CLASSIFYING 1-LATTICE MAXIMAL POLYTOPES

By

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ABSTRACT. In this senior thesis, we classify 1-Lattice Maximal Pyramids and Prisms up to unimodular transformation. 1-Lattice Maximal polytopes are important objects in convex geometry and find applications in algebra and optimization. To do this, we define a canonical position for polytopes to limit the search area. We then analyze the base of the polytopes and determine which are too big to be the bases. This creates a bounding box where the desired polytopes can be found. Finally we create two algorithms, one that searches through the bounding box to find all the possible polytopes inside and another that analyzes the results and deletes equivalent polytopes.

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CHAPTER 1

Introduction

1.1. Background and Motivation

Back in elementary school many of us learned the names of certain shapes—triangle, square, hexagon, etc. We learned that these shapes were called *polygons* and may have practiced drawing them at one time or another. Further on in math, we learn about three dimensional shapes such as pyramids and prisms, which fall under the category of shapes known as *polytopes*. The precise mathematical definition of a polytope in $\mathbb{R}^3$ is the convex hull of four or more affinely independent points in $\mathbb{R}^3$. A *lattice polytope* is one whose vertices are all points with integer coordinates. Polytopes can be beautiful geometric shapes, but their uses do not just stop at their appearance; studying their nature and behavior can aid us in numerous applications. There are many uses, both practical and theoretical, for lattice polytopes in various aspects of mathematics and applied mathematics that drive our research of this topic. In integer optimization, some algorithms use k-lattice bodies as representation of their solutions making these shapes valuable in helping to find integer solutions to these problems. Lattice polytopes also have uses in both algebraic and convex geometry. They aid in finding special ways to “triangulate” geometric bodies as well as easily estimating the number of integer points in a geometric body. An explicit example in number theory where lattice polytopes are useful is the *Frobenius problem*, which is the problem of giving change in different ways when you have coinages of various specific amounts.

1.2. Basic Definitions

Now that we have established some background information regarding polytopes, some basic definitions regarding lattice polygons will be provided. The *integer lattice* in $\mathbb{R}^3$ is the set of all points with integer coordinates and a *lattice point* is a point with integer coordinates. A *lattice pyramid* in $\mathbb{R}^3$ is a pyramid whose vertices are lattice points in $\mathbb{R}^3$. A *lattice prism* is the Minkowski sum of a polygon with a line segment. We say a lattice prism or pyramid is considered *maximal* if it contains at least one lattice point in the relative interior of each of its facets. This paper will give an explicit classification of both 1-lattice, which means it contains one lattice point in its interior, maximal prisms and pyramids up to integer unimodular transformations.

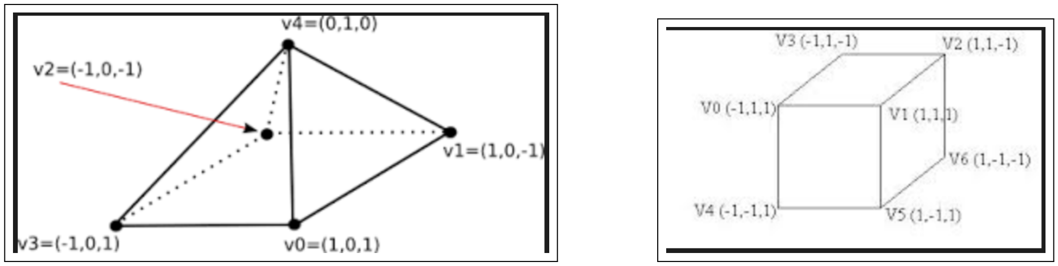

FIGURE 1. Lattice prism and lattice pyramid—Note: These are not in canonical position

An affine transformation $T(v) = Av + x_0$ is *integer unimodular* if both A and x_0 have integer entries and $\det(A) = \pm 1$. For example, the transformation

$$T_v = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{pmatrix}, + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{1.1}$$

is an integer unimodular transformation. Since integer unimodular transformations preserve both volume and the number of lattice points in a set, the classification of 1-lattice maximal pyramids and prisms will be done up to integer unimodular transformations. Given lattice pyramids or prisms S_1, S_2, we say that they are *lattice equivalent* if there exists an integer unimodular transformation T such that $T(S_1) = S_2$. Having lattice equivalent polytopes is essential to bounding the search area as it allows us to make use of canonical position.

If a polytope S has one of its facets contained in the $z = 0$ plane, then this facet will be called the *base* of the polytope. For pyramids, the vertex of S which is not contained in the base will be called the *apex* of S and the *height* of S will be the z-coordinate of its apex. For prisms, the height will be denoted the same as pyramids but there is no apex; instead there is a top facet the same shape as the base but not contained in the $z = 0$ plane.

