > d-g$, we have that $2(d-p) < g$ implies that $\widehat{\rho} < 0$. By Theorem 2.0.1, this in turn implies that the space $G^r_d(C,(P,\alpha_\bullet),(Q,\alpha_\bullet))$ must be empty. Since by Corollary 0.2.4.1, an inseparable $\mathfrak{g}^r_d$ on C is wildly ramified at P and Q, and is therefore contained in this space, we can conclude:

Proposition 2.2.1. *Assuming that $r < p \leq d$ and $2(d-p) < g \leq d-r$, there are no inseparable $\mathfrak{g}^r_d$s on C.*

In the case that $r \geq d-g$, we have that $\widehat{\rho} = \rho = \rho_0 - \sum_{i=0}^r(\alpha_i + \beta_i)$, and so only the respective ramification weights of $\alpha_\bullet$ and $\beta_\bullet$ matter for the calculation of $\widehat{\rho}$. Assuming that neither $\alpha_\bullet$ nor $\beta_\bullet$ is strongly minimal, we have the following from Theorem 1.2.4:

- If $p = 2$ and $d \geq r+2$, then either $\alpha_\bullet$ or $\beta_\bullet$ has ramification weight at most 2.
- If $p > 2$ and $d \geq r+p-1$, then either $\alpha_\bullet$ has ramification weight less than p or $\beta_\bullet$ has ramification weight 2.
- If $p > 2$ and $d \geq r+2$, then either $\beta_\bullet$ has ramification weight less than p or $\alpha_\bullet$ has ramification weight 2.

If we in fact have that $\alpha_\bullet$ is strongly minimal, then it either has ramification weight less than p or ramification weight 2.

CHAPTER 3

Limit Linear Series in Positive Characteristic

Given a projective nodal curve, its *dual graph* is the graph with vertices (respectively, edges) indexed by the components (respectively, nodes) of the curve. A projective nodal curve is *of compact type* if its dual graph is a tree.

Let C_0 be a projective nodal curve of compact type over an algebraically closed field K and let Γ be its dual graph. Given a vertex $v \in V(\Gamma)$ (respectively, an edge $e \in E(\Gamma)$), let C_v (respectively, Q_e) be the corresponding component (respectively, node). We can now generalize Definition 2.1.3 to curves of compact type:

DEFINITION 3.0.1. A *limit linear series* (or *limit* $\mathfrak{g}^r_d$) on C_0 is a tuple of $\mathfrak{g}^r_d$s on each C_v such that: Given $e \in E(\Gamma)$ with $v, v' \in V(\Gamma)$ the adjacent vertices, we have that $a_i^{(e,v)} + a_{r-i}^{(e,v')} \geq d$ for each $0 \leq i \leq r$, where $a_\bullet^{(e,v)}$ is the vanishing sequence at Q_e of the $\mathfrak{g}^r_d$ on C_v (we will refer to these as the *Eisenbud-Harris inequalities*).

A limit linear series is *refined* if $a_i^{(e,v)} + a_{r-i}^{(e,v')} = d$ for every $e \in E(\Gamma)$ and $0 \leq i \leq r$. Otherwise, it is *crude*. The *vanishing sequence* (respectively, *ramification sequence*) of a limit linear series at a nonsingular point is the vanishing sequence (respectively, ramification sequence) of the corresponding $\mathfrak{g}^r_d$ at that point. We observe that if a limit $\mathfrak{g}^r_d$ is refined, the vanishing sequences $a_\bullet^{(e,v)}$ and $a_\bullet^{(e,v')}$ at each node are either both tame or both wild. For this reason, we will refer to a refined limit $\mathfrak{g}^r_d$ being *tamely ramified* or *wildly ramified* at a node.

In this chapter, we will establish some generalizations of the Plücker formula that cover separable limit linear series and give a formula for the number of separable limit $\mathfrak{g}^1_d$s on a genus 0 curve of compact type.

3.1. Some Consequences of The Plücker Formula

We begin by recalling the statement of the Plücker formula (Theorem 0.2.4):

Theorem (Plücker formula). *Given a separable $\mathfrak{g}^r_d$ on a nonsingular projective curve C of genus g over K, we have that*

$$\sum_{P\in C}\sum_{i=0}^{r}\alpha_i(P) \leq (r+1)(d-r)+r(r+1)g$$

with equality if and only if the $\mathfrak{g}^r_d$ is everywhere tamely ramified.

In order to extend this result, we will need to define separability and inseparability in the context of limit $\mathfrak{g}^r_d$s:

Definition 3.1.1. A limit $\mathfrak{g}^r_d$ on C_0 is *separable* (respectively, *quasi-separable*) if it is separable on every (respectively, at least one) component of C_0. Otherwise, it is *inseparable* (respectively, *totally inseparable*).

Given an edge $e \in E(\Gamma)$ with adjacent vertices v and v' and a limit $\mathfrak{g}^r_d$ on C_0, we define the *crudeness index* at the corresponding node of C_0 to be

$$\mathrm{cr}(e) \coloneqq \sum_{i=0}^{r}\alpha_i^{(e,v)} + \sum_{i=0}^{r}\alpha_i^{(e,v')} - (r+1)(d-r)$$

where $\alpha_\bullet^{(e,v)}$ is the ramification sequence on the component of C_0 corresponding to v. We observe that $\mathrm{cr}(e) \geq 0$ with equality for every $e \in E(\Gamma)$ if and only if the limit $\mathfrak{g}^r_d$ is refined.

We can then extend the Plücker formula to C_0 as follows:

Corollary 3.1.2. *Given a separable limit $\mathfrak{g}^r_d$ on C_0, we have that*

$$\sum_{P}\sum_{i=0}^{r}\alpha_i(P) + \sum_{e\in E(\Gamma)}\mathrm{cr}(e) \leq (r+1)(d-r)+r(r+1)g$$

where the first sum is over all nonsingular points $P \in C_0$ and with equality if and only if there is no wild ramification, including at the nodes.

Proof. Given a vertex $v \in V(\Gamma)$, let $E_v(\Gamma)$ be the set of edges adjacent to v. Then if the corresponding component C_v has genus g_v, the Plücker formula gives that

$$\sum_P \sum_{i=0}^r \alpha_i(P) + \sum_{e \in E_v(\Gamma)} \sum_{i=0}^r \alpha_i^{(e,v)} \leq (r+1)(d-r) + r(r+1)g_v$$

where the first sum is over all nonsingular points $P \in C_v$. The result then follows by summing this inequality over $v \in V(\Gamma)$, using that $\sum_{v \in V(\Gamma)} g_v = g$ by [**10**, Corollary 3.1.9] and that $|V(\Gamma)| = |E(\Gamma)| + 1$, and observing that wild ramification at any point on a component (including at a node) gives strict inequality for the Plücker formula on that component and therefore for the sum over all components. $\square$

The proof of the Plücker formula from [**2**, Proposition 1.1] and [**8**, Proposition 2.4] gives that a separable $\mathfrak{g}^r_d$ on a nonsingular projective curve C of genus g over K given by a pair $(\mathscr{L}, V)$ induces a nonzero section s of the line bundle $\mathscr{L}^{\otimes(r+1)} \otimes (\Omega^1_C)^{\otimes\binom{r+1}{2}}$. As in the proof of [**4**, Theorem 3], the order of vanishing of s at a point $P \in C$ (we will call this the *differential weight* $w(P)$ of $(\mathscr{L}, V)$ at P) satisfies the following semicontinuity condition:

