Orthosymplectic Differential Geometric Operator Algebras with Conformal Symmetry

Senior Honors Thesis

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Abstract: We explore and expand a symplectic Lie algebra representation satisfied by commutations of several differential geometry operators acting on tensor fields defined on curved manifolds possessing a hypersurface-orthogonal conformal Killing vector. We use this Killing vector to produce new interesting operators, and give their algebra. We reinterpret these operators as a quantum mechanical model for a particle with intrinsic structure, and explore the rigid symmetries of its classical antecedent system using Noether's theorem. We generalize to the full $\mathfrak{osp}(Q|2p+2)$ Lie algebra with the addition of orthosymplectic commuting and anticommuting variables, and present the obstructions to our algebras due to curvature. We also review requisite background material, including tensor calculus, variational calculus, Noether's theorem, and Lie algebras.

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1. Introduction

The techniques of tensor calculus, spurred by Riemann's work in the mid-19th century and brought to fruition by Ricci and Levi-Civita at the turn of the 20th century, are an extremely useful tool in physics and other fields, particularly indispensable in Einstein's theory of gravity. We aim to study a set of operators important to tensor calculus, and in particular how they behave under commutation. We also demonstrate the correspondence between this operator algebra with both a quantum mechanical and classical model for a particle.

In order to introduce four new tensor operators which complete an algebra already studied in [1], we are forced to restrict the curved backgrounds under consideration. Specifically, we require the existence of a hypersurface-orthogonal conformal Killing vector, which yields manifolds equivalent, subject to a change of coordinates, to conical geometry over a base manifold [2]. We fully explain the geometric conditions just introduced in Section 2.

Our operator algebras use an indexless notation by means of contraction over coordinate differentials, encoded with the symmetry properties of a particular tensor's index structure. In Section 4.1 we discuss symmetric tensors in particular, and generalize to arbitrary tensors in Section 5. The operators we present correspond to various common differential geometric operations, such as the gradient, divergence, and (curvature-corrected) Laplacian. In particular, we generalize the results of [1] to include four new operators, which correspond to the inner and outer product with a vector field, the covariant derivative along a vector field, and the multiplication by a scalar field. It is for these operators that the homothety condition is necessary. Note that the representation suffers from several obstructions due to curvature of the underlying manifold, which we also present.

An equivalent representation of the algebra corresponding to these operators can be obtained by interpreting them as quantized Noether charges of a quantum mechanical model for a particle with intrinsic structure, and furthermore as the rigid symmetries of the classical action which generates these quantum mechanical operators upon first quantization. With the correspondence among these three algebras, we have a powerful lexicon with which to equate differential geometry with an interesting particle model. The latter two representations are presented in Sections 4.2 and 4.3. The algebras behind all of the various representations in Section 4 is the symplectic Lie algebra $\mathfrak{sp}(4)$, and with the generalization in Section 5, the full orthosymplectic $\mathfrak{osp}(Q|2p+2)$.

Lastly, we present in Section 2 background material to the subjects we will be working with, including curved space treated using the affine and spin connections, variational calculus, Noether's theorem, and Lie algebras.

2. Background

2.1. Curved Space. Our main results apply to general curved spaces, so here we briefly introduce the basic mathematics of curved space. A more thorough treatment can be found in, for example, [3]. First is the covariant derivative construction under the affine connection, and then the Riemann curvature tensor.

Although some concepts may be lost or unrecognizably distorted in the transition from flat to curved spaces, several important ideas persist. Most importantly, the idea of parallel transport—taking a vector and moving it in such a way that it does not change direction or length (now measured by the metric)—is still possible in curved space, but not as one might wish to define using the partial derivatives of its components. To this end, it is necessary to define the covariant derivative, which is the operator against which parallel transport is measured (no longer the partial derivative by itself), as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho, \tag{1}$$

where $\Gamma^\nu_{\mu\rho}$ is the affine or Christoffel connection, symmetric in μ and ρ, and defined as

$$\Gamma^\nu_{\mu\rho} = g^{\lambda\nu} \frac{1}{2} \left(\partial_\mu g_{\lambda\rho} + \partial_\rho g_{\mu\lambda} - \partial_\lambda g_{\mu\rho} \right). \tag{2}$$

In an equation, parallel transport of a vector v_ν is the statement that

$$\nabla_\mu v_\nu = 0. \tag{3}$$

By construction of the covariant derivative using the Christoffel connection, the metric always satisfies parallel transport:

$$\nabla_\mu g_{\nu\rho} = \nabla_\mu g^{\nu\rho} = 0. \tag{4}$$

This is mandatory so that, when taking derivatives of tensor fields, any contribution made by the change of the measurement device (the metric) is not considered. Each separate index of a tensor on which the covariant derivative is applied receives its own Christoffel contraction, where importantly contravariant (raised) indices, as in Equation (1), have Christoffel contraction added, whereas covariant (lowered) indices have such a contraction subtracted; i.e.,

$$\nabla_\mu v_\nu \equiv \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho. \tag{5}$$

The reason for this is clear from the equation

$$\begin{aligned} \nabla_\mu \left(v^\nu w_\nu \right) &= \left(\nabla_\mu v^\nu \right) w_\nu + v^\nu \left(\nabla_\mu w_\nu \right) \\ &= \left(\partial_\mu v^\nu + \Gamma^\nu_{\mu\sigma} v^\sigma \right) w_\nu + v^\nu \left(\partial_\mu w_\nu - \Gamma^\sigma_{\mu\nu} w_\sigma \right) \\ &= \partial_\mu \left(v^\nu w_\nu \right), \end{aligned} \tag{6}$$

since the covariant derivative of a scalar is just the partial derivative.[1]

Also relating to parallel transport, and in effect measuring its failure around closed simple curves, the Riemann curvature tensor $R_{\mu\nu\rho}{}^\sigma$ is defined by a commutator of covariant derivatives, as

$$R_{\mu\nu\rho}{}^\sigma v_\sigma = \left[\nabla_\mu, \nabla_\nu \right] v_\rho = 2 \left(\partial_{[\nu} \Gamma^\sigma_{\mu]\rho} + \Gamma^\lambda_{\rho[\mu} \Gamma^\sigma_{\nu]\lambda} \right) v_\sigma, \tag{7}$$

where $[\cdot\cdot]$ denotes antisymmetrization with unit weight. From this relation it is clear that the Riemann tensor is antisymmetric in its first two indices; it is likewise antisymmetric in its last two, and the first pair can be switched with the second pair. This important tensor is a precise measure of the curvature of a space. One interpretation, in the absence of torsion (the failure of parallelograms to close), follows the thought experiment of the parallel transport (in accordance with the covariant derivative) of a vector around a small (infinitesimal) closed loop, and then the measurement of the deviation the vector experiences from its initial value upon completion of its orbit (Figure 1). In this picture, the Riemann tensor's first two indices measure the area of the loop (when contracted with coordinate differentials), and its second two act as a rotation matrix (hence why they're antisymmetric) on the vector to determine the deviation.

[1] The fact that a minus sign goes with covariant indices and a plus sign with contravariant indices is simply a matter of convention in the definition of the Christoffel symbol; the unavoidable feature is that they have opposite signs.

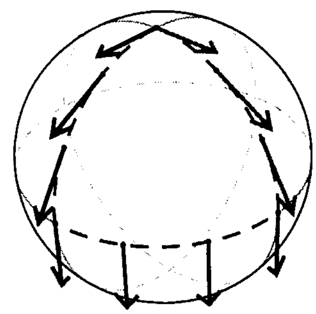


Figure 1. Diagram of a vector parallel-transported in a closed loop over the surface of a sphere, where a deviation results between its initial and final directions. The Riemann tensor's second pair of indices acts as a rotation matrix to determine such a deviation when considering an infinitesimal loop, and its first two indices, when contracted on coordinate differentials, account for the size of the loop.

2.2. The Spin Connection. An alternative to the mathematical description of curved space we have just introduced (the affine/Christoffel connection) is to use the spin connection. The idea is to introduce a tangent/flat space approximation at each point in a manifold; it is of course necessary to make these tangent planes compatible when moving from point to point in order to discuss continuity and derivatives. This approach is convenient for certain applications because it eliminates the metric tensor $g_{\mu\nu}$ in favor of the *constant* signature metric η^{mn}, which commutes with all differential operators (unlike $g_{\mu\nu}$). For this reason, we will use the spin connection extensively in Section 5.