1.3. Previous Research

Because of the myriad application that lattice polytopes have in mathematics, different mathematicians have been studying the existence and behaviors of lattice polytopes for centuries. Out of this compilation of research has come several historical milestones that were useful in formulating the methodology to find maximal one-lattice polytopes. Three important mathematicians that made discoveries specifically relevant to canonical position were Pick, Scott and Rabinowitz. In 1889 G. Pick related the area of a convex lattice polygon to the number of interior and boundary points resulting in Pick's formula. This, along with Scott's bound, which P.R. Scott discovered in 1979, and bounds the number of boundary points in a k-lattice polygon, were used in the proof for $2D$ canonical position

[**4**]. This proof was in turn part of the proof for canonical position in $3D$. The work of Rabinowitz, namely the x-axis lemma, was also useful in the proof of canonical position [**3**]. The other relevant discoveries have to do specifically with polytopes and were useful in shaping the thought process when in the beginning stages of classification. For example, in 1991 J.C. Lagarias and G.M. Ziegler proved that for each k there are finitely many k-lattice polyhedra up to unimodular transformation, an important fact as otherwise we could not even classify these polyhedra. Lastly, in 2011 G. Averkov et al showed that there are 12 different lattice-free polyhedra. This paper in particular was useful as it showed us what such lattice-free polyhedra looked like and gave an example of classifying polyhedra [**1**].

One important previous paper and research project was on lattice simplices. This research was conducted by a team of undergraduates that I worked with for part of the summer. Their work dealt with *lattice simplices*, the convex hull of four affinely independent lattice points. Much of their methodology is similar to classifying pyramids and prisms, though there are some differences by virtue of having different shapes. They were able to classify all one-lattice simplices. They came to the result that there are 11 1-Lattice simplices [**2**]. Here is a table of their results.

No.	p	h	k	μ
1	12	0	3	2
2	8	0	4	2
3	6	0	6	2
4	8	2	4	2
5	6	0	3	3
6	6	1	3	3
7	4	0	2	4
8	4	1	1	4
9	4	1	2	4
10	4	0	4	4
11	3	0	2	6

Given $p, h \in \mathbb{N}$ and $k \in \mathbb{Z}$ the triangle with vertices $(0,0,0), (p,0,0)$ and $(k,h,0)$ will be denoted as $\triangle(p,k,h)$. Also note that μ=height of the simplex. In addition to the classification of simplices, part of their proof for canonical position is needed to finish the 3-dimensional proof of canonical position.

LEMMA 1.3.1 (x-axis lemma). *Let P be a lattice polygon, e an edge of P, and A, B the endpoints of e. Let $\ell(e)$ be the interior length. If $\ell(e) = p$, then there exists an integer unimodular transformation that maps A into the origin, B into $(p,0)$ and maps all the remaining vertices of P into points above the x-axis. [3]*

Using Rabinowitz's x-axis lemma, it can be proven that

LEMMA 1.3.2. *Every lattice polygon P is lattice equivalent to a lattice polygon in canonical position. [3]*

Proof. Let p be the largest integer length of the edges of P. By the x-axis lemma, P is lattice equivalent to a polygon P' that has the segment with endpoints $(0,0)$ and $(0,p)$ as an edge and all its vertices above the x-axis. Since integral unimodular transformations preserve the number of lattice points in a given set, the base of P' has the biggest integer length of its sides. Applying the necessary shears about the x-axis, we can assume all the vertices of P' have non-negative y-coordinate.

Let (k,h) be the vertex of P' that follows clockwise from the origin. Let $a, r \in \mathbb{Z}$ with $r \in \{0,\ldots,h-1\}$ be such that $k = ah + r$ and let F be the shear about the x-axis of magnitude $-a$. Then

$$F(k,h) = (ah + r - ah, h) = (r,h).$$

All the vertices of the polygon $F(P')$ have non-negative x-coordinate because they are on the right of the line determined by the points $(0,0),(r,h)$. Then, P is lattice equivalent to $F(P')$, which is in canonical position.

□

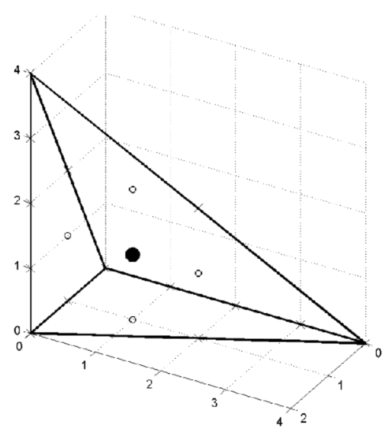

FIGURE 2. Example of Lattice Simplex

1.4. Canonical Position

Canonical Position is one of the most important aspects of the classification process as it is the foundation of our ability to create a bounding box that lattice polytopes will be found in. Thus, it merits its own section to properly define and also to prove that it is a valid position to shifts the lattice polytopes to.

