THE EUCLIDEAN VOLUME OF THE MODULI SPACE

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Abstract. A topological recursion formula is developed for the Euclidean volume of the Moduli Space of curves with n marked points. It is proven by the computation of the inverse Laplace Transform of another formula in [2]. Without reference to a symplectic volume recursion formula, this proof is the only proof for the volume recursion formula. In particular, there are no straightforward geometric arguments that lead to a proof. This fact is another indication of the effectiveness of the method developed in the current paper.

Contents

1. Introduction

The goal of this paper is to derive a topological recursion formula for the Euclidean volume of the Moduli space of curves with n marked points. By a topological recursion formula, it is meant a formula for computing a function $f_{g,n+1}(p_1, ..., p_{n+1})$ from previous functions $f_{g',n'}$, with $2g - 2 + (n + 1) > 2g' - 2 + n' > 0$. This has effectively been done already, because the symplectic and Euclidean volumes satisfy $v^S_{g,n}(\mathbf{p}) = 2^{5g-5+2n} v^E_{g,n}(\mathbf{p})$ from [2], and there is a topological recursion formula for the symplectic volume from [1]; substituting would give the desired recursion formula for the Euclidean volume. This paper will derive the recursion formula for the Euclidean volume without appeal to the symplectic recursion formula.

Why do the Euclidean and symplectic volume of the Moduli space of curves matter? The fact that the ratio of the two is a constant (ignoring g and n) is used in Kontsevich's proof of the Witten conjecture, a statement that two different models of two dimensional quantum gravity give the same results. The methods in [1] and [2] provide an elementary way to compute the ratio of these two functions, so it is worth understanding.

The main strategy to achieve the goal uses the relation between lattice point counting *quasi-polynomials* and Euclidean volume derived in [2]. This relation is between the Laplace transform of the lattice point count functions and the Laplace transform of the Euclidean volume, and it gives rise to a topological recursion formula for the Laplace transform of the Euclidean volume. From here, the inverse Laplace transform is taken to obtain the topological recursion formula for the Euclidean volume.

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2. Moduli Space and Ribbon Graphs

The Moduli space of Riemann surfaces is defined, as a set, as all complex structures on Σ_g, where Σ_g is a Riemann surface of genus g, modulo biholomorphic equivalence. Consider a slightly different object, the Moduli space with n marked points, $\mathcal{M}_{g,n}$, It is defined as the set of all complex structures, or complex coordinate systems, on a Riemann surface of genus g, Σ_g, with n marked points, modulo an equivalance relation. Two complex structures on Σ_g are equivalent if there is a biholomorphic mapping h (h and h^{-1} are holomorphic) between the two coordinate systems.

As an example, consider $\mathcal{M}_{0,3}$. This is the set of all complex structures on the Riemann sphere with three fixed points. Any complex structure on the Riemann sphere must be diffeomorphic to $\mathbb{C} \cup \{\infty\}$, so assume that the complex structure is of the form $\mathbb{C} \cup \{\infty\}$. First, pick three distinct points on the sphere, a_0, a_1, a_2, considered as points in $\mathbb{C} \cup \{\infty\}$. Suppose $a_i \neq \infty$ for each i. Define h by

$$h(x) = \frac{a_1 - a_2}{a_1 - a_0} \frac{x - a_0}{x - a_2}.$$

Then the function h maps a_0 to 0, a_1 to 1, and a_2 to ∞. If one of the $a_i = \infty$, then without loss of generality, let $a_2 = \infty$, and define g by

$$g(x) = \frac{x - a_0}{a_1 - a_0}.$$

Then g maps a_0 to 0, a_1 to 1, and a_2 to ∞. Since h and g are biholomorphic, any complex structure on the sphere with three distinct marked points can be brought to another complex structure with the marked points $\{0, 1, \infty\}$. Therefore, $\mathcal{M}_{0,3} = \{\text{one single point}\}$, since any complex structure on the Riemann sphere with three arbitrary marked points can be brought to another complex structure with $\{0, 1, \infty\}$ as the marked points.

Consider now $\mathcal{M}_{0,4}$ and start with four points, a_0, a_1, a_2, a_3. Map three of them to $\{0, 1, \infty\}$, using the map h. After mapping by h, there is one point left, b. The question now becomes, is there some biholomorphic map, fixing $\{0, 1, \infty\}$ pointwise, that could map any arbitrary $b \in \mathbb{C} \setminus \{0, 1\}$ to some fixed point $p \in \mathbb{C} \setminus \{0, 1\}$? The answer is no; the only biholomorphic function mapping the Riemann sphere to itself and fixing 0, 1, and ∞ is the identity map.

To show this, singularities and Laurent series are considered. Let f be a biholomorphic mapping of $\mathbb{C} \cup \{\infty\}$ into itself, fixing 0, 1, and ∞ pointwise. Then f restricted to $\mathbb{C}$ is entire, so f has a power series expansion

$$f(x) = \sum_{i=0}^{\infty} a_i x^i,$$

valid for all $x \in \mathbb{C}$. Define $g(x) = f(x^{-1})$ for $x \neq 0, \infty$. Then g has Laurent series expansion

$$g(x) = \sum_{i=0}^{\infty} a_i x^{-i},$$

valid for all $x \neq 0, \infty$. Since

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} f(x^{-1}) = \lim_{x \to \infty} f(x) = \infty,$$

g has an isolated singularity, in this instance a pole, at $x = 0$. A result from complex analysis states that if g has an isolated pole at $x = 0$ and g is holomorphic in a punctured neighborhood of 0, then the principal part of its Laurent series expansion around 0 has only finite terms; in other words, there is a N such that $n > N \implies a_n = 0$. Therefore, the original function f was a polynomial. Since f is biholomorphic, and since only $f(0) = 0$, it must be that $f(x) = x^n$, because the only root of f is at $x = 0$ and because $f(1) = 1$. But this function is only biholomorphic when $n = 1$, since an *nth* root of unity $\xi_n \neq 1$ for $n \neq 1$ gives $f(\xi_n) = 1$. Therefore, f is the identity. Thus, $\mathcal{M}_{0,4}$ is, as a set, $\mathbb{C} \setminus \{0, 1\}$.

General g and n cases are difficult to deal with. What kinds of spaces are they? To help our understanding, there is a theorem by [3, 5, 7] which relates the moduli space with n marked points to metric ribbon graphs. The isomorphism is an *orbifold* isomorphism. For more information, refer to [4].

Theorem 2.1. $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong RG_{g,n}$

$RG_{g,n}$ is a space of all metric ribbon graphs of type (g, n), defined as

$$\coprod_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)},$$

where $e(\Gamma)$ is the number of edges of a ribbon graph Γ, and $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ, which is finite for $2g - 2 + n > 0$. The automorphism group of Γ can be considered as the set of all homeomorphisms ψ: $\Sigma_g \to \Sigma_g$ which fix the set of vertices and edges V and E of Γ setwise, modulo the equivalence relation $\psi \sim \phi$ if ψ and ϕ agree on V and E.