Lemma 3.1.3. *Given a family of nonsingular projective curves C_t over K, each with a marked point $P(t) \in C_t$, and a family of separable $\mathfrak{g}^r_d$s on each C_t such that the differential weights $w(P(t))$ are constant for $t \neq 0$, we have that $w(P(0)) \geq w(P(t))$ with equality if and only if there is no ramification specializing to the point $P(0)$ other than along $P(t)$.*

Furthermore, we have that the differential weight at $P \in C$ satisfies $w(P) \geq \sum_{i=0}^r \alpha_i(P)$, with equality if and only if $(\mathscr{L}, V)$ is tamely ramified at P. In particular:

Corollary 3.1.4. *In the situation of Lemma 3.1.3, if the ramification along $P(t)$ is tame, then we achieve the result of Lemma 3.1.3 with the ramification weight $\sum_{i=0}^r \alpha_i(P(t))$ in place of the differential weight $w(P(t))$.*

Example 3.1.4.1. This result fails if the ramification along $P(t)$ is wild: Assuming that K has characteristic $p > 0$, let $\mathbb{P}^1_t$ be a genus 0 family over K and define a $\mathfrak{g}^1_{p+1}$ on each $\mathbb{P}^1_t$

by the map $x \mapsto x^{p+1} + tx^p$. We have that for $t \neq 0$, each $\mathfrak{g}^1_{p+1}$ is wildly ramified to order[1] p at 0, but tamely ramified to order $p+1$ at 0 on $\mathbb{P}^1_0$. In particular, the ramification weight at 0 increases while, as indicated by Lemma 3.1.3, the differential weight remains constant.

Remark 3.1.5. In the case of an inseparable $\mathfrak{g}^r_d$ on C, we have from [**8**, Proposition 2.4] that the section s is the zero section, and so we will consider $w(P) = \infty$ for every $P \in C$.

If we consider a limit $\mathfrak{g}^r_d$ on C_0 and an edge $e \in E(\Gamma)$ with adjacent vertices v and v', we can define the *differential crudeness index* at the node corresponding to e to be

$$\operatorname{dcr}(e) := w(e, v) + w(e, v') - (r+1)(d-r)$$

where $w(e, v)$ is the differential weight on the component corresponding to v. We observe that $\operatorname{dcr}(e) \geq \operatorname{cr}(e)$ with equality if and only if the limit $\mathfrak{g}^r_d$ is tamely ramified at that node on both components. In particular, this gives that $\operatorname{dcr}(e) \geq 0$, with equality for every $e \in E(\Gamma)$ if and only if the limit $\mathfrak{g}^r_d$ is refined and tamely ramified at every node. We can use the differential crudeness index to restate Corollary 3.1.2:

Lemma 3.1.6. *Given a separable limit $\mathfrak{g}^r_d$ on C_0, we have that*

$$\sum_P w(P) + \sum_{e \in E(\Gamma)} \operatorname{dcr}(e) = (r+1)(d-r) + r(r+1)g$$

where the first sum is over all nonsingular points $P \in C_0$.

Proof. The proof of the Plücker formula from [**2**, Proposition 1.1] shows that given a separable $\mathfrak{g}^r_d$ on a nonsingular projective curve C of genus g, we have that

$$\sum_{P \in C} w(P) = (r+1)(d-r) + r(r+1)g$$

As in the proof of Corollary 3.1.2, we can apply this result to each component of C_0 and then sum to achieve the final result. □

[1]We define the order of ramification of a $\mathfrak{g}^1_d$ at a point in the next section (3.2).

3.2. Separable Limit Linear Series on a Genus 0 Curve

If C is a nonsingular projective curve over K, a basepoint free $\mathfrak{g}^1_d$ on C with vanishing sequence $0, a_1(P)$ at a point $P \in C$ corresponds (cf. Section 0.2.2) to a morphism $C \to \mathbb{P}^1$ which is *ramified to order* $e_P \coloneqq a_1(P)$ at P. As such, given points $P_1, \ldots, P_n$ of C and positive integers $e_1, \ldots, e_n$ with $e_i \leq d$ for each i, we will use the notation $G^1_d(C, \{P_i, e_i\}_i)$ to denote the space of $\mathfrak{g}^1_d$s on C which are ramified to order e_i at P_i.

Lemma 3.2.1. *Given a refined limit $\mathfrak{g}^1_d$ on C_0 that is basepoint free except at the nodes, subtracting off the basepoints at the nodes induces a bijection between such limit $\mathfrak{g}^1_d$s and tuples of maps to $\mathbb{P}^1$ on each component such that:*

- *The order of ramification at each node is the same on both adjacent components;*
- *If d_v is the degree of the map on the component corresponding to the vertex $v \in V(\Gamma)$, we have that*

$$\sum_{v \in V(\Gamma)} d_v - \sum_{e \in E(\Gamma)} e_e = d$$

  where e_e is the order of ramification at the node corresponding to the edge $e \in E(\Gamma)$.

Proof. Consider a refined limit $\mathfrak{g}^1_d$ on C_0 with no basepoints other than at the nodes. Then if $e \in E(\Gamma)$ is an edge with adjacent vertices $v, v' \in V(\Gamma)$ and $a^{(e,v)}_\bullet$ is the corresponding vanishing sequence, we have that $a^{(e,v)}_0 + a^{(e,v')}_1 = d = a^{(e,v')}_0 + a^{(e,v)}_1$. As such, the node corresponding to e is ramified to order $e_e \coloneqq a^{(e,v)}_1 - a^{(e,v)}_0 = a^{(e,v')}_1 - a^{(e,v')}_0$ on each of the adjacent components. Furthermore, subtracting off the basepoints at the nodes gives that the degree d_v of the induced map on the component corresponding to the vertex v is given by $d_v = d - \sum_{e \in E_v(\Gamma)} a^{(e,v)}_0$, where the sum is over all edges e adjacent to v. Then

$$\sum_{v \in V(\Gamma)} d_v = \sum_{v \in V(\Gamma)} \left(d - \sum_{e \in E_v(\Gamma)} a^{(e,v)}_0 \right) = |V(\Gamma)| d - \sum_{v \in V(\Gamma)} \sum_{e \in E_v(\Gamma)} a^{(e,v)}_0$$

For every edge e, we can label one of its adjacent vertices as v and the other as v'. Then, using that $|V(\Gamma)| = |E(\Gamma)| + 1$, we have

$$\begin{aligned}
\sum_{v\in V(\Gamma)} d_v &= |V(\Gamma)|d - \sum_{v\in V(\Gamma)} \sum_{e\in E_v(\Gamma)} a_0^{(e,v)} \\
&= d + |E(\Gamma)|d - \sum_{e\in E(\Gamma)} \left(a_0^{(e,v)} + a_0^{(e,v')}\right) \\
&= d + \sum_{e\in E(\Gamma)} \left(d - a_0^{(e,v')}\right) - \sum_{e\in E(\Gamma)} a_0^{(e,v)} \\
&= d + \sum_{e\in E(\Gamma)} a_1^{(e,v)} - \sum_{e\in E(\Gamma)} a_0^{(e,v)} \\
&= d + \sum_{e\in E(\Gamma)} e_e
\end{aligned}$$

Conversely, consider a tuple of maps to $\mathbb{P}^1$ on each component with the given properties, and fix an edge e with adjacent vertices v and v'. If we remove v from Γ, we can associate every edge with one of its adjacent vertices and thus define a bijection $\varphi_v \colon V(\Gamma)\setminus\{v\} \to E(\Gamma)$ (in particular, $\varphi_v(v') = e$). Next, because Γ is a tree, we have that the subgraph obtained by removing the edge e consists of two components, $Y_{(e,v)}$ (which we can assume contains v) and $Z_{(e,v)}$ (which will then contain v'). We can then define

$$a_0^{(e,v)} := d - d_v - \sum_{w\in V\left(Y_{(e,v)}\right)\setminus\{v\}} \left(d_w - e_{\varphi_v(w)}\right) = \sum_{w\in V\left(Z_{(e,v)}\right)} \left(d_w - e_{\varphi_v(w)}\right)$$