The spin connection involves the use of vielbeine denoted $e_\mu{}^m$ or $e^\mu{}_m$ (each of which is the inverse of the other), which constitute a local orthonormal frame field, or in other words an orthogonal basis for the tangent space at each point which varies smoothly across the manifold considered. Other common names for vielbein include vierbein and tetrad (specific to the four-dimensional case). Contracting a "curved" tensorial index (denoted with Greek letters μ, ν, etc., as we have been using already) with the appropriate index of a vielbein in effect produces a "flattened" index (denoted m, n, etc.), e.g.,

$$v_m = e^\mu{}_m v_\mu. \tag{8}$$

The vielbeine satisfy

$$\eta_{mn} = g_{\mu\nu} e^\mu{}_m e^\nu{}_n, \tag{9}$$

$$g_{\mu\nu} = \eta_{mn} e_\mu{}^m e_\nu{}^n. \tag{10}$$

As such, contraction of two flattened tensorial indices is accomplished using η^{mn} since, considering for example the inner product of two vectors v^μ and w^μ,

$$\begin{aligned} g_{\mu\nu} v^\mu w^\nu &= \eta_{mn} e_\mu{}^m e_\nu{}^n v^\mu w^\nu \\ &= \eta_{mn} v^m w^n. \end{aligned} \tag{11}$$

It is important to keep curved and flattened indices separate, since their differentiation is treated differently as we shall see momentarily. We warn the reader that we reserve the Greek letters α, β, γ, δ, and ε to denote orthosymplectic superindices (appearing in Section 5 and Appendices A and B).

The covariant derivative can be defined for operation on vielbein-contracted flat indices. Curved indices are contracted with the usual Christoffel terms, but the spin connection $\omega_\mu{}^m{}_n$ instead contracts on flat indices. Given a vielbein set, the spin connection is defined as

$$\omega_\mu{}^m{}_n = e^\nu{}_n \left(\partial_\mu e_\nu{}^m - \Gamma^\rho_{\mu\nu} e_\rho{}^m\right). \tag{12}$$

Note that unlike the symmetric lower indices of Christoffel symbols, the spin connection has antisymmetry in its second and third indices; this results from the fact that it acts to quotient off a rotational invariance of the vielbeine, in loose terms, and these last two indices are thus in effect an antisymmetric rotation matrix. Additionally, although a Christoffel contraction on covariant indices is subtracted, while it is added for contravariant indices, the spin connection is always added, with contraction on the last index. For example,

$$\nabla_\mu v_m{}^\nu = \partial_\mu v_m{}^\nu + \Gamma^\nu_{\mu\rho} v_m{}^\rho + \omega_{\mu m}{}^n v_n{}^\nu. \tag{13}$$

The veilbeine then satisfy the *veilbein postulate* (which can alternatively be considered as a defining relation for $\omega_\mu{}^m{}_n$ and the covariant derivative):

$$0 = \nabla_\mu e^\nu{}_n = \partial_\mu e^\nu{}_n + \Gamma^\nu_{\mu\sigma} e^\sigma{}_n + \omega_{\mu n}{}^r e^\nu{}_r. \tag{14}$$

This condition is essential, just like the parallel transport of the metric from Section 2.1, so that any contribution to a derivative made by the change of our local tangent spaces is disregarded.

The Riemann tensor can also be formed using the spin connection, and remarkably is identical, up to flattening, to the affine Riemann tensor:

$$\begin{aligned} R_{mnr}{}^s v_s &= \left[\nabla_m, \nabla_n\right] v_r \\ &= 2\left(\partial_{[m}\omega_{n]r}{}^s + \omega_{[mn]}{}^t \omega_{tr}{}^s + \omega_{[mr}{}^t \omega_{n]t}{}^s\right) v_s, \end{aligned} \tag{15}$$

with antisymmetrization applying to only m and n.

2.3. Variational Calculus. In Section 4.3 we consider a representation based on rigid symmetries of an action for a particle; as such, here we will review variational calculus in curved space. This subject deals with the extremization of integrals subject to unknown functions, just as ordinary differential calculus essays to extremize functions with respect to unknown values. In physics, this most often corresponds to minimizing an action to yield equations of motion.

As a preliminary, the canonical Lagrangian action principle in flat space is

$$S = \int \frac{\dot{x}_\mu \dot{x}^\mu}{2} dt, \tag{16}$$

where x^μ are the coordinates for a particle parameterized by t, with $\dot{x}^\mu$ denoting the derivative of x^μ with respect to t. Varying the above corresponds to requiring that S be extremized by $x(t)$, so that adding any infinitesimal variation $\xi\delta x$ to x and then differentiating with respect to the parameter ξ must yield zero. This process itself is often denoted by the variation δ:

$$0 = \delta S = \int \dot{x}_\mu \, \delta\dot{x}^\mu \, dt = -\int \ddot{x}^\mu \, \delta x_\mu \, dt, \tag{17}$$

resulting in the geodesic equation

$$\ddot{x}^\mu = 0. \tag{18}$$

Upon turning our attention to curved space treated using the affine connection, it is useful to work with covariant versions of the variation δ or worldline derivative $\dot{x}$, because the inner product between vectors is no longer constant—use of the fact that $\nabla_\mu g_{\nu\sigma} = 0$ affords us more compact calculations. To this end, using ∇_μ we can define the covariant variation $\mathcal{D}$ and covariant worldline derivative ∇/dt as

$$\mathcal{D}v^\mu = \delta v^\mu + \Gamma^\mu_{\nu\rho}\delta x^\nu v^\rho \tag{19}$$

$$\frac{\nabla v^\mu}{dt} = \dot{v}^\mu + \Gamma^\mu_{\nu\rho} v^\nu \dot{x}^\rho. \tag{20}$$

In the particular case that the vector (or tensor) is a function of the coordinates $v^\mu(x^\sigma)$, the product rule then yields

$$\mathcal{D}v^\mu = \delta x^\sigma \nabla_\sigma v^\mu \tag{21}$$

$$\frac{\nabla v^\mu}{dt} = \dot{x}^\sigma \nabla_\sigma v^\mu. \tag{22}$$

We will presently list two useful curved-space identities. First, the relation between the covariant variation and worldline derivative is:

$$\begin{aligned} \mathcal{D}\dot{x}^\mu &= \delta \dot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \, \delta x^\rho \\ &= \frac{\nabla}{dt}\delta x^\mu. \end{aligned} \tag{23}$$

Second, the variation of a scalar is the same as the covariant variation:

$$\begin{aligned} \delta\left(v^\mu w_\mu\right) &= (\delta v^\mu + \Gamma^\mu_{\nu\rho}\delta x^\nu v^\rho) w_\mu + v^\mu(\delta w_\mu - \Gamma^\rho_{\mu\nu}\delta x^\nu w_\rho) \\ &= \mathcal{D}(v^\mu w_\mu). \end{aligned} \tag{24}$$

Now we are prepared to consider variational calculus in general curved spaces with the affine connection. The canonical Lagrangian action principle becomes

$$S = \frac{1}{2}\int \dot{x}^\mu g_{\mu\nu}\dot{x}^\nu dt; \tag{25}$$

varying it with the covariant variation (which is equal to the normal variation, since the action is a scalar), making use of Equation (23), produces

$$\begin{aligned} 0 = \delta S &= \int \mathcal{D}\left(\frac{\dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu}{2}\right) dt \\ &= \int (\mathcal{D}\dot{x}^\mu) g_{\mu\nu}\dot{x}^\nu dt \\ &= -\int \delta x^\mu \frac{\nabla \dot{x}^\mu}{dt} dt. \end{aligned} \tag{26}$$

From this we can extract the geodesic equation

$$\frac{\nabla \dot{x}^\mu}{dt} = 0. \tag{27}$$

This is the statement of parallel transport (see Section 2.1) of the tangent vector to a path, clearly a necessary (and sufficient) condition for geodesic motion.

2.4. Noether's Theorem. This famous theorem gives the explicit correspondence between a rigid symmetry and a conserved charge, under a dynamical model formulated with an action. For example, it equates temporal and spatial translation invariance with energy and momentum conservation, and rotational invariance with angular momentum conservation.