A ribbon graph of type (g, n), Γ, is the corresponding graph to the 2-skeleton of a cell decomposition of a genus g surface with n 2-cells, satisfying the following equation: $v - e + n = 2 - 2g$, where v and e are the number of vertices and edges of Γ, respectively. The equality arises because the ribbon graphs are cell decompositions of the Riemann surface, and so it is possible to compute the Euler characteristic of the surface in two different ways. Every vertex of the graph also is required to have valency greater than or equal to three. As an example, a figure eight is a ribbon graph of type $(0, 3)$, because it has two edges and one vertex, so $(1 - 2) + n = 2 - 0$, and so $n = 3$. The dumbbell shape is also a ribbon graph of type $(0, 3)$, because it has three edges and two vertices, and by the previous computation, this means $n = 3$.

What is the real dimension of $RG_{g,n}$? It is determined by the maximum number of edges a ribbon graph can have. This is the case when every vertex has valency three (the number of half-edges coming into the vertex is three). This is because if there is a vertex of valency four or more, an edge can be added to create two new vertices, both of which have less valency. As an example, consider the figure eight; it has a vertex of valency four. Adding an edge creates the dumbbell shape, which makes two vertices of valency three. But no

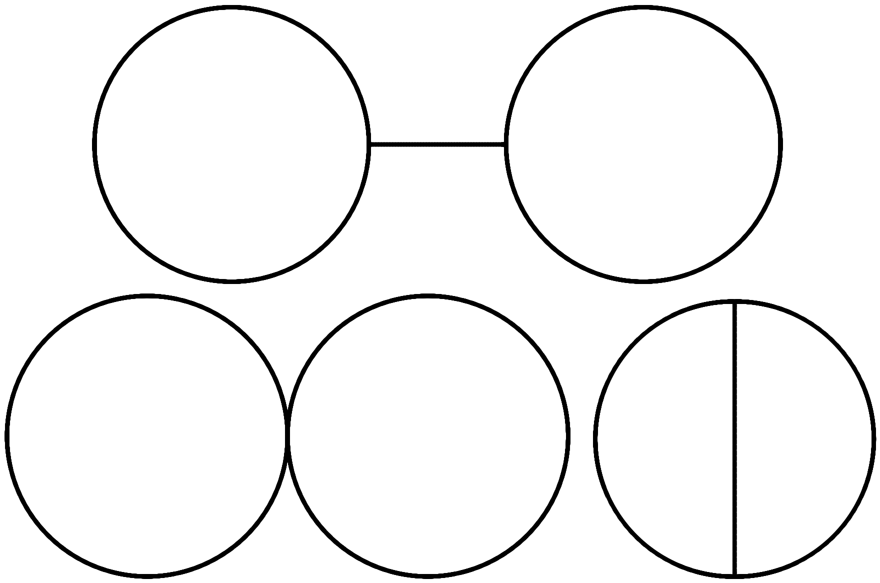

FIGURE 2.1. The dumbbell, the figure eight, and a double bubble shape, all ribbon graphs of type $(0,3)$. These are the only ribbon graphs of type $(0,3)$.

more edges can be added to the dumbbell, since then some vertex would have valency less than three.

Since the maximum number of edges occur in a trivalent ribbon graph, it is enough to know $e(\Gamma)$ for Γ trivalent. In this case, every vertex has valency three, and every edge touches two (possibly nondistinct) vertices, so $3v = 2e$. Plugging into the equation $v-e+n = 2-2g$ gives $2e-3e+3n = 6-6g$, or that $e = 6g-6+3n$ and $v = 4g-4+2n$. For the case $(0,3)$, $e = 3$ and $v = 2$, which is consistent with previous information. For the general case, this gives that the real dimension of $RG_{g,n}$ is $6g-6+3n$. Due to theorem 2.1, this implies that the Moduli space with n marked points has real dimension $6g-6+2n$ (although this is known through other methods). For the case $(0,3)$, the Moduli space is zero dimensional, which is consistent with what was shown earlier.

3. Euclidean Volume Functions and Lattice Point Count

A metric ribbon graph is a ribbon graph which has a positive real number assigned to every edge. By theorem 2.1, $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong RG_{g,n}$. There is a map $\pi : RG_{g,n} \to \mathbb{R}_+^n$, which is a projection map. This map takes a metric ribbon graph of type (g,n) and gives an n dimensional real vector, with entries given by the boundary lengths of the faces of the ribbon graph. As an example, consider the double bubble shaped ribbon graph of type $(0,3)$ in figure 3.1.

The edges have length 2, 1.5, and 7, and each face has a label, b_1, b_2, and b_3. Each face has boundary length 3.5, 8.5, and 9, respectively. Therefore, π would take this metric ribbon graph to the vector $(3.5, 8.5, 9)$.

A natural question to ask is, what is $\pi^{-1}(3.5, 8.5, 9)$? That is, what are the metric ribbon graphs of type $(0,3)$ that are taken to $(3.5, 8.5, 9)$ when π is applied? The double bubble described above is taken to $(3.5, 8.5, 9)$, but what about other graphs? For the figure eight, label the interior faces b_1 and b_2 and the external face b_3, with edge lengths e_1 and e_2; then the length of b_1 is e_1, the length of b_2 is e_2, and the length of b_e is $e_1 + e_2$. Consider

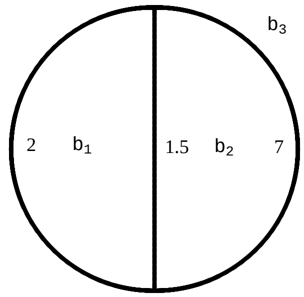

FIGURE 3.1

the system of equations $e_1 = 3.5$, $e_2 = 8.5$ and $e_1 + e_2 = 9$, which has no solution. Thus, no metric ribbon graph in the shape of a figure eight is taken to $(3.5, 8.5, 9)$. If the metric ribbon graph is of the form of a dumbbell, the following equations are appropriate: $e_1 = 3.5$, $e_2 = 8.5$, and $e_1 + e_2 + e_3 = 9$. But all the edge lengths are positive, so again there is no solution. Therefore, there is only one metric ribbon graph that is taken to $(3.5, 8.5, 9)$. Hence $\mathcal{M}_{0,3} \cong \pi^{-1}(3.5, 8.5, 9)$, because $\mathcal{M}_{0,3}$ is just one point. This kind of result is true in general; $\pi^{-1}(\mathbf{p}) \cong \mathcal{M}_{g,n}$ for $\mathbf{p} \in \mathbb{R}^n_+$.

For a specific ribbon graph Γ of type (g, n), label all of the edges and faces. Let A_Γ denote the edge-face incidence matrix of the matrix. That is, A_Γ is a $n \times e(\Gamma)$ dimensional matrix, with entries a_{ij} which count how many times an edge j appears in a face i. Since an edge can appear no times, one time, or twice in a face, it follows that a_{ij} is zero, one, or two. As an example, consider the double bubble ribbon graph as shown in figure 2.1. Let e_1, e_2, e_3 denote the edges of length 2, 1.5, and 7, respectively. Then the incidence matrix is

$$A_\Gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The incidence matrix will map the vector with the edge lengths to a vector with the lengths of the boundaries of the faces.