Since $e_{\varphi_v(w)} \leq d_w$ for every w, we have that $0 \leq a_0^{(e,v)} < d$. If we then define

$$a_1^{(e,v)} := e_e + a_0^{(e,v)} = d - (d_v - e_e) - \sum_{w\in V\left(Y_{(e,v)}\right)\setminus\{v\}} \left(d_w - e_{\varphi_v(w)}\right)$$

we have that $a_0^{(e,v)} < a_1^{(e,v)} \leq d$. Furthermore, we have that

$$a_0^{(e,v')} + a_1^{(e,v)} = d - (d_v - e_e) - \sum_{w\in V\left(Y_{(e,v)}\right)\setminus\{v\}} \left(d_w - e_{\varphi_v(w)}\right) + \sum_{w\in V\left(Z_{(e,v')}\right)} \left(d_w - e_{\varphi_{v'}(w)}\right)$$

Since $V\left(Z_{(e,v')}\right) = V\left(Y_{(e,v)}\right)$ and $\varphi_{v'}(v) = e$, this equation reduces to $a_0^{(e,v')} + a_1^{(e,v)} = d$.

Finally, given any vertex v, we have that

$$\begin{aligned} d_v + \sum_{e \in E_v(\Gamma)} a_0^{(e,v)} &= d_v + \sum_{e \in E_v(\Gamma)} \sum_{w \in V\left(Z_{(e,v)}\right)} \left(d_w - e_{\varphi_v(w)}\right) \\ &= d_v + \sum_{w \in V(\Gamma)\setminus\{v\}} \left(d_w - e_{\varphi_v(w)}\right) \\ &= d \end{aligned}$$

As such, by adding basepoints at the nodes such that the node corresponding to the edge e has vanishing sequence $a_\bullet^{(e,v)}$ on the component corresponding to the vertex v, we define a refined limit $\mathfrak{g}_d^1$ on C_0 which is basepoint free at all other points. $\square$

For the remainder of this section, assume that C_0 is a general n-marked curve of genus 0 (i.e. C_0 has marked points $P_1, \ldots, P_n$ such that the marked points and nodes on each component are general) and, given $r \leq d$, let $a_\bullet^1, \ldots, a_\bullet^n$ be (r,d)-sequences (with corresponding (r,d)-indices $\alpha_\bullet^j$) such that $\sum_{j=1}^n \sum_{i=0}^r \alpha_i^j = (r+1)(d-r)$. This condition is equivalent to the expected dimension of the space of limit linear series $G_d^r(C_0, \{(P_j, a_\bullet^j)\}_j)$ being $\rho = 0$.

Lemma 3.2.2. *A separable limit $\mathfrak{g}_d^r$ on C_0 with vanishing sequence $a_\bullet^j$ at P_j is refined and unramified at every point which is not a node or one of the P_j.*

Proof. We can apply Corollary 3.1.2 with $g = 0$ to obtain that

$$(r+1)(d-r) = \sum_{j=1}^n \sum_{i=0}^r \alpha_i^j \leq \sum_P \sum_{i=0}^r \alpha_i(P) + \sum_{e \in E(\Gamma)} \mathrm{cr}(e) \leq (r+1)(d-r)$$

where the second sum is over all nonsingular points $P \in C_0$. We can immediately conclude that all the above inequalities must in fact be equalities, which says exactly that $\mathrm{cr}(e) = 0$ for every $e \in E(\Gamma)$ and that if P is a nonsingular point other than one of the P_j, $\alpha_i(P) = 0$ for every $0 \leq i \leq r$. As such, the result follows. $\square$

Next, assume that K has characteristic $p > 2$ and let $e_1, \ldots, e_n$ be positive integers satisfying $e_i \leq d$, $e_i < p$, and $\sum_i (e_i - 1) = 2d - 2$.

THEOREM 3.2.3. *Assuming that $n \geq 3$, the number of separable limit $\mathfrak{g}^1_d$s on C_0 ramified to e_i at P_i is equal to the number of $(n-3)$-tuples of positive integers $e'_2, \ldots, e'_{n-2}$ such that if e, e', e'' is the triple e_1, e_2, e'_2, the triple e'_{n-2}, e_{n-1}, e_n, or a triple of the form e'_{i-1}, e_i, e'_i, then:*

- *$e + e' + e''$ is odd and less than $2p$;*
- *e, e', and e'' satisfy the triangle inequality.*

REMARK 3.2.3.1. This extends Osserman's result [**12**, Theorem 1.4] which establishes this formula as the number of separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$ ramified to e_i at general points $Q_i \in \mathbb{P}^1$.

We will prove this first in the case that C_0 is the curve C_n consisting of a chain of $n-2$ copies of $\mathbb{P}^1$ glued at nodes $P'_2, \ldots, P'_{n-2}$ such that P_1 and P_2 are on the first component, P_{n-1} and P_n are on the last component, and P_i is on the component containing P'_{i-1} and P'_i (i.e. each component contains exactly three of the P_i and P'_i).

PROPOSITION 3.2.4. *The result of Theorem 3.2.3 holds if C_0 is the chain C_n.*

PROOF. If $n = 3$, then the curve C_3 is just $\mathbb{P}^1$ with three marked points and the result follows from [**12**, Theorem 1.4]. Assuming that $n \geq 4$, consider a separable limit $\mathfrak{g}^1_d$ on C_n ramified to e_i at P_i. By Lemma 3.2.2, this limit $\mathfrak{g}^1_d$ is refined and unramified except at the points P_i and P'_i. In particular, by Lemma 3.2.1, each node P'_i is ramified to the same order e'_i on both adjacent components. We can then check that the tuple $e'_2, \ldots, e'_{n-2}$ satisfies the desired conditions: If d_i is the degree of the induced map on the component containing the point P_i for $3 \leq i \leq n-2$, the Riemann-Hurwitz formula [**10**, Example 2.3.6] gives that

$$(e'_{i-1} - 1) + (e_i - 1) + (e'_i - 1) = 2d_i - 2$$

so we have $e'_{i-1}+e_i+e'_i = 2d_i+1$. By [**12**, Theorem 4.2], there exists a separable $\mathfrak{g}^1_{d_i}$ ramified to e'_{i-1} at P'_{i-1}, e_i at P_i, and e'_i at P'_i on this component only if $d_i < p$. As such,

$$2d_i + 1 \leq 2(p-1) + 1 = 2p - 1$$

so $e'_{i-1} + e_i + e'_i$ is odd and less than $2p$. Furthermore, since e'_{i-1}, e_i, and e'_i each do not exceed d_i, we have that

$$e'_{i-1} + e_i - e'_i = (e'_{i-1} + e_i + e'_i) - 2e'_i = 2(d_i - e'_i) + 1 \geq 1$$

which establishes that $e'_i \leq e'_{i-1} + e_i$ (and similarly, $e_i \leq e'_{i-1} + e'_i$ and $e'_{i-1} \leq e_i + e'_i$). Finally, the equations $e_1 + e_2 + e'_2 = 2d_2 + 1$ and $e'_{n-2} + e_{n-1} + e_n = 2d_{n-1} + 1$ establish the corresponding result for the triples e_1, e_2, e'_2 and e'_{n-2}, e_{n-1}, e_n.