To prove a version of Noether's theorem, we first start with an action

$$S = \int L(q^\mu, \dot{q}^\mu, t)dt. \tag{28}$$

Applying arbitrary variations to it yields the field equations:

$$\frac{\partial L}{\partial q^\mu} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu}. \tag{29}$$

Suppose there is a rigid symmetry of the action in question. Mathematically this is the statement that there exists a particular variational symmetry $\xi\delta q^\mu$, where ξ is a "small" time-independent linear parameter and δq^μ is an arbitrary function, such that the variation of the action with respect to it is 0 (neglecting boundary terms):

$$\delta S = \int \left(\xi\frac{\partial L}{\partial q^\mu}\delta q^\mu + \xi\frac{\partial L}{\partial \dot{q}^\mu}\frac{d}{dt}(\delta q^\mu)\right) dt = 0, \quad \text{without field equations.} \tag{30}$$

Now we can temporarily pretend that the linear parameter ξ does in fact depend on time: $\xi(t)$. (We must additionally require that $\xi(t)$ strictly vanishes for $t = \pm\infty$, but not necessarily the same for $\dot{\xi}(t)$.) Calculating the variation,

$$\begin{aligned}\delta S &= \int \left(\frac{\partial L}{\partial q^\mu}\xi\delta q^\mu + \frac{\partial L}{\partial \dot{q}^\mu}(\dot{\xi}\delta q^\mu + \xi\delta\dot{q}^\mu)\right) dt \\ &= \int \left[\xi\left(\frac{\partial L}{\partial q^\mu}\delta q^\mu + \frac{\partial L}{\partial \dot{q}^\mu}\delta\dot{q}^\mu\right) + \dot{\xi}\left(\frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right)\right] dt.\end{aligned} \tag{31}$$

Now, δq^μ is a rigid symmetry by assumption, so the coefficient of ξ must have a total time antiderivative, F, so that it can be integrated by parts to vanish:

$$\frac{dF}{dt} = \frac{\partial L}{\partial q^\mu}\delta q^\mu + \frac{\partial L}{\partial \dot{q}_\mu}\delta\dot{q}^\mu. \tag{32}$$

Proceeding,

$$\begin{aligned}\delta S &= \int \dot{\xi}\left(-F + \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right) dt \\ &= \int \xi\frac{d}{dt}\left(F - \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu\right).\end{aligned} \tag{33}$$

Because δS must vanish subject to any arbitrary variation, it must also vanish under this particular symmetry. Moreover, at this point we can invoke the fact that $\xi(t)$ is a completely arbitrary function of t, so that the rest of the integrand must in fact separately vanish. If we define

$$\begin{aligned}Q &= -F + \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu \\ &= \frac{\partial L}{\partial \dot{q}^\mu}\delta q^\mu - \int_0^t \left(\frac{\partial L}{\partial q^\mu}\delta q^\mu + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}^\mu\right) dt,\end{aligned} \tag{34}$$

we find

$$\frac{d}{dt}Q = 0. \tag{35}$$

Q then does not change in time, which is another way of saying that it is a *conserved* quantity. We refer to it as the *Noether charge* of the symmetry.

2.5. Poisson Brackets and Quantization. The interplay between a classical model for a particle formulated with an action principle and its resultant (first-quantized) quantum mechanical theory is central to the discussion in Sections 4.2 and 4.3, so we précis the most important elements of first quantization here. We first remind the reader of the definition for classical Poisson brackets

$$\{A,\, B\}_{\mathrm{PB}} \equiv \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x^\mu}, \tag{36}$$

where in fact x and p here can of course be taken to be any set of coordinates and their momenta, respectively. Poisson brackets in particular can be used to determine time evolution:

$$\{H,\, A\}_{\mathrm{PB}} = \frac{dA}{dt}, \tag{37}$$

where A is any dynamical quantity and H is a system's Hamiltonian. For example, with $H = p^\mu p_\mu/2$ and $A = x^\rho$,

$$\begin{aligned}\left\{\frac{1}{2}p_\mu g^{\mu\nu}p_\nu,\, x^\rho\right\}_{\mathrm{PB}} &= \frac{1}{2}\frac{\partial x^\rho}{\partial x^\nu}\frac{\partial(p^\mu p_\mu)}{\partial p_\nu} - \frac{1}{2}\frac{\partial(p^\mu p_\mu)}{\partial x^\nu}\frac{\partial x^\rho}{\partial p_\nu} \\ &= \delta^\rho_\nu \delta^\nu_\mu p^\mu \\ &= \dot{x}^\rho,\end{aligned} \tag{38}$$

where we have used the first-order Hamiltonian field equation $p_\mu = \dot{x}_\mu$.

Under first quantization, going from a classical theory to a quantum mechanical one, the most fundamental shift is that Poisson brackets become quantum commutators in the following way:

$$\{\cdot\,,\cdot\}_{\mathrm{PB}} \;\rightarrow\; \frac{1}{i\hbar}\,[\cdot\,,\cdot]\,. \tag{39}$$

All measurable quantities of a system are replaced instead by operators acting on a Hilbert space, whose algebra is given by the algebra of the original classical Poisson brackets. Many of the fundamental relations of quantum mechanics follow from this principle.

The state of a system is given by an element of a Hilbert space $|\Psi\rangle$. Given an operator $\hat{Q}$ acting on a system's Hilbert space, corresponding to an observable quantity Q, its eigenvalues λ_i and eigenstate spectrum ϕ_i can be used to produce the probability that a measurement of Q will yield a particular value:

$$P(Q = \lambda_i) = \langle\phi_i|\Psi\rangle\,. \tag{40}$$

The expectation value of Q is

$$\langle Q\rangle = \langle\Psi|\hat{Q}\Psi\rangle. \tag{41}$$

2.6. Lie Algebras. We now present an interlude on the mathematical theory of Lie algebras, which applies to physics as the study of continuous symmetries. We will rely heavily on this subject in later sections, so this section summarizes the germane points; [4] gives a more thorough reference geared specifically towards physics.

A Lie group G is a group which depends smoothly on a set of continuous parameters, i.e., a group which is also a manifold. Its corresponding Lie algebra $\mathfrak{g}$ is the tangent space to G at the identity, which is simply a vector space such that there is a bijective continuous mapping from the $\mathfrak{g}$ to G, which exchanges vector addition for the group multiplication law. This is known as the exponential map:

$$(G \ni g) = \exp(\lambda \in \mathfrak{g}), \quad \exp\colon \mathfrak{g} \to G. \tag{42}$$

Since a Lie algebra is a vector space, it can be spanned by an orthonormal basis. The elements of such a basis are called *generators* of the Lie algebra. A Lie algebra must also come equipped with a Lie bracket binary operation: $[\cdot\,,\,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. The Lie bracket is motivated by commutators, and under representations of a Lie algebra the bracket indeed becomes a commutator. A Lie bracket must be antisymmetric, linear, and distributive, and must satisfy the Jacobi identity:

$$0 = [a,\, [b,\, c]] + [b,\, [c,\, a]] + [c,\, [a,\, b]]\,. \tag{43}$$

2.6.1. The adjoint representation. Let X_i be the ith generator of a particular Lie algebra. The Lie bracket operation must produce another element of the Lie algebra, which is spanned by the generators, so we can say

$$[X_i,\, X_j] = f_{ij}{}^k X_k, \tag{44}$$

where f_{ijk} are constants, called the structure constants of the Lie algebra. If we define

$$[T_i]_j{}^k = f_{ij}{}^k, \tag{45}$$

where each T_i is a matrix, these satisfy

$$[T_i,\, T_j] = f_{ij}{}^k T_k, \tag{46}$$

so the T matrices form a representation of the algebra, called the adjoint representation.

2.6.2. Roots and weights. It is worthwhile to determine the maximum set of commuting hermitian generators in a particular Lie algebra; we call these the *Cartan generators* of a particular Lie algebra, labeled H_i, and the portion of the Lie algebra spanned by these is called its Cartan subalgebra. In the adjoint representation, the Cartan generators become $n \times n$ matrices, which can be thought of as operators acting on vectors. Since they commute, they can be simultaneously diagonalized to yield a set of eigenvectors. We define the roots, α_i, of an eigenvector in this representation as its set of eigenvalues with respect to the Cartan generators H_i; $\alpha = (\alpha_i)$ is called a root vector. Each root vector α itself has a corresponding generator in the Lie algebra, labeled e_α, which satisfies

$$[H_i,\, e_\alpha] = \alpha_i e_\alpha. \tag{47}$$

Another fundamental equation is

$$[e_\alpha,\, e_\beta] \propto \begin{cases} e_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root vector} \\ 0 & \text{if } \alpha+\beta \text{ is not a root vector,} \end{cases} \tag{48}$$

where β is another root vector. Finally, with a semisimple Lie algebra, given a generator e_α, there must exist another generator $e_{-\alpha}$. There exist subalgebras, however, that do not satisfy this property.

Equation (47) allows us to grade the root vector generators e_α by the Cartan generators, essentially telling us the "component" of each e_α along each H_i "axis". A diagram showing this grading is called a root lattice, which, using Equation (48), gives an exceedingly efficient way of classifying a Lie algebra and exhibiting all its salient features. An example of a root lattice appears later in Figure 2.