For Γ of type (g, n), call $P_\Gamma(\mathbf{p}) = A_\Gamma^{-1}(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^n_+$, where A_Γ is considered as a map from $\mathbb{R}^{e(\Gamma)}_+$ to $\mathbb{R}^n_+$. The *Euclidean volume of the polytope* $P_\Gamma(\mathbf{p})$, denoted by $\mathrm{vol}(P_\Gamma(\mathbf{p}))$, is defined by the pushforward measure of A_Γ as $\left.\dfrac{(A_\Gamma)_*(\mu)}{\psi}\right|_{\mathbf{p}}$, where μ is the Lebesgue measure on $\mathbb{R}^{e(\Gamma)}$ and ψ is the Lebesgue measure on $\mathbb{R}^n$. The pushforward measure is denoted by $(A_\Gamma)_*(\mu)$ and defined by $(A_\Gamma)_*(\mu)(B) = \mu(A_\Gamma^{-1}(B))$ for all measurable $B \subset \mathbb{R}^n$. This definition is equivalent to imposing that, for all measurable B,

$$\int_B \mathrm{vol}(P_\Gamma(\mathbf{p}))d\psi = \Big| \int_{A_\Gamma^{-1}(B)} d\mu \Big|, \tag{3.1}$$

which is the method used to compute the first few volumes of polytopes.

For the double bubble ribbon graph, A_Γ was computed. In that case, $\psi = \mu$, and integration over $A_\Gamma^{-1}(B)$ is simply $\frac{1}{2} * \mu(B)$, since A_Γ has determinant two. Thus the volume function is simply $\frac{1}{2}$, since the relation must hold for all open sets B. Note that in the double bubble case, there is an implicit assumption that, for $\mathbf{p} = (p_1, p_2, p_3)$, he inequalities $p_1 + p_2 \leq p_3$, $p_2 + p_3 \leq p_1$, $p_3 + p_1 \leq p_2$ do not hold, so that the volume function is really $\frac{1}{2}$ when those conditions are not satisfied, and 0 otherwise. In the dumbbell case, the volume function is $\frac{1}{2}$ when one of $p_1+p_2<p_3$, $p_2+p_3<p_1$, or $p_3+p_1<p_2$ hold, and in the figure eight case, one of $p_1+p_2=p_3$, $p_2+p_3=p_1$, or $p_3+p_1=p_2$ holds. Note that for any measurable $B\subset\mathbb{R}^3$ satisfying $p_1+p_2=p_3$ for all $\mathbf{p}\in B$ is two-dimensional, so that $\mathrm{vol}(P_\Gamma(\mathbf{p}))$ is not well-defined by equation 3.1 in this case.

In general, this problem of being unable to compute certain volumes only occurs when $e(\Gamma)<n$. Since the smallest number of vertices possible is 1, $1-e+n=2-2g$, or that the smallest $e(\Gamma)$ can be is $2g-1+n$. Thus, this problem only occurs when $g=0$. To solve the problem, the volume when the integration is over the $n-1$ dimensional space is just defined to be 0.

With the Euclidean volumes of the polytopes, it is possible to define the *Euclidean volume of the Moduli space* to be, for $2g-2+n>0$,

$$v^E_{g,n}(\mathbf{p})=\sum_{\substack{\Gamma\text{ trivalent ribbon}\\ \text{graph of type }(g,n)}}\frac{\mathrm{vol}(P_\Gamma(\mathbf{p}))}{|\mathrm{Aut}(\Gamma)|}. \tag{3.2}$$

$v^E_{g,n}$ is a polynomial, a fact which will be proven later. By the previous computations, $v^E_{0,3}(\mathbf{p})=\frac{1}{2}$. Also, $v^E_{1,1}(p)=\frac{1}{96}p^2$.

To show this, first note that there is only one trivalent ribbon graph of type $(1,1)$ with automorphism group $\mathbb{Z}_6$. Since there is only one face and three edges, $A_\Gamma=[2\ 2\ 2]$. Let $b\geq 0$ and consider now $A_\Gamma^{-1}(b)$. This is the set of all (x,y,z) such that $2(x+y+z)=b$, with $x,y,z\geq 0$. Now, $A_\Gamma^{-1}([0,q])$ is the union of all $A_\Gamma^{-1}(b)$ with $b\in[0,q]$. This union is a tetrahedron with vertices $(0,0,0),(\frac{q}{2},0,0),(0,\frac{q}{2},0),(0,0,\frac{q}{2})$. If V is the volume of this tetrahedron, then $\int_0^q \mathrm{vol}(P_\Gamma(p))dp=V$. But the volume of the tetrahedron is $(\frac{q}{2})^3\frac{1}{3!}$. Differentiating both sides shows that $\mathrm{vol}(P_\Gamma(q))=\frac{q^2}{16}$. This gives that $v^E_{1,1}(p)=\frac{p^2}{96}$.

Consider now metric ribbon graphs with integer edge lengths. It is natural to consider the discrete analogue of the volume, the *lattice point count* functions. They are defined by, for $\mathbf{p}\in\mathbb{Z}^n_+$ and $2g-2+n>0$,

$$N_{g,n}(\mathbf{p})=\sum_{\Gamma\text{ of type }(g,n)}\frac{\#\{x\in\mathbb{Z}_+^{e(\Gamma)}|A_\Gamma x=\mathbf{p}\}}{|\mathrm{Aut}(\Gamma)|}. \tag{3.3}$$

These functions were first considered by Norbury in [6]. They have the property that, for $\mathbf{p}=(p_1,...,p_n)$, if $p_1+...+p_n\equiv 1 \mod 2$, then $N_{g,n}(\mathbf{p})=0$. This is because every edge appears uniquely in two faces (sometimes in the same face two times), and therefore for $e_1,...,e_m$ the edge lengths in a metric ribbon graph, $2(e_1+...+e_m)=p_1+...+p_n$. It is easy to check that $N_{0,3}(\mathbf{p})=1$ if $p_1+p_2+p_3$ is even and is 0 otherwise and that $N_{1,1}(p)=\frac{1}{48}p^2-\frac{1}{12}$ when p is even.

The lattice point count functions satisfy the following topological recursion relation, which was first proven in [2]:

Theorem 3.1. *The lattice point count functions satisfy the following recursion formula:*

$$p_1N_{g,n}(p_{[n]})=\frac{1}{2}\sum_{j=2}^{n}\left[\sum_{q=0}^{p_1+p_j}q(p_1+p_j-q)N_{g,n-1}(q,p_{[n]\setminus\{1,j\}})\right. \tag{3.4}$$

$$+ H(p_1 - p_j) \sum_{q=0}^{p_1-p_j} q(p_1 - p_j - q) N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})$$

$$+ H(p_j - p_1) \sum_{q=0}^{p_j-p_1} q(p_j - p_1 - q) N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})\Bigg]$$

$$+ \frac{1}{2} \sum_{0\le q_1+q_2\le p_1} q_1 q_2 (p_1 - q_1 - q_2) \Bigg[N_{g-1,n+1}(q_1, q_2, p_{[n]\setminus\{1\}}) +$$

$$\sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{1\}}}^{stable} N_{g_1,|I|+1}(q_1, p_I) N_{g_2,|J|+1}(q_2, p_J) \Bigg],$$

where $[n] = \{1, 2, ..., n\}$, $p_{[n]} = (p_1, p_2, ..., p_n)$, the stability condition is $2|I| - 1 + g_1 > 0$ and $2|J| - 1 + g_2 > 0$ and where

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

is the Heaviside step function. The equation is proven by consideration of edge removal of metric ribbon graphs with *cilium*, the details of which can be found in [2].