Conversely, consider a tuple of positive integers $e'_2, \ldots, e'_{n-2}$ with the desired properties. Given $3 \leq i \leq n-2$, we have that $e'_{i-1} + e_i + e'_i$ is at least 3 and odd, so we can write $e'_{i-1} + e_i + e'_i = 2d_i + 1$ for some positive integer d_i. By the triangle inequality, we have that

$$e'_i \leq e'_{i-1} + e_i = (e'_{i-1} + e_i + e'_i) - e'_i = 2d_i + 1 - e'_i$$

which implies that $e'_i \leq d_i$ (and similarly, $e'_{i-1} \leq d_i$ and $e_i \leq d_i$). Furthermore, since $e'_{i-1} + e_i + e'_i < 2p$, we have that $2d_i + 1 \leq 2p - 1$, so $d_i < p$. Then [**12**, Theorem 4.2] gives that there exists a unique separable $\mathfrak{g}^1_{d_i}$ ramified to e'_{i-1} at P'_{i-1}, e_i at P_i, and e'_i at P'_i on the component containing the point P_i. By repeating this process with the triples e_1, e_2, e'_2 and e'_{n-2}, e_{n-1}, e_n, we similarly achieve positive d_2 and d_{n-1} such that there exists a unique separable $\mathfrak{g}^1_{d_2}$ on the first component and a unique separable $\mathfrak{g}^1_{d_{n-1}}$ on the last component with the correct ramification.

Finally, by summing the equations

$$(e_1 - 1) + (e_2 - 1) + (e'_2 - 1) = 2d_2 - 2$$
$$(e'_{i-1} - 1) + (e_i - 1) + (e'_i - 1) = 2d_i - 2 \text{ for } 3 \leq i \leq n-2$$
$$(e'_{n-2} - 1) + (e_{n-1} - 1) + (e_n - 1) = 2d_{n-1} - 2$$

and using that $\sum_i (e_i - 1) = 2d - 2$, we obtain

$$2d - 2 + 2\sum_{i=2}^{n-2}(e'_i - 1) = \sum_{i=2}^{n-1}(2d_i - 2)$$

which can be rearranged to become

$$d + e'_2 + \cdots + e'_{n-2} = d_2 + \cdots + d_{n-1}$$

By Lemma 3.2.1, this says exactly that our $\mathfrak{g}^1_{d_i}$s on the components of C_n determine a refined limit $\mathfrak{g}^1_d$ which is separable on every component of C_n and ramified to e_i at P_i. We have therefore established a correspondence between such limit $\mathfrak{g}^1_d$s and tuples $e'_2, \ldots, e'_{n-2}$ with the desired properties, and the result follows. □

Remark 3.2.5. Because the number specified in Theorem 3.2.3 is the number of separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$ ramified to order e_i at Q_i, where $Q_1, \ldots, Q_n$ are any general marked points, we have that this number is invariant under any reordering of the e_i, and so we can assign it the notation $N(e_1, \ldots, e_n)$ without ambiguity.

Proof of Theorem 3.2.3. Assume that C_0 has a component $\mathbb{P}^1_v$ (corresponding to a vertex v of Γ) containing $m \geq 4$ combined marked points and nodes. We have by Lemma 3.2.2 that a separable limit $\mathfrak{g}^1_d$ on C_0 ramified to e_i at P_i is refined, and therefore has a well-defined ramification order at each node. For each tuple $\{e_e\}_{e \in E(\Gamma)}$ of such orders, we have by Proposition 3.2.4 and Remark 3.2.5 that there are the same number of separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1_v$ ramified to e_i at P_i and to e_e at every node corresponding to an edge $e \in E_v(\Gamma)$ as separable limit $\mathfrak{g}^1_d$s on C_m with the same ramification. In particular, we can replace $\mathbb{P}^1_v$ with the curve C_m without changing the number of separable limit $\mathfrak{g}^1_d$s on C_0. As such, we can assume that every component of C_0 contains exactly three marked points and nodes.

We can then complete the proof by induction on n: If $n = 3$, then the result follows by [**12**, Theorem 1.4]. Otherwise, choose a component of C_0 containing two marked points (by Remark 3.2.5, we can label these points as P_{n-1} and P_n) and a node P. As above, a separable limit $\mathfrak{g}^1_d$ on C_0 ramified to e_i at P_i has a well-defined ramification order e at P. We can then remove this component from C_0 (replacing it with a marked point at P) and use the inductive hypothesis to calculate the number of such limit $\mathfrak{g}^1_d$s as $\sum_e N(e_1, \ldots, e_{n-2}, e)$, where the sum is over all possible values of e. We have that $N(e_1, \ldots, e_{n-2}, e)$ is the number of tuples $e'_2, \ldots, e'_{n-3}$ satisfying the conditions listed in Theorem 3.2.3—in particular, that the triple e'_{n-3}, e_{n-2}, e has odd sum less than $2p$ and satisfies the triangle inequality. Since the proof of Proposition 3.2.4 shows that the triple e, e_{n-1}, e_n also has this property, we therefore have (by relabeling e as e'_{n-2}) that

$$\sum_e N(e_1, \ldots, e_{n-2}, e) = N(e_1, \ldots, e_n)$$

and the result follows. $\square$

CHAPTER 4

Results on Families of Linear Series

In this chapter, we will examine some specific results of Eisenbud and Harris regarding the behavior of families of linear series. We will begin by constructing a limit $\mathfrak{g}^1_d$ that cannot be obtained as the limit of any family of $\mathfrak{g}^1_d$s with given ramification (by [**3**, Proposition 3.1], such a limit $\mathfrak{g}^1_d$ does not exist in characteristic 0). We will then examine what happens when a family of linear series specializing to a limit linear series on a curve of compact type has ramification specializing to a node.

4.1. A Non-Smoothable Limit Linear Series

Let K be an algebraically closed field of characteristic $p > 2$, A a discrete valuation ring with residue field K, $B = \operatorname{Spec} A$, and $\pi\colon X \to B$ a family of curves such that:

- π is flat and proper and has sections $P_1, \ldots, P_n$;
- The total space X is nonsingular;
- If b_0 and b_1 are the closed and generic points of B, respectively, then the fiber $X_1 = X|_{b_1}$ is nonsingular and the fiber $X_0 = X|_{b_0}$ is a curve of compact type.

Then given integers $r \leq d$ and (r, d)-sequences $a^1_\bullet, \ldots, a^n_\bullet$, we have from [**10**, Theorem 4.1.5] that there exists a space $G^r_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ whose fiber over b_1 (respectively, b_0) is isomorphic to the space of $\mathfrak{g}^r_d$s (respectively, limit $\mathfrak{g}^r_d$s) on X_1 (respectively, X_0) with vanishing sequence at least $a^i_\bullet$ at P_i.