3. Homothetic Conformal Geometry

Our results apply to general, curved, torsion-free, d-dimensional, (pseudo)Riemannian manifolds. We will often specialize to spaces which possess a hypersurface-orthogonal homothetic conformal Killing vector, i.e., a vector field $\boldsymbol{\xi}$ satisfying

$$g_{\mu\nu} = \nabla_\mu \xi_\nu. \tag{49}$$

Henceforth we refer to the important condition of Equation (49) as *hyperhomothety.* Using $\boldsymbol{\xi}$, a homothetic potential ϕ can be formed:

$$\phi = \frac{\xi^\mu \xi_\mu}{2}, \tag{50}$$

which then satisfies

$$\nabla_\mu \phi = \xi_\mu, \tag{51}$$

and consequently

$$g_{\mu\nu} = \nabla_\mu \nabla_\nu \phi. \tag{52}$$

Under the conditions of hyperhomothety, the manifold admits coordinates (r, x^i) such that the metric is explicitly a light cone over some base manifold with metric h_{ij}, where in particular

$$\phi = \frac{r^2}{2}. \tag{53}$$

That is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 h_{ij} dx^i dx^j; \tag{54}$$

see for instance [2]. Note that because the Riemann tensor is made up from two covariant derivatives, a contraction of $\boldsymbol{\xi}$ with Riemann must summarily vanish:

$$R_{\mu\nu\rho}{}^{\sigma} \xi_\sigma = \left[\nabla_\mu, \nabla_\nu\right] \xi_\rho = 0. \tag{55}$$

The hyperhomothety condition has several interesting consequences. The most elementary flat space example is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \Rightarrow \quad \xi^\mu = x^\mu, \tag{56}$$

where $\boldsymbol{\xi}$ is the Euler vector field, which generates dilations. Spaces admitting such vectors $\boldsymbol{\xi}$ satisfying Equation (49) are thus very close to flat space.

Another condition with which to obtain spaces similar to flat space, in that they have many symmetries, is the locally symmetric space condition, the requirement that the covariant derivative of the Riemann tensor vanish:

$$\nabla_\lambda R_{\mu\nu\rho\sigma} = 0. \tag{57}$$

This condition is explored in [1] to obtain models with symmetries subject to a maximal parabolic subalgebra of $\mathfrak{osp}(Q|2p+2)$.

The two conditions discussed above—hyperhomothety from Equation (49), and locally symmetric space from Equation (57)—intersect only in flat space, and are otherwise *mutually exclusive*. For, applying only Equation (49), we can obtain

$$[\xi^\sigma \nabla_\sigma,\, [\nabla_\mu,\, \nabla_\nu]] = -2\,[\nabla_\mu,\, \nabla_\nu]\,. \tag{58}$$

However, if we selectively apply both Equations (49) and (57), we can instead obtain

$$[\xi^\sigma \nabla_\sigma,\, [\nabla_\mu,\, \nabla_\nu]] = 0; \tag{59}$$

hence, the Riemann tensor must be identically zero.

In the following sections we will use the geometric conditions just presented to explore elegant operator and symmetry algebras, focusing primarily on hyperhomothety.

4. $\mathfrak{sp}(4)$ Conformal Quantum Mechanics

Here we will present a set of geometric operators on totally symmetric tensors which form a representation of the $\mathfrak{sp}(4)$ Lie algebra. We will then show that these operators can instead be interpreted as quantized Noether charges of an action for a particle with intrinsic structure, the rigid symmetries of which satisfy the same $\mathfrak{sp}(4)$ algebra.

4.1. Symmetric Tensors. Totally symmetric tensors can be represented by completely contracting all indices with commuting coordinate differentials dx^μ. Given a rank-n symmetric tensor field $\phi_{\nu_1\nu_2\ldots\nu_n}$, introduce

$$\Phi(x^\mu, dx^\mu) = \phi_{\nu_1\nu_2\ldots\nu_n} dx^{\nu_1} dx^{\nu_2} \cdots dx^{\nu_n},$$

and hence interpret the original tensor ϕ instead as a function Φ of the coordinates and an *analytic* function of commuting coordinate differentials dx^μ; note that it is now possible to add symmetric tensors of different ranks. In this indexless notation, one can then define useful operators on such symmetric tensors. To this end, in addition to coordinate differentials it is useful to introduce dual objects dx^*_μ, which mutually commute but obey the Heisenberg algebra

$$\left[dx^*_\mu,\, dx^\nu\right] = \delta^\nu_\mu. \tag{60}$$

Since we now consider coordinate differentials themselves as coordinates, we can represent these dual objects as $dx^*_\mu = \partial/\partial dx^\mu$.

Importantly, we can produce an operator equivalent to the covariant derivative when acting on an indexless symmetric tensor, which we denote D_μ (distinct from the ordinary covariant derivative ∇, which acts on tensors with indices):

$$D_\mu = \partial_\mu - \Gamma^\sigma_{\mu\nu} dx^\nu dx^*_\sigma, \tag{61}$$

where ∂_μ denotes the partial derivative $\partial/\partial x^\mu$. That is to say,

$$D_\mu \Phi = (\nabla_\mu \phi_{\nu_1\nu_2\ldots\nu_n})\, dx^{\nu_1} dx^{\nu_2} \cdots dx^{\nu_n}. \tag{62}$$

We can also produce the Lorentz/rotation generators

$$M^{\mu\nu} = 2g^{\rho[\nu}dx^{\mu]}dx^*_\rho,$$

which satisfy

$$[M^{\mu\nu}, M^{\rho\sigma}] = 4M^{\mu\sigma}\eta^{\nu\rho}$$

(where the μ-ν and ρ-σ antisymmetries are preserved in the result). Note that although D_μ and $M^{\mu\nu}$ are meant to act on tensors contracted with coordinate differentials, their outputs *possess indices*. For this reason, they will not appear alone in the algebra we will discuss, but only as part of larger composite objects.

At this point we will introduce a set of operators which map symmetric tensors to symmetric tensors without appeal to indices. First, using just the operators dx and dx^*, we can construct three bilinears:

$$\mathsf{N} = dx^\mu dx^*_\mu, \quad \mathsf{g} = g_{\mu\nu}dx^\mu dx^\nu, \quad \mathsf{tr} = g^{\mu\nu}dx^*_\mu dx^*_\nu. \tag{63}$$

Geometrically, these operators can be interpreted as: N determines tensor rank (and returns the rank as its eigenvalue); g is a symmetrized outer product with the metric tensor; and tr is the symmetrized trace or contraction with the metric tensor.

Next, adding our covariant derivative operator D_μ, we can form another three bilinears:

$$\mathsf{grad} = dx^\mu D_\mu, \quad \mathsf{div} = g^{\mu\nu}dx^*_\mu D_\nu,$$
$$\Box = \Delta + \frac{1}{4}R_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma}, \tag{64}$$

where we are forced to add a quantum ordering term to form the Laplacian

$$\Delta = g^{\mu\nu}\left(D_\mu D_\nu - \Gamma^\sigma_{\mu\nu}D_\sigma\right),$$

and an additional curvature correction to Δ to make our operator $\Box$ (for symmetries reasons soon to be evident). These operators can be interpreted as: grad is the symmetrized gradient; div is the symmetrized divergence; and $\Box$ is the Lichnerowicz wave operator [5], essentially a curvature-corrected Laplacian. The operators from Equations (63) and (64) are discussed more thoroughly in [1].

Lastly, adding the hyperhomothetic Killing vector $\boldsymbol{\xi}$, we can form four additional bilinears:

$$\mathsf{ord} = -\xi^\mu D_\mu, \quad \mathsf{inner} = \xi^\mu dx^*_\mu, \quad \mathsf{outer} = \xi_\mu dx^\mu, \quad \blacksquare = \xi^\mu \xi_\mu. \tag{65}$$

These four operators can, of course, be defined whether or not the hyperhomothety condition (49) holds; if it does not, $\boldsymbol{\xi}$ can then be any arbitrary vector field. Given this, these operators can be interpreted as: ord functions as $\nabla_{\boldsymbol{\xi}}$, and, under hyperhomothety, counts derivatives (and returns the count as its eigenvalue); inner and outer are the symmetrized inner and outer products with a vector field, respectively; and $\blacksquare$ is the multiplication by a scalar field (the homothetic potential $2\phi = \xi \cdot \xi$). Another common notation for inner is $\iota_{\boldsymbol{\xi}}$, the symmetrized contraction of $\boldsymbol{\xi}$ onto a tensor.