There is a relation between the lattice point count and Euclidean volume, satisfied for all bounded and closed $B \subset \mathbb{R}_+^n$ and for continuous functions f from $\mathbb{R}^n$ into $\mathbb{R}$,

$$\lim_{k\to\infty} \sum_{\mathbf{p}\in B\cap\frac{1}{k}\mathbb{Z}_+^n} N_{g,n}(k\mathbf{p}) \frac{1}{k^{3(2g-2+n)}} f(\mathbf{p}) = \int_B v_{g,n}^E(\mathbf{p}) f(\mathbf{p}) dp_1 dp_2 \cdots dp_n. \tag{3.5}$$

This relation holds because of the definition of the Euclidean volume in terms of the push-forward measure. Because of this relation between the lattice point count functions and the Euclidean volume, it is natural to expect a recursion formula for the Euclidean volume functions. If one naively replaces the lattice point count functions with the Euclidean volume and substitutes appropriate summations with integrations, one obtains a false recursion formula. The naive formula has, instead of factors of $\frac{1}{4}$, factors of $\frac{1}{2}$.

Theorem 3.2. *The Euclidean volume satisfies the following recursion relation:*

$$p_1 v_{g,n}^E(p_{[n]}) = \frac{1}{4} \sum_{i=2}^{n} \Bigg(\int_0^{p_1+p_j} q(p_1 + p_i - q) v_{g,n-1}^E(q, p_{[n]\setminus\{1,i\}}) dq \tag{3.6}$$

$$+ H(p_1 - p_j) \int_0^{p_1-p_j} q(p_1 - p_j - q) v_{g,n-1}^E(q, p_{[n]\setminus\{1,i\}}) dq$$

$$- H(p_j - p_1) \int_0^{p_j-p_1} q(p_j - p_1 - q) v_{g,n-1}^E(q, p_{[n]\setminus\{1,i\}}) dq \Bigg)$$

$$+ \frac{1}{4} \iint_{0\le q_1+q_2\le p_1} q_1 q_2 (p_1 - q_1 - q_2) \Bigg(v_{g-1,n+1}^E(q_1, q_2, p_{[n]\setminus\{1\}})$$

$$+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}} v^E_{g_1,n-1}(q_1,p_I)v^E_{g_2,n-1}(q_2,p_J)\Bigg) dq_1dq_2.$$

The initial conditions $v^E_{0,3}(\mathbf{p}) = \frac{1}{2}$ and $v^E_{1,1}(p) = \frac{1}{96}p^2$ completely determine $v^E_{g,n}$ for $2g-2+n > 0$. It is obvious that $v^E_{g,n}$ is a polynomial, because the integration of a polynomial over each region is itself a polynomial.

4. The Laplace Transform of the Lattice Point Count Functions

Because $N_{g,n}$ is defined only over $\mathbb{Z}^n_+$, while the Euclidean volume is defined over $\mathbb{R}^n_+$, it is natural to consider the Laplace transform of $N_{g,n}$ and $v^E_{g,n}$, so as to extend the functions to the same domain for comparison. For any complex $w_{[n]} \in \mathbb{C}^n$, with $Re(w_j) > 0$ for each $j \in [n]$, define the Laplace transform of $N_{g,n}$ by

$$L_{g,n}(w_{[n]}) = \sum_{\mathbf{p}\in\mathbb{Z}^n_+} N_{g,n}(\mathbf{p})e^{-\langle \mathbf{p}, w_{[n]}\rangle}, \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$, and define $V^E_{g,n}$ by

$$V^E_{g,n}(t_{[n]}) = \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_n}\left(\int_{\mathbb{R}^n_+} v^E_{g,n}(\mathbf{p})e^{-\langle \mathbf{p}, w_{[n]}\rangle}dp_{[n]}\right), \tag{4.2}$$

where the variable substitution

$$e^{-w_j} = \frac{t_j+1}{t_j-1} \tag{4.3}$$

is used. Define

$$\mathcal{L}_{g,n}(t_{[n]}) = \frac{\partial_1}{\partial t_1}\cdots\frac{\partial_n}{\partial t_n}L_{g,n}(w(t_{[n]})), \tag{4.4}$$

where $w(t_{[n]}) = (w_1(t_1), ..., w_n(t_n))$. The variable substitution turns $\mathcal{L}_{g,n}$ into a *Laurent polynomial* in the t_j^2 variables and $V^E_{g,n}$ into a *polynomial* in t_j^2, as will be shown. From here, the Laplace transform of the lattice point count recursion formula is taken to obtain a new recursion formula.

Theorem 4.1. *The Laplace transform of $N_{g,n}$ satisfies the following recursion formula:*

$$\begin{aligned} \mathcal{L}_{g,n}(t_{[n]}) &\\ = -\frac{1}{16}\sum_{j=2}^{n}\frac{\partial}{\partial t_j}&\left[\frac{t_j}{t_1^2-t_j^2}\left(\frac{(t_1^2-1)^3}{t_1^2}\mathcal{L}_{g,n-1}(t_{[n]\setminus\{j\}}) - \frac{(t_j^2-1)^3}{t_j^2}\mathcal{L}_{g,n-1}(t_{[n]\setminus\{1\}})\right)\right] \\ -\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}&\left[\mathcal{L}_{g-1,n+1}(t_1,t_1,t_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}}\mathcal{L}_{g_1,|I|+1}(t_1,t_I)\mathcal{L}_{g_2,|J|+1}(t_1,t_J)\right]. \end{aligned} \tag{4.5}$$

Proof: Multiply each side of equation (3.4) by $p_{[n]\setminus\{1\}}$ and take the Laplace transform. The left hand side of the equation becomes $\frac{\partial}{\partial w_{[n]}}L_{g,n}(w_{[n]}) = \frac{\partial}{\partial w_1}\cdots\frac{\partial}{\partial w_n}L_{g,n}(w_{[n]})$, which will now be denoted by $\widehat{L}_{g,n}(w_{[n]})$. Consider for now the first line, which becomes

$$\frac{1}{2}\sum_{j=2}^{n}\sum_{\mathbf{p}\in\mathbb{Z}^n_{\geq 0}}\sum_{q=0}^{p_1+p_j} p_j(p_1+p_j-q)(qp_2\cdots\widehat{p_j}\cdots p_n)N_{g,n-1}(q,p_{[n]\setminus\{1,j\}})e^{-\langle\mathbf{p},w_{[n]}\rangle}(q,p_{[n]\setminus\{1,j\}}),$$

where $\widehat{p_j}$ indicates the absence of p_j. Summation over nonnegative $\mathbf{p}$ is allowed and changes nothing, since multiplication by p_i for each i was performed. This can be rewritten as

$$\sum_{j=2}^{n}\sum_{q=0}^{\infty}\sum_{p_{[n]\setminus\{1,j\}}\in\mathbb{Z}^{n-2}_{\geq 0}}\left[qp_2\cdots\widehat{p_j}\cdots p_n N_{g,n-1}(q,p_{[n]\setminus\{1,j\}})\right]e^{-\langle p_{[n]\setminus\{1,j\}},w_{[n]\setminus\{1,j\}}\rangle}e^{-qw_1}$$

$$\times\sum_{\ell=0}^{\infty}\ell e^{-2\ell w_1}\sum_{p_j=0}^{q+2\ell}p_j e^{p_j(w_1-w_j)}\ ,$$