Lemma 4.1.1. *The space $G^r_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ has an open subspace $G^{r,\mathrm{sep}}_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ (respectively, $G^{r,\mathrm{qs}}_d(X/B, \{(P_i, a^i_\bullet)\}_i)$) whose fiber over b_1 contains separable $\mathfrak{g}^r_d$s on X_1 and whose fiber over b_0 contains separable (respectively, quasi-separable) limit $\mathfrak{g}^r_d$s on X_0.*

Proof. Since by [**10**, Theorem 4.1.5] the space $G^r_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ is compatible with base-change, we can assume by [**1**, Proposition 2.2.14] that the family X/B has a section specializing to every component of the special fiber X_0. Then, following the proof of [**11**, Theorem II.4.3], we can define the inseparable (respectively, totally inseparable) subspace of $G^r_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ fiber-by-fiber as the union (respectively, intersection) of the closed subspaces obtained by imposing enough ramification along the sections specializing to each component of the fiber X_0 such that the Plücker formula forces inseparability on that component. We therefore obtain closed subspaces which contain inseparable $\mathfrak{g}^r_d$s on the fiber X_1 and inseparable (respectively, totally inseparable) limit $\mathfrak{g}^r_d$s on the fiber X_0. Since the complement of this space is then the space $G^{r,\text{sep}}_d(X/B, \{(P_i, a^i_\bullet)\}_i)$ (respectively, $G^{r,\text{qs}}_d(X/B, \{(P_i, a^i_\bullet)\}_i)$), the result follows. □

Proposition 4.1.2. *Assume that X_1 is isomorphic to $\mathbb{P}^1$ and that X_0 is a general n-marked curve of genus 0, and let $e_1, \dots, e_n$ be positive integers satisfying $e_i \leq d$, $e_i < p$, and $\sum_i (e_i - 1) = 2d - 2$. Then if $\overline{G^1_d}$ denotes the closure of the fiber $G^{1,\text{sep}}_d(X_1, \{(P_i, e_i)\}_i)$ inside $G^1_d(X/B, \{(P_i, e_i)\}_i)$, the fiber of $\overline{G^1_d}$ over b_0 contains only separable limit $\mathfrak{g}^1_d$s.*

Proof. We begin by observing the following:

- By Lemma 4.1.1, $G^{1,\text{sep}}_d(X/B, \{(P_i, e_i)\}_i)$ is an open subspace of $G^1_d(X/B, \{(P_i, e_i)\}_i)$.
- By [**12**, Theorem 1.4], the fiber $G^{1,\text{sep}}_d(X_1, \{(P_i, e_i)\}_i)$ contains a finite number of $\mathfrak{g}^1_d$s, and by Theorem 3.2.3, the fiber $G^{1,\text{sep}}_d(X_0, \{(P_i, e_i)\}_i)$ contains the same (finite) number of limit $\mathfrak{g}^1_d$s.
- In particular, each fiber of $G^{1,\text{sep}}_d(X/B, \{(P_i, e_i)\}_i)$ has the expected dimension $\rho = 0$.

We can then apply "smoothing" [**10**, Corollary 4.1.7] along with [**7**, Proposition 3.2.7] to see that the space $G^{1,\text{sep}}_d(X/B, \{(P_i, e_i)\}_i)$ is universally open and flat over B. In particular, this implies that none of the separable limit $\mathfrak{g}^1_d$s on X_0 are isolated, and so each is contained in the fiber of $\overline{G^1_d}$ over b_0. By [**5**, Corollary 15.5.2], the number of limit $\mathfrak{g}^1_d$s in the fiber of $\overline{G^1_d}$ over b_0 is at most the number of $\mathfrak{g}^1_d$s in the fiber over b_1, that is, the number of separable $\mathfrak{g}^1_d$s on X_1 ramified to e_i at P_i. Since this number is equal to the number of separable limit $\mathfrak{g}^1_d$s on X_0 with the same ramification conditions, we can therefore conclude that they are the only limit $\mathfrak{g}^1_d$s in the fiber of $\overline{G^1_d}$ over b_0. □

Remark 4.1.2.1. Since $\overline{G^1_d}$ is also proper over B (this follows from [**10**, Theorem 4.1.5]), we have in fact that every separable $\mathfrak{g}^1_d$ on X_1 specializes to a limit $\mathfrak{g}^1_d$ on X_0.

We have from Lemma 4.1.1 that the totally inseparable subspace of $G^1_d(X/B, \{(P_i, e_i)\}_i)$ is closed, so an inseparable $\mathfrak{g}^1_d$ on X_1 must specialize to a totally inseparable limit $\mathfrak{g}^1_d$ on X_0. We can therefore conclude the following:

Corollary 4.1.3. *Under the assumptions of Proposition 4.1.2, the fiber over b_0 of the closure of $G^1_d(X_1, \{(P_i, e_i)\}_i)$ contains only separable or totally inseparable limit $\mathfrak{g}^1_d$s.*

Example 4.1.4 (A Non-Smoothable Limit $\mathfrak{g}^1_d$). Let C_0 be a curve consisting of two copies of $\mathbb{P}^1$ glued at 0 (we will denote these as $\mathbb{P}^1_s$ and $\mathbb{P}^1_i$). Then, assuming that $p > 3$, choose general points $P_1, \ldots, P_n$ on $\mathbb{P}^1_s$ and Q_1, Q_2 on $\mathbb{P}^1_i$ (with none of the P_i or Q_j equal to 0 or ∞) and integers $e_1, \ldots, e_n$ such that $2 \leq e_i < p$ and $\sum_i (e_i - 1) \leq p - 2$.

If we let $g(x)$ be a formal antiderivative of the polynomial $(x - P_1)^{e_1 - 1} \cdots (x - P_n)^{e_n - 1}$ (g is then a nonconstant polynomial of degree at most $p - 1$), we can define $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ to be the Frobenius map $x \mapsto x^p$ and $G\colon \mathbb{P}^1 \to \mathbb{P}^1$ the map $x \mapsto \frac{g(x)}{x^p}$. Then F is an inseparable map of degree p ramified to order p at every point of $\mathbb{P}^1$, while by [**11**, Lemma I.4.1], G is a separable map of degree p ramified to order p at 0, to order e_i at P_i, and to order $e_\infty \coloneqq p - \deg g$ at ∞. As such, by Lemma 3.2.1 we can define a refined limit $\mathfrak{g}^1_p$ on C_0 by using the map F on $\mathbb{P}^1_i$ and the map G on $\mathbb{P}^1_s$ (this limit $\mathfrak{g}^1_p$ is, in particular, quasi-separable but not separable).

Since $\sum_i (e_i - 1) = \deg g - 1$, setting $e'_1 = p - 1$ and $e'_2 = 3$ gives that

$$\sum_{i=1}^{n} (e_i - 1) + (e_\infty - 1) + \sum_{j=1}^{2} (e'_j - 1) = 2p - 2$$

Then Corollary 4.1.3 gives that for any genus 0 family of curves degenerating to C_0 with sections specializing to the P_i (including $P_\infty = \infty$) and Q_j, it is not possible for a family of $\mathfrak{g}^1_p$s which are ramified to e_i and e'_j along the sections corresponding to P_i and Q_j, respectively, to specialize to this limit $\mathfrak{g}^1_p$.

Remark 4.1.4.1. One example of such a genus 0 family of curves is the family of $\mathbb{P}^1$s given by hyperbolas $xy = t$ inside $\mathbb{P}^1 \times \mathbb{P}^1$ which degenerates to the union of the axes at $t = 0$. Specifically, each $\mathbb{P}^1$ is a fiber of the projection $X \to \mathbb{A}^1$, where $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$ is defined by the equation $x_1y_1 = tx_0y_0$. Each $\mathbb{P}^1$ is isomorphic to each axis by projection to that axis, and we can use these isomorphisms to construct sections corresponding to the marked points on the axes.

Remark 4.1.4.2. Each of the assumptions of Proposition 4.1.2 are needed for this result, most importantly that the family of $\mathfrak{g}^1_d$s preserves imposed ramification. In Section 4.3, we give an example of a quasi-separable limit $\mathfrak{g}^1_d$ of the type constructed above that can be obtained as the limit of a family of $\mathfrak{g}^1_d$s that does not satisfy this assumption. Furthermore, in the next section (4.2), we show that the result fails for a family of higher genus curves, thus establishing the necessity of the genus 0 assumption. Finally, we have by [**3**, Example 3.2] that the result fails for families of $\mathfrak{g}^r_d$s if $r > 1$, even in characteristic 0.