DEFINITION 1.4.1. A lattice polygon P will be in *canonical position* if

(1) Every point $(x,y) \in P$ is such that $0 \le x$, $0 \le y$.

(2) The edge of P with largest integer length lies on the positive x-axis with one endpoint on the origin.
(3) The vertex that follows clockwise from the origin has coordinates (r, h) with $h \in N$, $r \in \{0, \ldots, h-1\}$.

The edge that lies on the x-axis will be called the *base* of the polygon.

DEFINITION 1.4.2. A lattice polytope P will be in *canonical position* if
(1) Its base is a polygon in canonical position.
(2) The apex of the polytope has the coordinates (i, j, u) where i and j are between 0 and $u-1$.

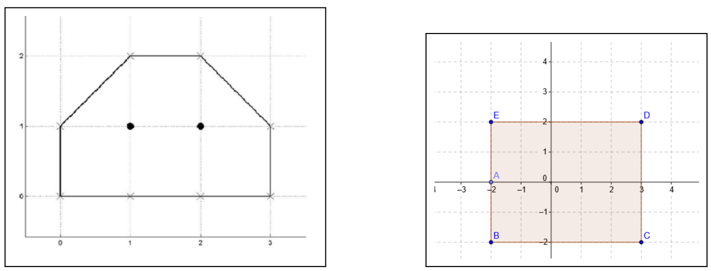

FIGURE 3. Left-Canonical Position, Right-Not Canonical Position

It can be shown that every pyramid and prism is integer equivalent to one in our canonical position. This then makes it possible to just look at the first quadrant when bounding the bases of the polytopes. This occurs in two steps: the first step was done by the simplices group and is a proof that the shape can be rotated and translated to the desired position from a $2d$ perspective. It was shown above. The second step addresses the issue of adding a third dimension and shows that a base floating in $\mathbb{R}^3$ can be brought down to the xy axis.

LEMMA 1.4.3. *Suppose u an v are two vectors in $\mathbb{Z}^3$ with integer length one. Then there exists an integer unimodular transformation T such that $Tu = (1, 0, 0)$ and $Tv = (p, q, 0)$ for some $p, q \in \mathbb{Z}$.*

Proof.
Write $u = (x, y, z)$. If $z \neq 0$, then consider the matrix

$$T_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -c & d \end{pmatrix},$$

where $c = \frac{z}{\gcd(y,z)}$, $d = \frac{y}{\gcd(y,z)}$, and a, b are the Bézout coefficients for d and c, respectively. Then

$$\det T_u = ad + bc = \gcd(c, d) = 1.$$

That is, T_u is an integer unimodular transformation Note that the z-coordinate of $T_u u$ is $-\frac{zy}{\gcd(y,z)} + \frac{yz}{\gcd(y,z)} = 0$.

Thus we may assume without loss of generality that $z = 0$. Similarly, we may assume that $y = 0$. Because u has integer length one, it follows that $u = (1, 0, 0)$.

We must now find a unimodular integer transformation T such that $Tu = u$ and Tv has z-coordinate zero.

Write $v = (x, y, z)$. Let

$$T_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -c & d \end{pmatrix},$$

where $c = \frac{z}{\gcd(y,z)}$, $d = \frac{y}{\gcd(y,z)}$, and a, b are the Bézout coefficients for d and c, respectively. Then

$$\det T_v = ad + bc = \gcd(c, d) = 1.$$

That is, T_v is an integer unimodular transformation Note that the z-coordinate of $T_v v$ is $-\frac{zy}{\gcd(y,z)} + \frac{yz}{\gcd(y,z)} = 0$. Note also that $T_v(1, 0, 0) = (1, 0, 0)$. Thus T_v is the desired transformation.

□

The following sections will use canonical position for their classifications.

CHAPTER 2

Pyramids

2.1. Methodology

We now begin by finding the bounds or the base of our polytopes, starting with pyramids. We do this by looking at slices of our pyramid at different levels. These slices will be smaller than the base of our pyramid. If any of these slices contains more than one integer point, it is too big, implying the base will be too big as well, since we are looking for pyramids with only one lattice point in their interiors. We will start with some definitions in regards to this methodology of slices.