where $p_1+p_j-q=2l$. The first summation $\sum_{p_j=0}^{q+2\ell}p_j e^{p_j(w_1-w_j)}$ is just

$$e^{-qw_1}\ell e^{-2\ell w_1}\frac{\partial}{\partial w_1}\ \frac{e^{w_j}-e^{w_j}e^{(1+q+2\ell)(w_1-w_j)}}{e^{w_j}-e^{w_1}}\ ,$$

and is equal to

$$\frac{e^{-qw_1}\ell e^{-2\ell w_1}}{(e^{w_1}-e^{w_j})^2}\left[e^{w_1+w_j}+(q+2\ell)e^{2w_1}e^{(q+2\ell)(w_1-w_j)}-(1+q+2\ell)e^{w_1+w_j}e^{(q+2\ell)(w_1-w_j)}\right]\ ,$$

which is equal to $\frac{1}{(e^{w_1}-e^{w_j})^2}\Bigg[e^{w_1+w_j}e^{-qw_1}\ell e^{-2\ell w_1}+\ell(q+2\ell)e^{2w_1}e^{-qw_j}e^{-2\ell w_j}$

$-\ell(1+q+2\ell)e^{w_1+w_j}e^{-qw_j}e^{-2\ell w_j}\Bigg].$

Summation over l and q is performed, which gives

$$\sum_{j=2}^{n}\left[\frac{e^{w_1+w_j}}{(e^{w_1}-e^{w_j})^2}\left(\frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{j\}})}{(e^{w_1}-e^{-w_1})^2}-\frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2}\right)-\frac{e^{w_1}}{e^{w_1}-e^{w_j}}\ \frac{\partial}{\partial w_j}\frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2}\right].$$

Consider now the second line of equation (3.4):

$$\sum_{j=2}^{n} \sum_{\mathbf{p}\in\mathbb{Z}^n_{\geq 0}} H(p_1-p_j) \sum_{q=0}^{p_1-p_j} p_j \, \frac{p_1-p_j-q}{2} \big[q p_2 \cdots \widehat{p_j} \cdots p_n N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})\big] e^{-\langle p,w\rangle}$$
$$= \sum_{j=2}^{n}\sum_{l=0}^{\infty} l e^{-2lw_1} \sum_{p_j=0}^{\infty} p_j e^{-p_j(w_1+w_j)}$$
$$\times \sum_{q=0}^{\infty} e^{-qw_1} \sum_{p_{[n]\setminus\{1,j\}}\in\mathbb{Z}^{n-2}_{\geq 0}} \big[q p_2 \cdots \widehat{p_j} \cdots p_n N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})\big] e^{-\langle p_{[n]\setminus\{1,j\}}, w_{[n]\setminus\{1,j\}}\rangle}$$
$$= \sum_{j=2}^{n} \frac{e^{w_1+w_j}}{(1-e^{w_1+w_j})^2} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{j\}})}{(e^{w_1}-e^{-w_1})^2},$$

where $p_1 - p_j - q = 2l$ is used. For the third line,

$$-\sum_{j=2}^{n} \sum_{\mathbf{p}\in\mathbb{Z}^n_{\geq 0}} H(p_j-p_1) \sum_{q=0}^{p_j-p_1} p_j \, \frac{p_j-p_1-q}{2} \big[q p_2 \cdots \widehat{p_j} \cdots p_n N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})\big] e^{-\langle p,w\rangle}$$
$$= -\sum_{j=2}^{n}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p_1=0}^{\infty} (p_1+q+2l) l e^{-p_1(w_1+w_j)} e^{-2lw_j} e^{-qw_j}$$
$$\times \sum_{p_{[n]\setminus\{1,j\}}\in\mathbb{Z}^{n-2}_{\geq 0}} \big[q p_2 \cdots \widehat{p_j} \cdots p_n N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})\big] e^{-\langle p_{[n]\setminus\{1,j\}}, w_{[n]\setminus\{1,j\}}\rangle}$$
$$= -\sum_{j=2}^{n} \frac{e^{w_1+w_j}}{(1-e^{w_1+w_j})^2} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2} + \sum_{j=2}^{n} \frac{e^{w_1}}{e^{w_1}-e^{-w_j}} \, \frac{\partial}{\partial w_j} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2},$$

where the substitution $p_j - p_1 - q = 2l$ is used.

Summing the first three lines gives

$$\sum_{j=2}^{n} \Bigg[\frac{e^{w_1+w_j}}{(e^{w_1}-e^{w_j})^2} \left(\frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{j\}})}{(e^{w_1}-e^{-w_1})^2} - \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2} \right)$$
$$- \frac{e^{w_1}}{e^{w_1}-e^{w_j}} \, \frac{\partial}{\partial w_j} \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2} \Bigg]$$
$$+ \sum_{j=2}^{n} \frac{e^{w_1+w_j}}{(1-e^{w_1+w_j})^2} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{j\}})}{(e^{w_1}-e^{-w_1})^2}$$
$$- \sum_{j=2}^{n} \frac{e^{w_1+w_j}}{(1-e^{w_1+w_j})^2} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2} + \sum_{j=2}^{n} \frac{e^{w_1}}{e^{w_1}-e^{-w_j}} \, \frac{\partial}{\partial w_j} \, \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2}$$
$$= \sum_{j=2}^{n} \frac{\partial}{\partial w_j} \left[\left(\frac{e^{w_1}}{e^{w_1}-e^{w_j}} - \frac{e^{w_1+w_j}}{e^{w_1+w_j}-1} \right) \left(\frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{j\}})}{(e^{w_1}-e^{-w_1})^2} - \frac{\widehat{L}_{g,n-1}(w_{[n]\setminus\{1\}})}{(e^{w_j}-e^{-w_j})^2} \right) \right].$$

For the fourth line, first note that, for any f,

$$\frac{1}{2}\sum_{p_1=0}^{\infty}\sum_{0\leq q_1+q_2\leq p_1} q_1q_2(p_1-q_1-q_2)e^{-p_1w_1}f(q_1,q_2)$$
$$=\frac{1}{2}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{l=0}^{\infty}2le^{-2lw_1}e^{-(q_1+q_2)w_1}q_1q_2f(q_1,q_2)$$
$$=\frac{1}{(e^{w_1}-e^{-w_1})^2}\widehat{f}(w_1,w_1),$$

where $p_1-q_1-q_2=2l$, and

$$\widehat{f}(w_1,w_2)=\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}q_1q_2f(q_1,q_2)e^{-(q_1w_1+q_2w_2)}.$$

The Laplace transform of the fourth line, then, is

$$\frac{1}{(e^{w_1}-e^{-w_1})^2}\left[\widehat{L}_{g-1,n+1}(w_1,w_1,w_{[n]\setminus\{1\}})+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}}\widehat{L}_{g_1,|I|+1}(w_1,w_I)\widehat{L}_{g_2,|J|+1}(w_1,w_J)\right].$$

The relations $\dfrac{1}{(e^{w_j}-e^{-w_j})^2}=\dfrac{1}{16}\dfrac{(t_j^2-1)^2}{t_j^2}$, $\dfrac{e^{w_1}}{e^{w_1}-e^{w_j}}-\dfrac{e^{w_1+w_j}}{e^{w_1+w_j}-1}=\dfrac{t_j(t_1^2-1)}{t_1^2-t_j^2}$, and $\widehat{L}_{g,n}\big(w_1(t),\dots,w_n(t)\big)=(-1)^n2^{-n}\mathcal{L}_{g,n}(t_1,\dots,t_n)(t_1^2-1)\cdots(t_n^2-1)$ hold, when changing from $w_{[n]}$ to $t_{[n]}$ using formula 4.3. By replacing each expression in w by one in t, formula 4.5 is obtained.