4.2. Fully Ramified Linear Series on Curves of Higher Genus

Given positive integers $d \geq 2$ and $g = 2d - 2$, we have from [**10**, Theorem 2.6.2] that the number of $\mathfrak{g}^1_d$s on a general curve of genus g over K is given by[1]

$$\frac{(2d-2)!}{(d-1)!d!} = \frac{1}{d}\binom{2d-2}{d-1} \tag{4.1}$$

This situation corresponds to a Brill-Noether number of $\rho_0 = 0$, so we furthermore have by Corollary 2.1.10 that they all must be separable.

[1]This quantity is the Catalan number $C_{d-1} = \binom{2d-2}{d-1} - \binom{2d-2}{d}$ and therefore an integer.

We next consider the nodal curve C_0 consisting of a copy of $\mathbb{P}^1$ with g elliptic curves glued at general points $Q_1, \dots, Q_g$.

Proposition 4.2.1. *A limit $\mathfrak{g}^1_d$ on C_0 is separable if and only if it is separable on the $\mathbb{P}^1$, in which case it is refined and ramified to order 2 at each node (and otherwise unramified) on the $\mathbb{P}^1$, while each elliptical tail has vanishing sequence $d-2, d$ at the node and is ramified to order 2 at three other points.*

Proof. By Remark 0.1.1.1, the degree of the basepoint-free linear series on each elliptical tail must be at least 2, and so the vanishing sequence at the node can be at most $d-2, d$. The Eisenbud-Harris inequalities then give that each node on the $\mathbb{P}^1$ must be ramified to order at least 2. By the Plücker formula, (tame) ramification to order 2 at $g = 2d-2$ nodes uses all the allowed ramification for a separable $\mathfrak{g}^1_d$ on $\mathbb{P}^1$. As such, separability on the $\mathbb{P}^1$ forces a vanishing sequence of exactly $d-2, d$ at the node, and therefore separability, on each elliptical tail, with the remaining ramification following from the Plücker formula. $\square$

We can observe further that a $\mathfrak{g}^1_2$ on an elliptic curve C ramified to order 2 at a point Q_j must be the complete linear series given by the line bundle $\mathscr{O}_C(2Q_j)$ (together with all of its global sections) and is therefore unique. This gives that, as in the proof of [**13**, Theorem 2.5], we can identify the space $G^{1,\mathrm{sep}}_d(C_0)$ with the space $G^{1,\mathrm{sep}}_d(\mathbb{P}^1, \{(Q_i, 2)\}_i)$ of separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$ that are ramified to order 2 at g general points.

Corollary 4.2.2. *The number of separable limit $\mathfrak{g}^1_d$s on C_0 does not exceed the quantity given in (4.1), above, with equality if and only if $d < p$.*

Proof. By [**12**, Theorem 1.4], the number of separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$ ramified to order 2 at g general points is given by $N(\underbrace{2, \dots, 2}_{g})$ from Remark 3.2.5. By Theorem 3.2.3, we can describe $N(\underbrace{2, \dots, 2}_{g})$ as the number of $(g-3)$-tuples of positive integers $e'_2, \dots, e'_{g-2}$ such that the triples $2, 2, e'_2$, $e'_{g-2}, 2, 2$, and $e'_{i-1}, 2, e'_i$ for $3 \le i \le g-2$ have odd sum less than $2p$

and satisfy the triangle inequality. In particular, we can use the triangle inequality to see that e'_2 and e'_{g-2} can be at most 3, which gives in turn that e'_3 and e'_{g-3} can be at most 4. Because $g = 2d-2$, continuing this process inductively gives maximum possible values of $e'_{\frac{g}{2}\pm 1} = d-1$ and $e'_{\frac{g}{2}} = d$. As such, the triple with the largest sum is $e'_{\frac{g}{2}-1} + 2 + e'_{\frac{g}{2}} = 2d+1$, which is smaller than $2p$ exactly when $d < p$.

In the case that $d < p$, there are no inseparable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$, and so we can apply the characteristic 0 result from [**13**, Corollary 5.6] to achieve the equality of $N(\underbrace{2,\ldots,2}_{g})$ and (4.1). We can therefore conclude that (4.1) counts all separable $\mathfrak{g}^1_d$s on $\mathbb{P}^1$ ramified to order 2 at g general points when $d < p$, but that there are strictly fewer if $d \geq p$. $\square$

We can now use this result to give a higher-genus counterexample to Proposition 4.1.2. By Winters's theorem [**10**, Theorem 3.4.4], we can construct a genus g family of curves X/B such that the total space X, the curve B, and the generic fiber X_1 are nonsingular and the special fiber X_0 (over a point $b_0 \in B$) is isomorphic to the curve C_0.

Proposition 4.2.3. *In the case that $d \geq p$, the fiber over b_0 of the closure $\overline{G^1_d(X_1)}$ inside $G^1_d(X/B)$ contains an inseparable limit $\mathfrak{g}^1_d$.*

Proof. Because the base B is a nonsingular curve, we have that the closure $\overline{G^1_d(X_1)}$ is finite and flat, and therefore that its fiber over b_0 has the same degree as its generic fiber, given by (4.1). As above, we can identify the space $G^{1,\text{sep}}_d(X_0)$ with the space $G^{1,\text{sep}}_d(\mathbb{P}^1, \{(Q_i, 2)\}_i)$, which by [**12**, Theorem 1.4] contains $N(\underbrace{2,\ldots,2}_{g})$ reduced points. We can then use Corollary 4.2.2 to conclude that the degree of $G^{1,\text{sep}}_d(X_0)$ is strictly smaller than (4.1), and therefore that $G^{1,\text{sep}}_d(X_0)$ cannot contain the special fiber of $\overline{G^1_d(X_1)}$. As such, the result follows. $\square$

Because there are no inseparable $\mathfrak{g}^1_d$s on X_1, we can therefore conclude:

Corollary 4.2.4. *Assuming that $d \geq p$, there exists a family of separable $\mathfrak{g}^1_d$s on X_1 which specializes to an inseparable limit $\mathfrak{g}^1_d$ on X_0.*

4.3. When Ramification Specializes to a Node

If we again consider the family of curves $\pi\colon X \to B$ from Section 4.1, we will now seek to determine what happens when a family of $\mathfrak{g}^r_d$s specializing to a limit $\mathfrak{g}^r_d$ on X_0 has ramification specializing to a node of X_0.

Because it is not possible for ramification to specialize to a node of X_0 along a section of π if X is nonsingular, we will use the following result (cf. [**3**, Theorem 2.6]):

Lemma 4.3.1. *By making a base change and blowing up a node Q of X_0 to resolve the resulting singularities, we obtain a family $\pi'\colon X' \to B'$ whose generic fiber X'_1 is isomorphic to X_1 and whose special fiber X'_0 is derived from X_0 by inserting of a chain of $\mathbb{P}^1$s at the node Q such that any ramification initially specializing to Q now specializes along a section of π' to one of the inserted components (with all other components of X_0 unaffected).*

Our main result is then the following:

Proposition 4.3.2. *If the limit $\mathfrak{g}^r_d$ on X_0 is refined and tamely ramified at a node Q, then the limit $\mathfrak{g}^r_d$ on X'_0 resulting from blowing up Q as in Lemma 4.3.1 is unramified on every inserted component except at the nodes. Conversely, if it is wildly ramified at Q, the limit $\mathfrak{g}^r_d$ on X'_0 is inseparable on every inserted component.*