Thus far, we have simply introduced an operator notation for standard geometric operations on symmetric tensors. Remarkably, subject to certain conditions stipulated in Section 3, these operators constitute a deformed representation of the symplectic Lie algebra $\mathfrak{sp}(4)$. A general root lattice showing the identification of our operators with $\mathfrak{sp}(4)$ roots is displayed in Figure 2, and the various different subalgebras that we will

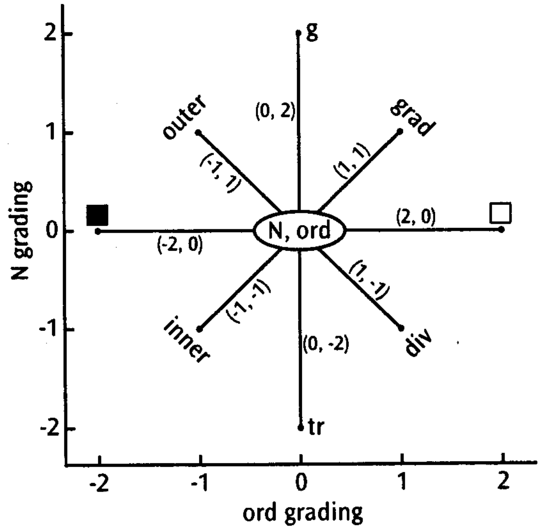


Figure 2. Root lattice for the ten bilinear symmetric-tensor operators as a representation of $\mathfrak{sp}(4)$, graded by the Cartan generators N and ord. Their explicit commutation algebra is given in Appendix B.

Name	*Root*	*Operator*	*Interpretation*
N	Cartan	$dx^\mu dx^*_\mu$	Counts tensor rank
g	$(0,2)$	$g_{\mu\nu}dx^\mu dx^\nu$	Metric outer product
tr	$(0,-2)$	$g^{\mu\nu}dx^*_\mu dx^*_\nu$	Trace
grad	$(1,1)$	$dx^\mu D_\mu$	Gradient
div	$(1,-1)$	$g^{\mu\nu}dx^*_\mu D_\nu$	Divergence
□	$(2,0)$	$g^{\mu\nu}\left(D_\mu D_\nu - \Gamma^\sigma_{\mu\nu}D_\sigma\right) + R_\mu{}^\nu{}_\rho{}^\sigma dx^\mu dx^*_\nu dx^\rho dx^*_\sigma$	Lichnerowicz wave operator
ord	Cartan	$-\xi^\mu D_\mu$	Counts derivatives
inner	$(-1,-1)$	$\xi^\mu dx^*_\mu$	Vector field inner product
outer	$(-1,1)$	$g_{\mu\nu}\xi^\mu dx^\nu$	Vector field outer product
■	$(-2,0)$	$g_{\mu\nu}\xi^\mu\xi^\nu$	Scalar field multiplication

Table 1. List of geometric symmetric-tensor operators, complete with root vectors for their corresponding $\mathfrak{sp}(4)$ root lattice displayed in Figure 2.

currently discuss are exhibited in Figure 3. We additionally tabulate all possible commutation relations in Appendix B. First we note that the operators from Equation (63) form a representation of $\mathfrak{sp}(2)$ in any background (Figure 3.i), with the index-counting operator N as its Cartan generator, since

$$[\mathsf{N}, \mathsf{g}] = 2\mathsf{g}, \quad [\mathsf{N}, \mathsf{tr}] = -2\mathsf{tr}, \quad [\mathsf{g}, \mathsf{tr}] = -4(\mathsf{N} + d/2). \tag{66}$$

Next, also in any background (and thus arbitrary vector field $\boldsymbol{\xi}$) we have the operators N, g, tr, inner, outer, and ■ forming a subalgebra of $\mathfrak{sp}(4)$. To add ord, which is the

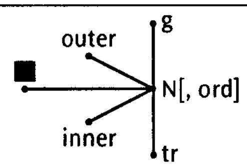
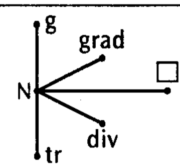
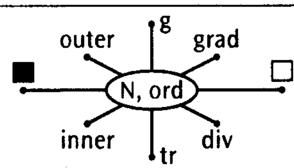
	Root lattice	*Conditions*	*Algebra*
i.		Any background	$\mathfrak{sp}(2)$
ii.		Any background without ord (ξ is then an arbitrary vector field); hyperhomothety with ord	Subalgebra of $\mathfrak{sp}(4)$; maximal parabolic with ord
iii.		Symmetric space	Subalgebra of $\mathfrak{sp}(4)$; obstruction given by (67)
iv.		Hyperhomothety	$\mathfrak{sp}(4)$; obstructions given by (67) and (68)

Figure 3. Root lattices for potential subalgebras of $\mathfrak{sp}(4)$, individually conditional upon application of hyperhomothety (49) or locally symmetric space (57) conditions.

other requisite Cartan generator, we need to invoke hyperhomothety, but we arrive with a maximal parabolic subalgebra (Figure 3.ii).

Next, under locally symmetric space (57), we have N, g, tr, grad, div, and □ forming a representation of a subalgebra of $\mathfrak{sp}(4)$ (Figure 3.iii), with an obstruction given by

$$[\mathsf{div}, \mathsf{grad}] = \Box - \frac{1}{2} R_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} \neq \Box; \tag{67}$$

this computation was already performed in [1].

Lastly, our ultimate algebra requires hyperhomothety, and with all ten operators from Equations (63)–(65) forms a complete representation of $\mathfrak{sp}(4)$ (Figure 3.iv), up to obstructions given by Equation (67) together with

$$[\Box, \mathsf{grad}] = -\frac{1}{4} \left(\nabla_\lambda R_{\mu\nu\rho\sigma}\right) \left(M^{\mu\nu} dx^\lambda M^{\rho\sigma}\right) \neq 0, \tag{68}$$

as well as a similar result for □ with div. This result is new in this work, and a generalized version applying to arbitrary tensors and spinors is given in Section 5. Note that Equation (68) importantly vanishes in symmetric space, and that indeed all of the obstructions listed identically vanish together exclusively *in flat space*, where $\mathfrak{sp}(4)$ is fully realized; any relaxation of flat space to more general backgrounds immediately yields a deformation of $\mathfrak{sp}(4)$. However, the obstruction in Equation (67) is in some sense more accommodating than that from Equation (68), since it still allows $\Box$ to be central among the operators with nonnegative ord grading (except ord and N), whereas (68) does not, with repercussions to appear in following sections.

4.2. Quantum Mechanics. Next we would like to interpret the geometric system described in Section 4.1 as a quantum mechanical one, with the symmetric tensors becoming symmetric wavefunctions of a model for a particle possessing internal structure:

$$\Phi(x^\mu, dx^\mu) \to |\Psi\rangle . \tag{69}$$

We moreover interpret the coordinate differentials dx and their duals dx^* from the preceding section as raising and lowering operators, respectively:

$$dx^\mu \to a^{\dagger\mu}, \quad dx^*_\mu \to a_\mu .$$

In this way the wavefunctions $|\Psi\rangle$ can be written

$$|\Psi\rangle = \psi_{\nu_1\nu_2\ldots\nu_n}(x^\mu) a^{\dagger\nu_1} a^{\dagger\nu_2} \cdots a^{\dagger\nu_n} |0\rangle , \tag{70}$$

where we have introduced the Fock vacuum state $|0\rangle$.

As expected, $a^\dagger$ and a obey

$$\left[a_\nu, a^{\dagger\mu}\right] = \delta^\mu_\nu , \tag{71}$$

similarly to the momentum $p_\mu = -i\partial_\mu$ and position x^μ:

$$[p_\mu, x^\nu] = -i\delta^\nu_\mu . \tag{72}$$

Also, just as the partial derivative is replaced by the canonical momentum, so too the operator D_μ from Equation (61) is replaced by the covariant canonical momentum

$$-\, iD_\mu \to \pi_\mu = p_\mu + i\Gamma^\sigma_{\mu\nu} a^{\dagger\nu} a_\sigma . \tag{73}$$

An appropriate norm for this system is given by first forming dual bras

$$\langle\Psi| = \psi^{*\nu_1\nu_2\ldots\nu_n}(x^\mu) \langle 0| a_{\nu_1} a_{\nu_2} \cdots a_{\nu_n} , \tag{74}$$

where integration is implicitly assumed and ψ^* denotes complex conjugation, and then using the normalization $\langle 0 \mid 0\rangle = 1$ to produce

$$|\Psi|^2 = \langle\Psi \mid \Psi\rangle = n! \int \cdots \int \sqrt{\pm g}\, (\psi_{\nu_1\nu_2\ldots\nu_n} \psi^{*\nu_1\nu_2\ldots\nu_n})\, dx^1 \cdots dx^d . \tag{75}$$

Here dx is clearly the standard integral notation (not the coordinate differentials from before), $\sqrt{\pm g}$ is the square root of the metric determinant, and $+$ or $-$ is chosen to ensure the quantity beneath the radical is positive (depending on the signature of $g_{\mu\nu}$).

The symmetric-tensor operators from the previous section are manifest in this representation as quantized Noether charge operators acting on wavefunctions, with the appropriate replacements for dx, dx^*, and D given above. The algebra satisfied by each charge in this system is exactly identical to that of its geometric vis-à-vis from Section 4.1. To gain further insight, we will next analyze the antecedent classical system behind the quantized quantum mechanical model just described.