Now, notice that $\mathcal{L}_{g,n}(t_{[n]})$ is a Laurent polynomial in the t_j^2 variables. This can be proven by induction over $2g-2+n$, using the base case $(g,n)=(0,3)$ or $(1,1)$. The initial values are $\mathcal{L}_{0,3}(t_{[3]})=-\dfrac{1}{16}\big(1-\dfrac{1}{t_1^2t_2^2t_3^2}\big)$ and $\mathcal{L}_{1,1}(t)=-\dfrac{1}{128}\dfrac{(t^2-1)^3}{t^4}$. The proof that it is a Laurent polynomial is as follows: check that each term being summed over in formula 4.3 is a Laurent polynomial. The only potential difficulty arises in checking for the term

$$\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\Big(\frac{(t_1^2-1)^3}{t_1^2}\mathcal{L}_{g,n-1}(t_{[n]\setminus\{j\}})-\frac{(t_j^2-1)^3}{t_j^2}\mathcal{L}_{g,n-1}(t_{[n]\setminus\{1\}})\right].$$

First, denote $\mathcal{L}_{g,n-1}(t_{[n]\setminus\{j\}})$ by $f(t_1)$ and $\mathcal{L}_{g,n-1}(t_{[n]\setminus\{1\}})$ by $f(t_j)$. Then the expression being considered is

$$\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\Big(\frac{(t_1^2-1)^3}{t_1^2}f(t_1)-\frac{(t_j^2-1)^3}{t_j^2}f(t_j)\right].$$

By induction hypothesis, $f(t_1)$ and $f(t_j)$ are Laurent polynomials in the t_i^2 variables. Therefore, $f(t_1)-f(t_j)$ is a Laurent polynomial in the t_i^2 variables. As $t_1\to t_j$, the expression tends to zero. Thus, it is divisible by $t_1^2-t_j^2$, and the quotient is a Laurent polynomial in the t_i^2 variables. For P a Laurent polynomial in the t_i^2 variables, $\frac{\partial}{\partial t_j}t_jP(t_1,t_j)$ is still a Laurent polynomial in the t_i^2 variables. Thus $\mathcal{L}_{g,n}(t_{[n]})$ is a Laurent polynomial in the t_i^2 variables.

Since $\mathcal{L}_{g,n}(t_{[n]})$ is a Laurent polynomial, the degree is well-defined; denote the top degree terms by $\overline{\mathcal{L}}_{g,n}(t_{[n]})$. It is not difficult to show (again by induction) that the following formula holds.

Theorem 4.2. *The top degree terms of the Laplace transform of $N_{g,n}$ satisfies the following recursion formula:*

$$\begin{aligned}(4.6)\quad & \overline{\mathcal{L}}_{g,n}(t_{[n]}) \\ &= -\frac{1}{16}\sum_{j=2}^{n}\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\left(t_1^4\overline{\mathcal{L}}_{g,n-1}(t_{[n]\setminus\{j\}})-t_j^4\overline{\mathcal{L}}_{g,n-1}(t_{[n]\setminus\{1\}})\right)\right] \\ &\quad -\frac{1}{32}\,t_1^4\left[\overline{\mathcal{L}}_{g-1,n+1}(t_1,t_1,t_{[n]\setminus\{1\}})+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}}\overline{\mathcal{L}}_{g_1,|I|+1}(t_1,t_I)\overline{\mathcal{L}}_{g_2,|J|+1}(t_1,t_J)\right].\end{aligned}$$

From here, it is easy to see that the degree of $\mathcal{L}_{g,n}(t_{[n]})$ is $2(3g-3+n)$.

5. The Laplace transform of the Euclidean Volume

Due to the relation of the Euclidean volume and the lattice point count functions, given by equation 3.5, $\overline{\mathcal{L}}_{g,n}(t_{[n]}) = V^E_{g,n}(t_{[n]})$.
Proof:
From (3.5),

$$\begin{aligned}\int_{\mathbb{R}^n_+} v^E_{g,n}(\mathbf{p})e^{-\langle w,\mathbf{p}\rangle}dp_1\cdots dp_n &= \lim_{k\to\infty}\sum_{\mathbf{p}\in\frac{1}{k}\mathbb{Z}^n_+} N_{g,n}(k\mathbf{p})e^{-\langle w,\mathbf{p}\rangle}\frac{1}{k^{3(2g-2+n)}} \\ &= \lim_{k\to\infty}\sum_{\mathbf{p}\in\mathbb{Z}^n_+} N_{g,n}(\mathbf{p})e^{-\frac{1}{k}\langle w,\mathbf{p}\rangle}\frac{1}{k^{3(2g-2+n)}} \\ &= \lim_{k\to\infty} L_{g,n}\left(\frac{w_1}{k},\cdots,\frac{w_n}{k}\right)\frac{1}{k^{3(2g-2+n)}}.\end{aligned}$$

The coordinate transformation, given by (4.3), has the expansion near $w=0$

$$\begin{aligned} t = t(w) &= -\frac{2}{w}-\frac{w}{6}+\frac{w^3}{360}-\frac{w^5}{15120}+\cdots, \\ w = w(t) &= -\frac{2}{t}-\frac{2}{3t^3}-\frac{2}{5t^5}-\cdots. \end{aligned}\tag{5.1}$$

Since

$$\mathcal{L}_{g,n}(t_{[n]}) = \frac{\partial^n}{\partial t_1\cdots\partial t_n}L_{g,n}\big(w(t_1),\ldots,w(t_n)\big)$$

is a Laurent polynomial of degree $2(3g-3+n)$, and since $w\mapsto w/k$ makes

$$t\longmapsto k\,t+\mathcal{O}\left(\frac{1}{k}\right)$$

for a fixed t,

$$\frac{\partial^n}{\partial t_1 \cdots \partial t_n} \int_{\mathbb{R}_+^n} v_{g,n}^E(\mathbf{p}) e^{-\langle w(t), \mathbf{p}\rangle} dp_1 \cdots dp_n$$
$$= \lim_{k\to\infty} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \ L_{g,n}\left(\frac{w(t_1)}{k}, \cdots, \frac{w(t_n)}{k}\right) \ \frac{1}{k^{3(2g-2+n)}}$$
$$= \lim_{k\to\infty} \mathcal{L}_{g,n}\left(kt_1 + \mathcal{O}\left(\frac{1}{k}\right), \cdots, kt_n + \mathcal{O}\left(\frac{1}{k}\right)\right) \ \frac{k^n}{k^{3(2g-2+n)}} = V_{g,n}^E(t_{[n]}).$$