Definition 2.1.1. Let S be a polytope with height μ, then the *slice of S at level j*, for $j \in \{0, \ldots, \mu\}$, is the intersection of S and the plane $z = j$. It will be denoted by T_j.

In particular, T_0 is the base of the polytope. It is also important to note that for a given j,

$$T_j = (1 - \frac{j}{\mu})T_0 + \frac{j}{\mu}A, \tag{2.1}$$

where A is the apex of S. Logistically, we know all 1-lattice polytopes have height greater than or equal to 2 since otherwise there would be no interior lattice point.

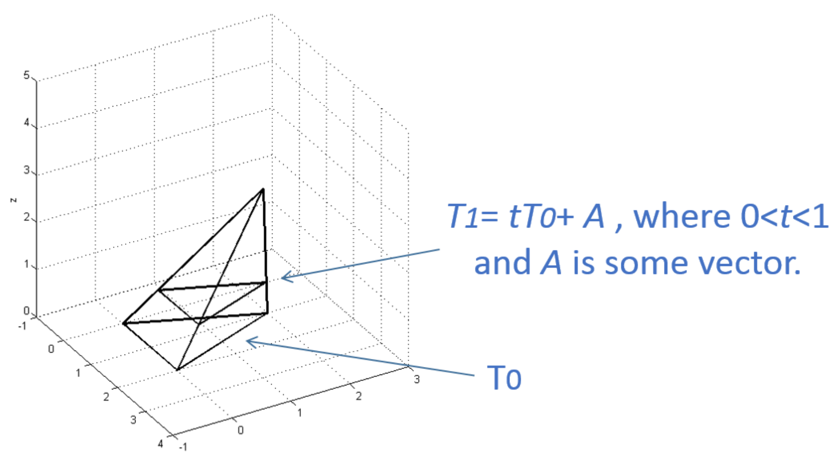

Figure 1. A Simplex and its Level One Slice [2]

The method for obtaining the possible bases for a quadrilateral pyramid is similar to that for simplices. Bounds will be determined by fitting either a triangle or parallelogram that is too big inside the quadrilateral base, or the level one slice. This will creates bounds for the base of the polytope. This gives rise to several different cases depending on the shape of the base.

THEOREM 2.1.2. *Every maximal, 1-lattice pyramid is lattice equivalent to one in the box with the dimensions* $(30 + (31\sqrt{2})) \times 31 \times 18$

This theorem is based off of the following lemmas which divide the theorem into cases based off of the placement of the vertices of the base of the polytope. Specifically, the four cases depend on where the vertices are located. These proofs hold for a quadrilateral with a base of any shape. Though many of the pictures are quadrilaterals, only one lemma is specific to a quadrilateral.

2.2. Lemmas

These lemmas are used to prove our main theorem. They are based off of the possible shapes for the base.

LEMMA 2.2.1. *Given a base of a pyramid, P in canonical position, figure 2, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with $v_0 = (0,0)$. If v_1 is on the y-axis and $v_{2_y} = v_{1_y}$, then $x + 2b \leq 12$ and $h \leq 31$ where x= righthand bound and b=base.*

Proof. This case is a trapezoid as $\overline{v_1 v_2}$ is parallel to the x-axis. Let ABCD be the parallelogram located inside the trapezoid with one side on the y axis and the top side the same as the top of the trapezoid. If this parallelogram has dimension $3 + \epsilon\ \times 3{+}\epsilon$, it will contain two lattice points inside it. Then, when scaled by $1/2$, the smallest T_1 slice, there will be a lattice points inside making the base too big. This situation will happen if the point $2/3$ from the x-axis on $\overline{CD}$ is greater than four or, in other words, if

$$\frac{x + 2b}{3}\ 4. \tag{2.2}$$

Therefore, to prevent this

$$\frac{2b + x}{3} 4,$$
$$2b + x \leq 12,$$

Thus $x + 2b$ must be less than or equal to 12. □

LEMMA 2.2.2. *Given a quadrilateral base of a pyramid, P seen in figure 3, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with $v_0 = (0,0)$. If v_1 is on the*