4.3. Classical Mechanics. We now work in a classical variational theory for a particle, where we choose to interpret the coordinate differentials dx and their duals dx^* from the geometrical representation in Section 4.1, or alternatively the quantum mechanical raising and lowering operators $a^\dagger$, a from Section 4.2, as comprising a complex-valued vector carried by the particle, i.e.,

$$dx^\mu \Big/ a^{\dagger\mu} \;\rightarrow\; \bar{z}^\mu, \qquad dx^*_\mu \Big/ a_\mu \;\rightarrow\; z_\mu. \tag{76}$$

The quantum mechanical commutators, under "reverse quantization", instead become classical Poisson brackets:

$$\{p_\mu,\, x^\nu\}_{\mathrm{PB}} = \delta^\nu_\mu, \quad \{\bar{z}^\mu,\, z_\nu\}_{\mathrm{PB}} = -i\delta^\mu_\nu. \tag{77}$$

Throughout this section we assume the hyperhomothety condition (49). Since it is central (up to obstructions), we take as our Hamiltonian $H = -\Box/2$, dropping the $g^{\mu\nu}\Gamma^\sigma_{\mu\nu}D_\sigma$ quantum ordering term for the classical system:

$$H = \frac{1}{2}\left(\pi^\mu\pi_\mu - R_\mu{}^\nu{}_\rho{}^\sigma\bar{z}^\mu z_\nu \bar{z}^\rho z_\sigma\right). \tag{78}$$

From (77) it is evident that we already have Darboux coordinates x and p, z and $\bar{z}$, so we can perform a Legendre transformation to obtain a suitable action principle for our particle:

$$S = \int \left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + i\bar{z}^\mu\frac{\nabla z_\mu}{dt} + \frac{1}{2}R_\mu{}^\nu{}_\rho{}^\sigma\,\bar{z}^\mu\, z_\nu\,\bar{z}^\rho\, z_\sigma\right) dt. \tag{79}$$

The three terms in this Lagrangian all have interesting geometric interpretations. The first is the usual energy integral, the extremization of which yields simple parametrized geodesic motion. The second ensures parallel transport of the vector z_μ. The third is in effect a coupling between the first two; including it in our model results in many more symmetries than with its omission, which we discuss below.

The ten operators forming our representation of $\mathfrak{sp}(4)$ in Section 4.1, listed in Table 1, correspond via Noether's theorem to conserved quantities of the above action, with explicit symmetries listed in Table 2. There are, however, three interesting caveats to make. First, because there is an obstruction to the algebra between the $\Box$ charge and both the grad and div charges (Equation (68)), grad and div technically *are not* conserved charges of our action—performing the grad variation, we find

$$\delta S = i\int \left(\nabla_\lambda R_\mu{}^\nu{}_\rho{}^\sigma\bar{z}^\lambda\bar{z}^\mu z_\nu\bar{z}^\rho z_\sigma\right) dt \quad \text{subject to} \quad \delta x^\mu = i\bar{z}^\mu,\ \mathcal{D}z_\mu = \dot{x}_\mu. \tag{80}$$

The result of this variation, which is exactly the classical equivalent of Equation (68), shows the failure of the grad and div symmetries due to the curvature of the underlying manifold.

Second, it is important to note that the symmetries corresponding to the hyperhomothetic operators—ord, inner, outer, and $\blacksquare$—do not commute with the Hamiltonian H; consequently, they are off-shell symmetries and moreover possess *time dependence*. The procedure to take a quantized Noether charge and produce a (possibly time-dependent) symmetry is as follows. First, given a quantized Noether charge operator Q_0, its time dependence is determined using the system's Hamiltonian operator H:

$$\frac{dQ}{dt} = [Q_0,\, H], \tag{81}$$

which can be integrated to yield $Q(t)$. For example, to obtain the ord charge's time dependence,

$$\frac{d}{dt}\mathsf{ord} = [\mathsf{ord},\, H] = 2H \quad \Rightarrow \quad \mathsf{ord}(t) = \mathsf{ord} + 2Ht, \tag{82}$$

or for ■,

$$\frac{d}{dt}\blacksquare = [\blacksquare,\, H] = 2\mathsf{ord} \quad \Rightarrow \quad \blacksquare(t) = \blacksquare + 2(\mathsf{ord})t + 2Ht^2. \tag{83}$$

From here, the rigid symmetry corresponding to Q for the coordinate q^μ is

$$\delta_Q q^\mu = [Q(t),\, q^\mu]\,. \tag{84}$$

Proceeding in this way one can obtain all the symmetries from Table 2. It is the convenient algebra satisfied by these quantized charges which allows integrations of this kind to be analytically evaluated.

Lastly, it is worthwhile to notice that there are actually two separate symmetries which both correspond to the □ Noether charge—the first is obtained using the procedure just described, and is quoted in Table 2, and the second is

$$\delta x^\mu = \dot{x}^\mu, \quad \delta \bar{z}^\mu = \dot{\bar{z}}^\mu, \quad \delta z_\mu = \dot{z}_\mu. \tag{85}$$

A variation of this kind corresponding to translation of the coordinates is in fact a symmetry of any time-independent action, and always produces the Hamiltonian as its Noether charge.

The geometric interpretations of the symmetries from Table 2 can be described as: N increases the magnitude of z_μ for a commensurate decrease in the magnitude of $\bar{z}^\mu$, or vice versa; g effects the admixture of z_μ into $\bar{z}^\mu$, and tr the opposite; grad and div are couplings between the coordinates x^μ and the vector z_μ, effecting exchanges between the two; □ produces translations; ord produces dilations; inner and outer couple the hyperhomothetic conformal Killing vector ξ_μ with z_μ; and finally ■ produces worldline conformal boosts.

5. $\mathfrak{osp}(Q|2p+2)$ Conformal Quantum Mechanics

We forthwith proceed to generalize the algebra from Section 4.1 to the general orthosymplectic Lie algebra $\mathfrak{osp}(Q|2p+2)$. This allows us to study tensors without the restriction of total symmetry. We will always assume the hyperhomothety condition from Equation (49) in this section, and will exclusively use vielbeine and the spin connection $\omega_m{}^n{}_r$. In place of the coordinate differentials dx^μ and dx^*_μ from before, we now denote a general orthosymplectic generator by X_α, where α is a superindex taking values $1 \le \alpha \le 2p+Q$ which can be thought of indexing the species of coordinate differential X_α corresponds to. For $1 \le \alpha \le 2p$, X_α is a bosonic variable, and otherwise it is fermionic. These Xs satisfy the following supercommutation relation:

$$\left[X^m_\alpha,\, X^n_\beta\right\} = \eta^{mn} J_{\alpha\beta}, \tag{86}$$

Name	*Symmetry*	*Noether charge*
N	$\delta z_\mu = -z_\mu$ $\delta \bar{z}^\mu = \bar{z}^\mu$	$\bar{z}^\mu z_\mu$
g	$\delta z_\mu = g_{\mu\nu} \bar{z}^\nu$	$g_{\mu\nu} \bar{z}^\mu \bar{z}^\nu$
tr	$\delta \bar{z}^\mu = g^{\mu\nu} z_\nu$	$g^{\mu\nu} z_\mu z_\nu$
grad	$\delta x^\mu = i\bar{z}^\mu$ $\mathcal{D} z_\mu = \dot{x}_\mu$	$\dot{x}^\mu \bar{z}_\mu$
div	$\delta x^\mu = i\bar{z}^\mu$ $\mathcal{D} \bar{z}^\mu = \dot{x}^\mu$	$\dot{x}^\mu z_\mu$
□	$\delta x^\mu = \dot{x}^\mu$ $\mathcal{D} \bar{z}^\mu = -iR_\alpha{}^\mu{}_\gamma{}^\delta \, \bar{z}^\alpha \, \bar{z}^\gamma \, z_\delta$ $\mathcal{D} z_\mu = iR_\mu{}^\beta{}_\gamma{}^\delta \, z_\beta \, \bar{z}^\gamma \, z_\delta$	$\frac{1}{2}\dot{x}^\mu \dot{x}_\mu - \frac{1}{2} R_\mu{}^\nu{}_\rho{}^\sigma \bar{z}^\mu z_\nu \bar{z}^\rho z_\sigma$
ord	$\delta x^\mu = 2t\dot{x}^\mu - \xi_\mu$	$\dot{x}^\mu \xi_\nu - t\dot{x}^\mu \dot{x}_\mu$
inner	$\delta x^\mu = -itz^\mu$ $\mathcal{D} \bar{z}^\mu = t\dot{x}^\mu - \xi^\mu$	$x^\mu z_\mu - t\dot{x}^\mu z_\mu$
outer	$\delta x^\mu = it\bar{z}^\mu$ $\mathcal{D} z_\mu = t\dot{x}_\mu - \xi_\mu$	$x^\mu \bar{z}_\mu - t\dot{x}^\mu \bar{z}_\mu$
■	$\delta x^\mu = t^2 \dot{x}^\mu + t\xi^\mu$	$\xi^\mu \xi_\mu - 2t\dot{x}^\mu \xi_\mu + t^2 \dot{x}^\mu \dot{x}_\mu$