This result then gives

Theorem 5.1. $V_{g,n}^E$ *satisfies the following recursion formula:*

$$(5.2) \quad V_{g,n}^E(t_{[n]})$$
$$= -\frac{1}{16} \sum_{j=2}^{n} \frac{\partial}{\partial t_j} \left[\frac{t_j}{t_1^2 - t_j^2}\left(t_1^4 V_{g,n-1}^E(t_{[n]\setminus\{j\}}) - t_j^4 V_{g,n-1}^E(t_{[n]\setminus\{1\}})\right)\right]$$
$$- \frac{1}{32}\, t_1^4 \left[V_{g-1,n+1}^E(t_1, t_1, t_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}} V_{g_1,|I|+1}^E(t_1, t_I) \overline{\mathcal{L}}_{g_2,|J|+1}(t_1, t_J)\right].$$

At this point, the Laplace transform of equation (3.6) can be computed (after multiplying both sides by $p_2 \cdots p_n$), and since every step is invertible, the previous recursion formula proves that formula (3.6) holds.
To compute the Laplace transform of (3.6), first consider the symmetric function $\widehat{V}_{g,n}^E(w_N)$ defined by the Laplace transform

$$\widehat{V}_{g,n}^E(w_1, \ldots, w_n) = \frac{\partial}{\partial w_1} \cdots \frac{\partial}{\partial w_n} \int_{\mathbb{R}_+^n} v_{g,n}^E(\mathbf{p}) e^{-\langle w, \mathbf{p}\rangle} dp_1 \cdots dp_n.$$

It satisfies the topological recursion

$$(5.3) \quad \widehat{V}_{g,n}^E(w_{[n]}) = -\frac{1}{2} \sum_{j=2}^{\infty} \frac{\partial}{\partial w_j} \left[\frac{w_j}{w_1^2 - w_j^2}\left(\frac{\widehat{V}_{g,n-1}^E(w_{[n]\setminus\{j\}})}{w_1^2} - \frac{\widehat{V}_{g,n-1}^E(w_{[n]\setminus\{1\}})}{w_j^2}\right)\right]$$
$$- \frac{1}{2w_1^2}\left(\widehat{V}_{g-1,n+1}^E(w_1, w_1, w_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g, \\ I\sqcup J=[n]\setminus\{1\}}} \widehat{V}_{g_1,|I|+1}^E(w_1, w_I) \widehat{V}_{g_2,|J|+1}^E(w_1, w_J)\right).$$

Proof:
Since

$$\widehat{V}_{g,n}^E(w_{[n]}) = (-1)^n \int_{\mathbb{R}_+^n} p_1 \cdots p_n v_{g,n}^S(\mathbf{p}) e^{-\langle w_{[n]}, \mathbf{p}\rangle} dp_1 \cdots dp_n,$$

to obtain the formula, multiply both sides of (3.6) by $(-1)^n p_2 \cdots p_n$ and take its Laplace transform. The left-hand side gives $\widehat{V}_{g,n}^E(w_{[n]})$.

For a continuous function $f(q)$, by putting $p_1+p_j-q=l$,

$$\begin{aligned}(5.4)\quad &\int_0^\infty dp_1\int_0^\infty dp_j\int_0^{p_1+p_j}dq\; p_jq(p_1+p_j-q)f(q)e^{-(p_1w_1+p_jw_j)}\\
&=\int_0^\infty dq\int_0^\infty dl\int_0^{q+l}dp_j\; qlf(q)e^{-qw_1}e^{-lw_1}p_je^{p_j(w_1-w_j)}\\
&=\frac{1}{(w_1-w_j)^2}\int_0^\infty dq\int_0^\infty dl\; qlf(q)\Big[e^{-(q+l)w_1}-e^{-(q+l)w_j}+(q+l)(w_1-w_j)e^{-(q+l)w_j}\Big]\\
&=\frac{1}{(w_1-w_j)^2}\left(\frac{\widehat{f}(w_1)}{w_1^2}-\frac{\widehat{f}(w_j)}{w_j^2}\right)-\frac{1}{w_1-w_j}\frac{\partial}{\partial w_j}\left(\frac{\widehat{f}(w_j)}{w_j^2}\right)\\
&=\frac{\partial}{\partial w_j}\left[\frac{1}{w_1-w_j}\left(\frac{\widehat{f}(w_1)}{w_1^2}-\frac{\widehat{f}(w_j)}{w_j^2}\right)\right],\end{aligned}$$

where $\widehat{f}(w)=\int_0^\infty qf(q)e^{-qw}dq$. Set $p_1-p_j-q=l$. Then

$$\begin{aligned}(5.5)\quad &\int_0^\infty dp_1\int_0^\infty dp_jH(p_1-p_j)\int_0^{p_1-p_j}dq\; p_jq(p_1-p_j-q)f(q)e^{-(p_1w_1+p_jw_j)}\\
&=\int_0^\infty dq\int_0^\infty dl\int_0^\infty dp_j\; qlf(q)e^{-qw_1}e^{-lw_1}p_je^{-p_j(w_1+w_j)}=\frac{1}{(w_1+w_j)^2}\frac{\widehat{f}(w_1)}{w_1^2},\end{aligned}$$

and similarly,

$$\begin{aligned}(5.6)\quad &-\int_0^\infty dp_1\int_0^\infty dp_jH(p_j-p_1)\int_0^{p_j-p_1}dq\; p_jq(p_j-p_1-q)f(q)e^{-(p_1w_1+p_jw_j)}\\
&=-\int_0^\infty dq\int_0^\infty dl\int_0^\infty dp_1\; qlf(q)e^{-qw_j}e^{-lw_j}(p_1+q+l)e^{-p_1(w_1+w_j)}\\
&=-\int_0^\infty dq\int_0^\infty dl\; qlf(q)e^{-qw_j}e^{-lw_j}\left[\frac{1}{(w_1+w_j)^2}+\frac{q+l}{w_1+w_j}\right]\\
&=-\frac{1}{(w_1+w_j)^2}\frac{\widehat{f}(w_j)}{w_j^2}+\frac{1}{w_1+w_j}\frac{\partial}{\partial w_j}\left(\frac{\widehat{f}(w_j)}{w_j^2}\right).\end{aligned}$$

Adding (5.5) and (5.6) gives

$$\begin{aligned}&\int_0^\infty dp_1\int_0^\infty dp_jH(p_1-p_j)\int_0^{p_1-p_j}dq\; p_jq(p_1-p_j-q)f(q)e^{-(p_1w_1+p_jw_j)}\\
&\quad-\int_0^\infty dp_1\int_0^\infty dp_jH(p_j-p_1)\int_0^{p_j-p_1}dq\; p_jq(p_j-p_1-q)f(q)e^{-(p_1w_1+p_jw_j)}\\
&=\frac{1}{(w_1+w_j)^2}\left(\frac{\widehat{f}(w_1)}{w_1^2}-\frac{\widehat{f}(w_j)}{w_j^2}\right)+\frac{1}{w_1+w_j}\frac{\partial}{\partial w_j}\left(\frac{\widehat{f}(w_j)}{w_j^2}\right)\\
&=-\frac{\partial}{\partial w_j}\left[\frac{1}{w_1+w_j}\left(\frac{\widehat{f}(w_1)}{w_1^2}-\frac{\widehat{f}(w_j)}{w_j^2}\right)\right].\end{aligned}$$