Proof. Assume that the limit $\mathfrak{g}^r_d$ on X_0 is refined and let $\alpha_\bullet$ and $\beta_\bullet$ be the ramification sequences at Q. Assuming that blowing up the node Q inserts a chain of n copies of $\mathbb{P}^1$ glued at nodes $P_1, \ldots, P_{n-1}$, let $\alpha^1_\bullet, \ldots, \alpha^n_\bullet$ and $\beta^1_\bullet, \ldots, \beta^n_\bullet$ be (r,d)-indices such that the limit $\mathfrak{g}^r_d$ on X'_0 has ramification sequences $\alpha^j_\bullet$ at P_j and $\beta^j_\bullet$ at P_{j-1} on the jth component (where we let P_0 and P_n be the nodes on the ends of the chain). If we set $\alpha^0_\bullet \coloneqq \alpha_\bullet$ and $\beta^{n+1}_\bullet \coloneqq \beta_\bullet$, the Eisenbud-Harris inequalities at the node P_j for $0 \le j \le n$ give us that

$$\sum_{i=0}^{r} \alpha^j_i + \sum_{i=0}^{r} \beta^{j+1}_i \ge (r+1)(d-r) \tag{4.2}$$

Furthermore, if we let ρ_j be the Brill-Noether number associated to the jth component for $1 \leq j \leq n$, we have that

$$\rho_j \leq (r+1)(d-r) - \sum_{i=0}^{r} \alpha_i^j - \sum_{i=0}^{r} \beta_i^j$$

and therefore that the space $G_d^r(\mathbb{P}^1, (P_j, \alpha_\bullet^j), (P_{j-1}, \beta_\bullet^j))$ is empty by Theorem 2.0.1 unless

$$\sum_{i=0}^{r} \alpha_i^j + \sum_{i=0}^{r} \beta_i^j \leq (r+1)(d-r) \tag{4.3}$$

Combining inqualities (4.2) and (4.3) gives that

$$(r+1)(d-r) - \sum_{i=0}^{r} \beta_i \leq \sum_{i=0}^{r} \alpha_i^n \leq \cdots \leq \sum_{i=0}^{r} \alpha_i^1 \leq \sum_{i=0}^{r} \alpha_i = (r+1)(d-r) - \sum_{i=0}^{r} \beta_i$$

and that

$$(r+1)(d-r) - \sum_{i=0}^{r} \alpha_i \leq \sum_{i=0}^{r} \beta_i^1 \leq \cdots \leq \sum_{i=0}^{r} \beta_i^n \leq \sum_{i=0}^{r} \beta_i = (r+1)(d-r) - \sum_{i=0}^{r} \alpha_i$$

where in each case the last equality comes from the fact that our limit $\mathfrak{g}_d^r$ is refined. This shows that each $\alpha_\bullet^j$ has the same ramification weight as $\alpha_\bullet$ and that each $\beta_\bullet^j$ has the same ramification weight as $\beta_\bullet$ and, in particular, that inequality (4.2) is an equality for each $0 \leq j \leq n$. As such, the Eisenbud-Harris inequalities $\alpha_i^j + \beta_{r-i}^{j+1} \geq d-r$ become equalities for all $0 \leq i \leq r$, and we can conclude that $\alpha_\bullet^j = \alpha_\bullet$ and $\beta_\bullet^j = \beta_\bullet$ for each j.

If $\alpha_\bullet$ and $\beta_\bullet$ are tame, then the $\mathfrak{g}_d^r$ on each inserted component must be separable. Because $\sum_{i=0}^r \alpha_i + \sum_{i=0}^r \beta_i = (r+1)(d-r)$, we have by the Plücker formula that all the ramification on each component must be at the nodes. Conversely, if $\alpha_\bullet$ and $\beta_\bullet$ are wild, then the Plücker formula gives that separability on any of the inserted components is not possible. $\square$

By Lemma 4.3.1, any ramification specializing to a node of X_0 specializes after blowup to a nonsingular point on an inserted component. However, Lemma 3.1.3 gives that this can't happen if the inserted components are unramified at all nonsingular points. Therefore:

Corollary 4.3.3. *If the limit $\mathfrak{g}^r_d$ on X_0 is refined, it is only possible for ramification to specialize to a node of X_0 if the limit $\mathfrak{g}^r_d$ is wildly ramified at that node.*

This result indicates that it should be possible for ramification to specialize to a node of X_0 even if the limit $\mathfrak{g}^r_d$ on X_0 is refined as long as there is wild ramification at that node. The following example shows that this is indeed the case:

Example 4.3.4. Consider a genus 0 family of curves degenerating to a nodal curve C_0 consisting of two copies of $\mathbb{P}^1$ glued at 0. Then, assuming that $p > 3$, we can define a limit $\mathfrak{g}^1_p$ on C_0 by using the map $x \mapsto \frac{1-x^{p-1}}{x^p}$ on each component. Because this limit $\mathfrak{g}^1_p$ is (wildly) ramified to order p at the node on both components and unramified at all other points (cf. the Artin-Schreier map [**10**, Example 2.3.12]), we have from Lemma 3.1.3 that any family of $\mathfrak{g}^1_p$s specializing to this limit $\mathfrak{g}^1_p$ must have all ramification specializing to the node. Specifically, if we use the family of $\mathbb{P}^1$s given by hyperbolas $xy = t$ inside $\mathbb{P}^1 \times \mathbb{P}^1$ degenerating to the union of the axes at $t = 0$ (cf. Remark 4.1.4.1), we can define a $\mathfrak{g}^1_p$ on each $\mathbb{P}^1$ by the map

$$x \mapsto \frac{1 - x^{p-1} + tx}{x^p + t^{p-2}}$$

If we take $t \to 0$, we get the desired $\mathfrak{g}^1_p$ on the x-axis. Furthermore, writing $x = \frac{t}{y}$ and clearing denominators gives

$$\frac{1 - x^{p-1} + tx}{x^p + t^{p-2}} = \frac{y^p - t^{p-1}y + t^2y^{p-1}}{t^p + t^{p-2}y^p}$$

Factoring out t^{p-2} from the denominator, subtracting the denominator from the numerator, and factoring out $-t^2$ from the numerator then gives the equivalent $\mathfrak{g}^1_p$

$$y \mapsto \frac{1 - y^{p-1} + t^{p-3}y}{y^p + t^2}$$

Since we assumed that $p > 3$, taking $t \to 0$ gives the desired $\mathfrak{g}^1_p$ on the y-axis.

REMARK 4.3.4.1. In this situation, Proposition 4.3.2 gives that the limit $\mathfrak{g}_p^1$ resulting from blowing up the origin is inseparable on each inserted component. In particular, since the ramification at each node matches that of the original limit $\mathfrak{g}_p^1$ at the origin, we have that the $\mathfrak{g}_p^1$ on each inserted component is the Frobenius map $x \mapsto x^p$.

4.3.1. Comparison with Characteristic 0**.** In [**3**, Proposition 2.5], Eisenbud and Harris proved that in characteristic 0, the limit $\mathfrak{g}_d^r$ on X_0 is refined if and only if there is no ramification specializing to a node of X_0. Example 4.3.4 provides a positive characteristic counterexample, but by strengthening what we mean by 'refined', we can rephrase Corollary 4.3.3 to recover one direction of this statement in arbitrary characteristic:

PROPOSITION 4.3.5. *It is not possible for ramification to specialize to a node of* X_0 *whose differential crudeness index is* 0*.*

Their proof simply uses the Plücker formula together with the semicontinuity statement from Lemma 3.1.3 to show that it is only possible to get less than the expected amount of ramification at the nonsingular points of X_0 if some ramification is specializing to a node. As such, if we add a separability condition, we can use the statement of the Plücker formula from Lemma 3.1.6 to recover the biconditional result in arbitrary characteristic:

PROPOSITION 4.3.6. *If the limit* $\mathfrak{g}_d^r$ *on* X_0 *is separable, then it has differential crudeness index* 0 *at every node if and only if there is no ramification specializing to a node of* X_0*.*

4.3.2. Non-Smoothability Revisited. We conclude this section with an example that simultaneously provides a counterexample to the converse of Proposition 4.3.5 and demonstrates the necessity in Proposition 4.1.2 that the family of $\mathfrak{g}_d^1$s preserve imposed ramification.