Table 2. Symmetries and corresponding Noether charges of the action from (79). Unspecified variations are zero.

where $[\cdot\,,\,\cdot\}$ is an anticommutator if both arguments are fermionic, and a commutator otherwise. Here $J_{\alpha\beta}$ is the orthosymplectic bilinear form, given by

$$(J_{\alpha\beta}) = \begin{cases} \left(\begin{array}{cc|cc} & -\mathbb{1}_{p\times p} & & \\ \mathbb{1}_{p\times p} & & & \\ \hline & & & \mathbb{1}_{q\times q} \\ & & \mathbb{1}_{q\times q} & \end{array} \right) & Q = 2q \text{ even}, \\ \left(\begin{array}{cc|ccc} & -\mathbb{1}_{p\times p} & & & \\ \mathbb{1}_{p\times p} & & & & \\ \hline & & & \mathbb{1}_{q\times q} & \\ & & \mathbb{1}_{q\times q} & & \\ & & & & 1 \end{array} \right) & Q = 2q+1 \text{ odd}. \end{cases} \tag{87}$$

We denote the inverse of $J_{\alpha\beta}$ by $J^{\beta\alpha}$, so that $J_{\alpha\beta} J^{\gamma\beta} = \delta_\alpha^\gamma$ (while $J^{\beta\alpha} J_{\gamma\beta} = -T^{\alpha\kappa}_{\kappa\gamma}$; see Appendix A).

We can now, using the orthosymplectic generators X_α, form the $SO(d)$ rotation/Lorentz generators

$$M^{mn} = J^{\beta\alpha} X_\alpha^{[m} X_\beta^{n]} \;\rightarrow\; [M^{mn},\, M^{rs}] = 4M^{ms}\eta^{nr} \tag{88}$$

(where the m-n and r-s antisymmetries are preserved in the result), as well as the spin connection covariant derivative operator acting on X-contracted tensors:

$$D_m = \partial_m + \omega_{mnr} M^{nr}, \tag{89}$$

where we employ the operator ordering $\partial_m = e^\mu{}_m \partial_\mu$.

Now we will build geometric operators equivalent to those from Figure 2. First, from the space of all possible orthosymplectic bilinears, we define

$$f_{\alpha\beta} = \eta_{mn} X^m_{(\alpha} X^n_{\beta]}, \tag{90}$$

where $(\alpha\beta]$ denotes antisymmetrization if α and β are both fermionic indices, and otherwise denotes symmetrization (with unit weight); we again refer the reader to Appendix A, where a computational scheme for working with such symmetrization is detailed. The supermatrix $f_{\alpha\beta}$ subsumes N, g, and tr from Section 4.1, as detailed below.

Next we can define

$$v_\alpha = X^m_\alpha D_m, \tag{91}$$

which subsumes grad and div, and which can be viewed as a generalized Dirac operator. We also clearly need the Lichnerowicz wave operator from before

$$\Box = \Delta + \frac{1}{4} R_{mnrs} M^{mn} M^{rs}, \tag{92}$$

where the Laplacian again requires an ordering term

$$\Delta = D^m D_m - \omega^{nm}{}_n D_m. \tag{93}$$

To complete our set, we presently add the hyperhomothetic operators. ord and ■ remain unchanged, except of course that ord now uses the new covariant derivative operator from Equation (89):

$$\text{ord} = -\xi^m D_m, \quad \blacksquare = \xi^m \xi_m. \tag{94}$$

Lastly, we produce the generalization of inner and outer:

$$w_\alpha = \xi_m X^m_\alpha. \tag{95}$$

The explicit correspondence between the $\mathfrak{osp}(Q|2p+2)$ operators $f_{\alpha\beta}$, v_α, and w_α and their $\mathfrak{sp}(4)$ equivalents is

$$(f_{\alpha\beta}) \rightleftarrows \begin{pmatrix} \mathsf{g} & \mathsf{N} + d/2 \\ \mathsf{N} + d/2 & \mathsf{tr} \end{pmatrix}, \quad (v_\alpha) \rightleftarrows \begin{pmatrix} \mathsf{grad} \\ \mathsf{div} \end{pmatrix}, \quad (w_\alpha) \rightleftarrows \begin{pmatrix} \mathsf{outer} \\ \mathsf{inner} \end{pmatrix}. \tag{96}$$

This new set of geometric operators are meant to act on the equivalent of multiforms, multispinors, and tensors with arbitrary combinations of symmetric and antisymmetric indices, with each index contracted on the appropriate species of X_α; groups of symmetric or antisymmetric indices should all use the same α value. The potential algebras which can be formed by our $\mathfrak{osp}(Q|2p+2)$ operator set are exactly analogous with the $\mathfrak{sp}(4)$ case (Figure 3), with $\mathfrak{sp}(2) \to \mathfrak{osp}(Q|2p)$ and $\mathfrak{sp}(4) \to \mathfrak{osp}(Q|2p+2)$. Specifically:

i. $f_{\alpha\beta}$ in and of itself forms $\mathfrak{osp}(Q|2p)$ in any background;
ii. $f_{\alpha\beta}$, w_α, and ■ form a subalgebra of $\mathfrak{osp}(Q|2p+2)$ in any background (arbitrary vector field $\boldsymbol{\xi}$), which becomes maximal and parabolic with the addition of ord and the hyperhomothety condition, Equation (49);
iii. $f_{\alpha\beta}$, v_α, and □ form a subalgebra of $\mathfrak{osp}(Q|2p+2)$ under the locally symmetric space condition, Equation (57), with an obstruction (97);
iv. all of $f_{\alpha\beta}$, ord, v_α, □, w_α, and ■ together form $\mathfrak{osp}(Q|2p+2)$, subject to obstructions (97) and (98).

There are again two curvature obstructions to the complete $\mathfrak{osp}(Q|2p+2)$ algebra otherwise formed by our operator set under hyperhomothety:

$$[v_\alpha, v_\gamma\} = J_{\alpha\gamma}\Delta + \frac{1}{2}R_{mnrs}X^m_\alpha X^n_\gamma M^{rs} \qquad \neq J_{\alpha\gamma}\Box, \tag{97}$$

$$[\Box, v_\alpha] = -\frac{1}{4}\left(\nabla_t R_{mnrs}\right)\left(M^{mn}X^t_\alpha M^{rs}\right) \ \neq 0, \tag{98}$$

with Δ as defined in Equation 93. These equations respectively generalize their $\mathfrak{sp}(4)$ counterparts in Equations (67) and (68). Note that (97) is again in some sense a much less severe obstruction, since by itself it still allows $\Box$ to be central among v_α and many of the entries of the supermatrix $f_{\alpha\beta}$ (those corresponding to the g and tr operators from $\mathfrak{sp}(4)$), whereas (98) is an actual commutativity failure.

New in this work, Equation (98) is an exceedingly general result, showing the failure of arbitrary Lichnerowicz wave operators to commute with arbitrary generalized Dirac/gradient/divergence operators. The remainder of the algebra satisfied by the operator set is explicitly tabulated in Appendix B.

6. Conclusions

We used the condition of the existence of a hypersurface-orthogonal homothetic conformal Killing vector to present several interesting representations (up to obstructions) of the $\mathfrak{sp}(4)$ Lie algebra, corresponding to indexless differential geometric operators on symmetric tensors (Section 4.1), the quantized Noether charges of a quantum mechanical model (Section 4.2), and the rigid symmetries of a classical action principle for a particle transporting a complex-valued vector (Section 4.3). We furthermore generalized the symmetric tensor operator algebra to more general $\mathfrak{osp}(Q|2p+2)$, corresponding to operators acting on multispinors, multiforms, and tensors with arbitrary index symmetry structure (Section 5).

This project naturally suggests further investigation. In the quantum mechanical representation, it would be a worthwhile task to consider the spectra of such operators, and indeed to ascertain entirely the precise nature of such spectra; further restrictions on the allowed curved backgrounds, so as to admit many symmetries, would likely be necessary. From the perspective of the antecedent classical system, it would be interesting to determine the trajectories of the action given in Equation 79, and in turn to investigate whether determination of the classical system aids in the determination of the quantum mechanical one, or vice versa. It is also possible to generalize our results from these representations into the full $\mathfrak{osp}(Q|2p+2)$, in which case the anticommuting "flavors" of coordinate differentials correspond classically to Grassmannian variables. Full determination of such models would likely be even more difficult, but likewise very general. Furthermore, second quantization of the quantum mechanical model into a quantum field theory is another feasible generalization.