The sum of the right-hand sides of (5.4)-(5.6) thus becomes

$$\frac{\partial}{\partial w_j}\left[\left(\frac{1}{w_1-w_j}-\frac{1}{w_1+w_j}\right)\left(\frac{\widehat{f}(w_1)}{w_1^2}-\frac{\widehat{f}(w_j)}{w_j^2}\right)\right].$$

Therefore, the first three lines of (3.6) give

$$-\frac{1}{2}\sum_{j=2}^{\infty}\frac{\partial}{\partial w_j}\left[\frac{w_j}{w_1^2-w_j^2}\left(\frac{\widehat{V}^E_{g,n-1}(w_{[n]\backslash\{j\}})}{w_1^2}-\frac{\widehat{V}^E_{g,n-1}(w_{[n]\backslash\{1\}})}{w_j^2}\right)\right].$$

For a continuous function f,

$$\begin{aligned}\int_0^\infty dp_1\iint_{0\le q_1+q_2\le p_1} & q_1q_2(p_1-q_1-q_2)f(q_1,q_2)e^{-p_1w_1}dq_1dq_2\\ &=\int_0^\infty dq_1\int_0^\infty dq_2\int_0^\infty dl\; q_1q_2lf(q_1,q_2)e^{-lw_1}e^{-(q_1+q_2)w_1}=\frac{\widehat{f}(w_1,w_1)}{w_1^2},\end{aligned}$$

where $\widehat{f}(w_1,w_2)=\int_{\mathbb{R}^2_+}p_1p_2f(p_1,p_2)e^{-(p_1w_1+p_2w_2)}dp_1dp_2$. Thus the last two lines of (3.6) give

$$-\frac{1}{2}w_1^2\left(\widehat{V}^E_{g-1,n+1}(w_1,w_1,w_{[n]\backslash\{1\}})+\sum_{\substack{g_1+g_2=g,\\ I\sqcup J=[n]\backslash\{1\}}}\widehat{V}^E_{g_1,|I|+1}(w_1,w_I)\widehat{V}^E_{g_2,|J|+1}(w_1,w_J)\right).$$

Hence (5.3) is proven.

Now, change coordinates by a new coordinate change, $w_j=-\frac{2}{t_j}$. Then

$$dw_j=\frac{2}{t_j^2}dt_j,\text{ and }\qquad \frac{\partial}{\partial w_j}=\frac{t_j^2}{2}\frac{\partial}{\partial t_j}\text{ hold.}$$

Then

$$V^E_{g,n}(t_{[n]})dt_{[n]}=\widehat{V}^E_{g,n}(w_{[n]})dw_{[n]},$$

or

$$V^E_{g,n}(t_{[n]})=2^n\frac{\widehat{V}^E_{g,n}(w_{[n]})}{t_1^2\cdots t_n^2}\text{ holds.}$$

So, multiply both sides of (5.3) by $\frac{2^n}{t_1^2\cdots t_n^2}$ to get, for the first term of the first line,

$$\begin{aligned}&-\frac{1}{2}\frac{2^n}{t_1^2\cdots t_n^2}\sum_{j=2}^{\infty}\frac{\partial}{\partial w_j}\left[\frac{w_j}{w_1^2-w_j^2}\,\frac{\widehat{V}^E_{g,n-1}(w_{[n]\backslash\{j\}})}{w_1^2}\right]\\ &=-\frac{1}{2}\sum_{j=2}^{\infty}\frac{\partial}{\partial t_j}\left[\frac{1}{2}\,\frac{t_1^2\,t_j}{t_1^2-t_j^2}\,\frac{t_1^2}{4}\,V^E_{g,n-1}(t_{[n]\backslash\{j\}})\right]=-\frac{1}{16}\sum_{j=2}^{\infty}\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\,t_1^4\,V^E_{g,n-1}(t_{[n]\backslash\{j\}})\right].\end{aligned}$$

Similarly, the second term of the first line becomes

$$\frac{1}{2}\frac{2^n}{t_1^2\cdots t_n^2}\sum_{j=2}^{\infty}\frac{\partial}{\partial w_j}\left[\frac{w_j}{w_1^2-w_j^2}\,\frac{\widehat{V}^E_{g,n-1}(w_{[n]\setminus\{1\}})}{w_j^2}\right]$$
$$=\frac{1}{2}\sum_{j=2}^{\infty}\frac{\partial}{\partial t_j}\left[\frac{1}{2}\,\frac{t_1^2\,t_j}{t_1^2-t_j^2}\,\frac{t_j^2}{4}\,\frac{t_j^2 V^E_{g,n-1}(t_{[n]\setminus\{1\}})}{t_1^2}\right]=\frac{1}{16}\sum_{j=2}^{\infty}\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\,t_j^4\,V^E_{g,n-1}(t_{[n]\setminus\{1\}})\right].$$

The second line of (5.3) is changed similarly. Since $V^E_{g,n}(t_{[n]})$ is known to be continuous, every step in the Laplace transform of (3.6) is reversible, and so the formula for the Euclidean volume holds.

6. Examples

By using the recursion formulas, it is possible to compute $v^E_{g,n}$ and $N_{g,n}$. Here are some of the results for small (g,n), including the initial data. The results for $N_{g,n}$ will only hold when the sum of its arguments is even; otherwise it will be zero. These values (except the initial conditions, of course) were computed using Mathematica.

Lattice Point Count and Euclidean Volumes	
$v^E_{0,3}(x,y,z)$	$\frac{1}{2}$
$v^E_{0,4}(x,y,z,w)$	$\frac{1}{8}(x^2+y^2+z^2+w^2)$
$v^E_{0,5}(x,y,z,w,r)$	$\frac{1}{64}(x^4+y^4+z^4+w^4+r^4+4x^2y^2+4x^2z^2+4x^2w^2+$
	$4x^2r^2+4y^2z^2+4y^2w^2+4y^2r^2+4z^2w^2+4z^2r^2+4w^2r^2)$
$v^E_{1,1}(x)$	$\frac{1}{96}x^2$
$v^E_{1,2}(x,y)$	$\frac{1}{768}(x^4+2x^2y^2+y^4)$
$v^E_{1,3}(x,y,z)$	$\frac{1}{9216}(x^6+y^6+z^6+6x^4y^2+6x^4z^2+$
	$6y^4z^2+6x^2y^4+6x^2z^4+6y^2z^4+12x^2y^2z^2)$
$v^E_{2,1}(x)$	$\frac{13}{46448640}x^8$
$N_{0,3}(x,y,z)$	1
$N_{0,4}(x,y,z,w)$	$\frac{1}{4}(x^2+y^2+z^2+w^2-4)$
$N_{0,5}(x,y,z,w,r)$	$\frac{1}{32}(x^4+y^4+z^4+w^4+r^4+4x^2y^2+4x^2z^2+4x^2w^2+$
	$4x^2r^2+4y^2z^2+4y^2w^2+4y^2r^2+4z^2w^2+4z^2r^2+4w^2r^2$
	$-20x^2-20y^2-20z^2-20w^2-20r^2+64)$
$N_{1,1}(x)$	$\frac{1}{48}x^2-\frac{1}{12}$
$N_{1,2}(x,y)$	$\frac{1}{384}(x^4+y^4+2x^2y^2-12x^2-12y^2+32)$
$N_{2,1}(x)$	$\frac{1}{23224320}\,(13x^8-804x^6+15204x^4-89536x^2+163008)$

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