EXAMPLE 4.3.7. Consider the genus 0 family inside $\mathbb{P}^1 \times \mathbb{P}^1$ given by hyperbolas $xy = t^3$ which degenerates to the union of the axes at $t = 0$ (we will again call this nodal curve C_0).

We can then define a $\mathfrak{g}^1_p$ on each $\mathbb{P}^1$ using the map

$$x \mapsto \frac{x+t}{x^p + t^{3p-1}}$$

Writing $x = \frac{t^3}{y}$ and manipulating as in Example 4.3.4 gives the equivalent $\mathfrak{g}^1_p$

$$y \mapsto \frac{1 - ty^{p-1}}{y^p + t}$$

Taking $t \to 0$ gives a limit $\mathfrak{g}^1_p$ on C_0 defined by the map $x \mapsto \frac{x}{x^p}$ on the x-axis and the map $y \mapsto \frac{1}{y^p}$ on the y-axis. The resulting $\mathfrak{g}^1_p$ on the x-axis is separable and ramified to order $p-1$ at 0 and ∞, while the $\mathfrak{g}^1_p$ on the y-axis is inseparable and ramified to order p at every point. In particular, this limit $\mathfrak{g}^1_p$ is crude, with vanishing sequence $1, p$ at the node on the x-axis and $0, p$ on the y-axis. Furthermore, each $\mathfrak{g}^1_p$ in the family is (wildly) ramified to order p at the point $(x, y) = (\sqrt[p]{-t^{3p-1}}, \sqrt[p]{-t})$, which specializes to the origin as $t \to 0$.

We can determine further the behavior of the family by blowing up the origin as in Lemma 4.3.1. In this case, the exceptional divisor is contained in $\mathbb{P}^2$ and given by equations $xy = t^3$, $xY = yX$, $xZ = tX$, and $yZ = tY$, where X, Y, and Z are the corresponding homogeneous coordinates. In particular, the exceptional divisor corresponds to the coordinate axes $XY = 0$, which gives that the blowup inserts two copies of $\mathbb{P}^1$ at the node. If we consider the affine chart given by $Z \neq 0$, we obtain coordinates $\bar{x} = X/Z$ and $\bar{y} = Y/Z$. These in turn give new equations $x = t\bar{x}$ and $y = t\bar{y}$, and let us rewrite our family of $\mathfrak{g}^1_p$s as

$$\bar{x} \mapsto \frac{\bar{x}+1}{\bar{x}^p + t^{2p-1}}$$

or, using that $\bar{x}\bar{y} = t$, equivalently as

$$\bar{y} \mapsto \frac{\bar{y}^p + t\bar{y}^{p-1}}{1 + t^{p-1}\bar{y}^p}$$

Then taking $t \to 0$ gives a limit $\mathfrak{g}^1_p$ on the new components defined by the map $\bar{x} \mapsto \frac{\bar{x}+1}{\bar{x}^p}$ on the $\bar{x}$-axis and the map $\bar{y} \mapsto \bar{y}^p$ on the $\bar{y}$-axis. The resulting $\mathfrak{g}^1_p$ on the $\bar{x}$-axis is separable and ramified to order p at 0 and to order $p-1$ at ∞, while the $\mathfrak{g}_p^1$ on the $\bar{y}$-axis is inseparable and ramified to order p at every point. We also note that each $\mathfrak{g}_p^1$ in this family is (wildly) ramified to order p at $(\bar{x}, \bar{y}) = (\sqrt[p]{-t^{2p-1}}, \sqrt[p]{-t^{1-p}})$, which specializes to $(0, \infty)$ as $t \to 0$.

As such, following blowup of the node of the curve C_0, we obtain a curve C_0' consisting of a chain of four copies of $\mathbb{P}^1$, with the internal components glued at 0 and each external component glued at 0 to its adjacent internal component at ∞, and a limit $\mathfrak{g}_p^1$ on C_0' defined in order by the maps $x \mapsto \frac{x}{x^p}$, $x \mapsto \frac{x+1}{x^p}$, $x \mapsto x^p$, and $x \mapsto \frac{1}{x^p}$. In particular, we get vanishing sequences $1, p$ and $0, p-1$ at the first node and $0, p$ on each side of the second and third nodes. We also note that our family of $\mathfrak{g}_p^1$s has ramification specializing to the third node. As such, we have a refined limit $\mathfrak{g}_p^1$ such that the second and third nodes have nonzero differential crudeness index and with ramification specializing to one, but not both, of these nodes.

Remark 4.3.7.1. Because the limit $\mathfrak{g}_d^1$ on C_0' is refined, we have that the the result of a subsequent blowup of any of the nodes can now be completely determined by Proposition 4.3.2. In particular, as in Remark 4.3.4.1, blowing up the second or third nodes will only result in the insertion of more components on which the $\mathfrak{g}_p^1$ is the inseparable Frobenius map.

Remark 4.3.7.2. Recall that the limit $\mathfrak{g}_p^1$ defined by the maps on the middle two components of C_0', considered as the union of the axes in $\mathbb{P}^1 \times \mathbb{P}^1$, is given by $x \mapsto \frac{x+1}{x^p}$ on the x-axis and the map $y \mapsto y^p$ on the y-axis. By setting $g(x) = x + 1$, we see that this limit $\mathfrak{g}_p^1$ is of exactly the form constructed to be non-smoothable in Example 4.1.4. Because this limit $\mathfrak{g}_p^1$ can in fact be obtained as the limit of a family of $\mathfrak{g}_p^1$s, we will briefly discuss why this does not present a counterexample to Proposition 4.1.2. The key point is the fact that Proposition 4.1.2 requires the family to not only preserve imposed ramification, but in particular that the imposed ramification be to order strictly less than p. Because our family of $\mathfrak{g}_p^1$s has a point that is wildly ramified, it does not satisfy this condition.

With the above in mind, we will conclude with the following result:

Corollary 4.3.8. *Under the assumptions of Proposition 4.1.2, any family of* $\mathfrak{g}^1_d$*s on* X_1 *specializing to an inseparable limit* $\mathfrak{g}^1_d$ *on* X_0 *must have a point ramified to order at least* p*. If* $d = p$*, this is equivalent to the family being wildly ramified at at least one point.*

Proof. Because the result is trivial in the case that the family is inseparable, we will assume that we have a family of separable $\mathfrak{g}^1_d$s. By the Plücker formula, the total amount of ramification of each $\mathfrak{g}^1_d$ is $2d-2$ and, assuming that none of the ramification is specializing to a node of X_0, we can make a base change such that all of the ramification specializes to nonsingular points of X_0 along sections of π. In particular, if it is the case that all of the ramification is to order less than p, Proposition 4.1.2 gives that the resulting limit $\mathfrak{g}^1_d$ on X_0 must be separable.

In the case that there is ramification specializing to a node of X_0, we can blow up that node as in Lemma 4.3.1 such that all ramification specializes along sections to nonsingular points of the new special fiber X_0'. Because this process preserves the original limit $\mathfrak{g}^1_d$ on X_0, we can conclude the result as above. $\square$

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