We also wish to consider extending the homothety condition, which corresponds to a scale symmetry or conformal structure, to complex or quaternionic structures. For example, the Kähler condition imparts complex structure:

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}}\phi\left(z, \bar{z}\right). \tag{99}$$

It is also worthwhile to consider manifolds with multiple hyperhomothetic vector fields ξ_i. Manifolds with such additional structures lend themselves to possessing many more interesting symmetries, and could produce exceedingly rich operator algebras from their quantized Noether charges. Lastly, it is possible to search for a Fourier-Jacobi algebra among the conformal operators introduced in this paper, as was done with the g, tr, and N operators in [6]. All of these are viable avenues of future study.

A. Supercommutation Computation

A particularly convenient method of carrying out computations using supercommutators, and other such sometimes symmetric and sometimes antisymmetric objects, is to define a tensor $T^{\delta\varepsilon}_{\alpha\beta}$, similar to a four-index Kronecker delta, contraction over the δ and ε indices of which produces the desired sign depending on the values of the α and β indices. T provides the advantages of enabling one to use standard Einstein-notation tensor calculus for otherwise ungainly symmetry/antisymmetry, and of being easily manipulated using a computer—many identities for T can easily be derived ad hoc.

Specifically, we define

$$T^{\delta\varepsilon}_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \delta \text{ or } \beta \neq \varepsilon, \\ -1 & \text{if } (\alpha = \beta) > 2p, \\ +1 & \text{otherwise.} \end{cases} \tag{100}$$

T can be used to produce a Leibniz rule for supercommutators (here we are working in $\mathfrak{osp}(Q|2p)$ with generators X_α, where indices are orthosymplectic superindices—see Section 5):

$$[X_\alpha, X_\beta X_\gamma\} = [X_\alpha, X_\beta\} X_\gamma + T^{\delta\varepsilon}_{\alpha\beta} X_\varepsilon [X_\delta, X_\gamma\}, \tag{101}$$

or even a Leibniz rule for supercommutators of bilinears:

$$[A_{\alpha\beta}, X_\gamma X_\delta\} = [A_{\alpha\beta}, X_\gamma\} X_\delta + T^{\mu\rho}_{\alpha\sigma} T^{\nu\sigma}_{\beta\gamma} X_\rho [A_{\mu\nu}, X_\delta\}. \tag{102}$$

It can also produce symmetry or antisymmetry in two orthosymplectic indices depending on their value:

$$A_{(\alpha} B_{\beta]} = \frac{1}{2}\left(A_\alpha B_\beta + T^{\delta\varepsilon}_{\alpha\beta} A_\varepsilon B_\delta\right). \tag{103}$$

T replaces the $(-)_{\alpha\beta}$ symbol sometimes seen employed to similar effect, but without any contraction implied on its indices.

T satisfies many identities, as well as interrelations with the orthosymplectic form $J_{\alpha\beta}$ defined in Section 5. We list several here.

$$T^{\mu\nu}_{\alpha\gamma} T^{\rho\sigma}_{\mu\nu} = \delta^\rho_\alpha \delta^\sigma_\gamma, \qquad J_{\alpha\beta} T^{\beta\kappa}_{\kappa\gamma} = -J_{\gamma\alpha}, \qquad T^{\mu\nu}_{\beta\gamma} J_{\alpha\nu} = T^{\mu\nu}_{\beta\alpha} J_{\nu\gamma},$$

$$\tau^{\mu\nu}_{\alpha\beta} J_{\mu\nu} = -J_{\beta\alpha}, \qquad J^{\beta\alpha} J_{\gamma\beta} = -T^{\alpha\kappa}_{\kappa\gamma}, \qquad J^{\beta\alpha} J_{\beta\gamma} = \delta^\alpha_\gamma,$$

$$T^{\omega\lambda}_{\gamma\delta} T^{\nu\sigma}_{\beta\lambda} T^{\mu\rho}_{\alpha\sigma} J_{\nu\omega} = T^{\mu\rho}_{\alpha\delta} J_{\beta\gamma}, \qquad T^{\varepsilon\kappa}_{\beta\gamma} T^{\beta\zeta}_{\zeta\kappa} = \delta^\varepsilon_\gamma, \qquad T^{\mu\nu}_{\alpha\beta} T^{\sigma\rho}_{\nu\gamma} J_{\mu\rho} = \delta^\sigma_\beta J_{\alpha\gamma}.$$

B. Explicit Algebras

We present the explicit commutation algebras of the $\mathfrak{sp}(4)$ operators from Sections 4.1 and the $\mathfrak{osp}(Q|2p+2)$ operators of Section 5 in Figures 4 and 5, respectively. Note that the operators $\underline{\mathsf{N}}$ and $\underline{\mathsf{ord}}$ used here are shifted from the originals defined earlier in the following way:

$$\underline{\mathsf{N}} = \mathsf{N} + d/2, \quad \underline{\mathsf{ord}} = \mathsf{ord} - d/2,$$

where d represents the dimension of the manifold in question—we opt to absorb this $d/2$ term into the operators, which is necessary algebraically.

$[\cdot,\cdot]$	ord	g	tr	grad	div	$\square$	outer	inner	$\blacksquare$
N	0	2g	−2tr	grad	−div	0	outer	−inner	0
ord		0	0	grad†	div†	$2\square^\dagger$	−outer†	−inner†	$-2\blacksquare^\dagger$
g			$-4\underline{\text{N}}$	0	−2grad	0	0	−2outer	0
tr				2div	0	0	2inner	0	0
grad					Eq. 67	Eq. 68	g†	N + ord†	outer†
div						cf. Eq. 68	$\underline{\text{N}} - \underline{\text{ord}}^\dagger$	tr†	2inner†
$\square$							2grad†	2div†	$-4\underline{\text{ord}}^\dagger$
outer								$-\blacksquare$	0
inner									0

Figure 4. Explicit commutation algebra of the operators from Section 4.1. All results apply to arbitrary curved manifolds, except those labeled †, which are contingent upon application of the hyperhomothety condition from Equation (49). Commutators are of the form [left column, top row]. Note the definitions $\underline{\text{N}} = \text{N} + d/2$, $\underline{\text{ord}} = \text{ord} - d/2$.

$[\cdot,\cdot\}$	$f_{\gamma\delta}$	ord	v_γ	$\square$	w_γ	$\blacksquare$
$f_{\alpha\beta}$	$4J_{(\beta(\gamma}f_{\alpha]\delta]}$	0	$2v_{(\alpha}J_{\beta]\gamma}$	0	$2w_{(\alpha}J_{\beta]\gamma}$	0
ord	0	0	${v_\gamma}^\dagger$	$2\square^\dagger$	$-{w_\gamma}^\dagger$	$-2\blacksquare^\dagger$
v_α	$2J_{\alpha(\gamma}v_{\delta]}$	$-{v_\alpha}^\dagger$	Eq. 97	Eq. 98	$-J_{\alpha\gamma}\underline{\text{ord}} + {f_{\alpha\gamma}}^\dagger$	$2{w_\alpha}^\dagger$
$\square$	0	$-2\square^\dagger$	Eq. 98	0	$2{v_\gamma}^\dagger$	$-4\underline{\text{ord}}^\dagger$
w_α	$2J_{\alpha(\gamma}w_{\delta]}$	${w_\alpha}^\dagger$	$-J_{\alpha\gamma}\underline{\text{ord}} - {f_{\alpha\gamma}}^\dagger$	$-2{v_\alpha}^\dagger$	$J_{\alpha\gamma}\blacksquare$	0

Figure 5. Explicit supercommutation algebra of the operators from Section 5. All results apply to arbitrary curved manifolds, except those labeled †, which are contingent upon application of the hyperhomothety condition from Equation (49). Supercommutators are of the form [left column, top row}. Note that the symmetrization on the result of $[f_{\alpha\beta}, f_{\gamma\delta}\}$ is meant to carry symmetry of the form $(\beta\alpha]$ and $(\gamma\delta]$. Also note the definition $\underline{\text{ord}} = \text{ord} - d/2$.

References

1. K. Hallowell and A. Waldron, Commun. Math. Phys. **278**, 775 (2008), hep-th/0702033.
2. G. W. Gibbons and P. Rychenkova, Phys. Lett. B **443**, 138 (1998), hep-th/9809158.
3. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, 2003).
4. H. Georgi, *Lie Algebras in Particle Physics*, 2nd ed. (Perseus Books, 1999).
5. A. Lichnerowicz, Bull. Soc. Math. Fr. **92**, 11 (1964).
6. K. Hallowell and A. Waldron, (2007), 0707.3164.