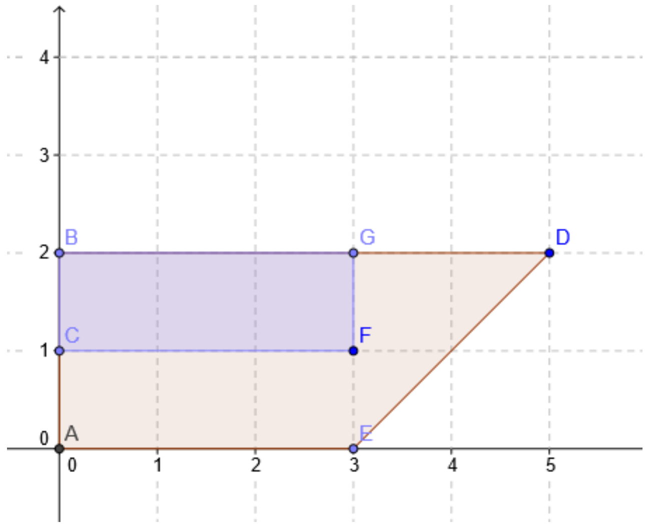

FIGURE 2. Lemma 2.2.1

y-axis and $v_{2_y} \neq v_{1_y}$ then the slope of the line $BC \geq 1$ and $x \leq g$ where g is the intersection of the slope and the line y=31.

Proof. If B is on the y axis it must be on an integer point. Thus, the y value will be ≥ 1 and slope will both be $\neq 1$. If positive, the slope will eventually intersect the height bound of $y = 31$ for the bases. If negative, it will intersect the x axis. The x-value of this intersection will be the bound for the x-value of the vertices. If positive, the least value the slope can then be is 1. This will then yield the furthest possible bound for x since other slopes will intersect $y = 31$ at a smaller x-value. Using the formula for slope, with slope equal to 1, and known points $(0, y)$ and $(x, 31)$ yields

$$1 = \frac{31 - y}{x},$$
$$x = 31 - y,$$

$y \geq 0$ so $x \leq 31$. □

LEMMA 2.2.3. *Given a base of a pyramid, P, figures 4,5, and 6, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with $v_0 = (0, 0)$. If v_1 is not on the y-axis then $x \leq 30 + (31\sqrt{2})$.*

Proof. Let quad ABCD be the base of the quadrilateral pyramid with A on the origin. Let AD be the line segment from A to D. Consider the triangle ABC. Let d be the distance from C to AB. By the x-axis lemma, there exists a single unimodular transformations such

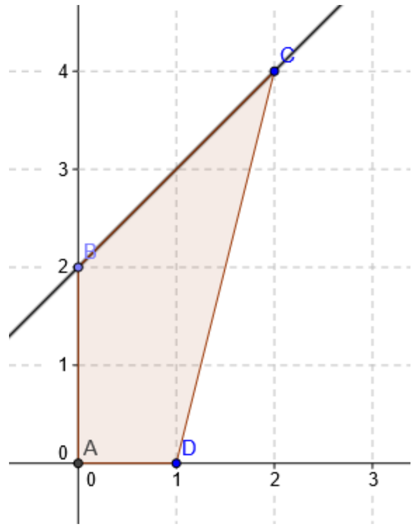

FIGURE 3. Lemma 2.2.2

that AB is on the x-axis and C is the apex of the triangle. This new triangle, $\triangle(A'B'C')$, will be lattice equivalent to $\triangle(ABC)$ and d will become d', the height of the new triangle. Since AB is getting mapped to the x-axis, the length will be the same or shrink as unimodular transformations preserve integer length. Since they also preserve area, d will either stay the same or grow larger, to preserve the area of $\triangle(ABC)$. This yields the inequality

$$d \leq d' \leq 31. \tag{2.3}$$

The bound for the x distance between C and B can be derived by looking at $\triangle(BCE)$ which is the triangle. C and B will be farthest apart when $\angle CBE$ is at a 45 degrees which would give a 45,45,90 triangle as $\overline{CE}$ is perpendicular to $\overline{BC}$. The inequality is now

$$\frac{C_x}{\sqrt{2}} \leq d \leq d' \leq 31, \tag{2.4}$$

$C_x \leq 31\sqrt{2}$. This is combined with the fact that the furthest x-value of B is 30 because of the nature of canonical position. Thus, the right-most bound for the base is $30 + (31\sqrt{2})$. $\square$

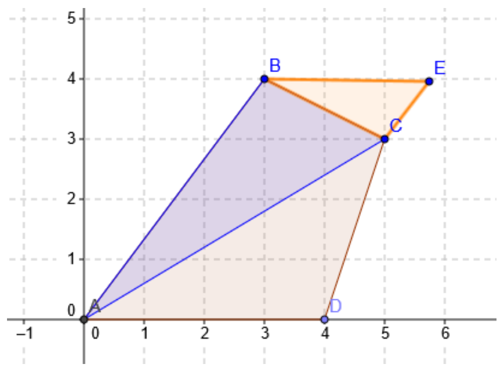

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FIGURE 4. Lemma 2.2.3

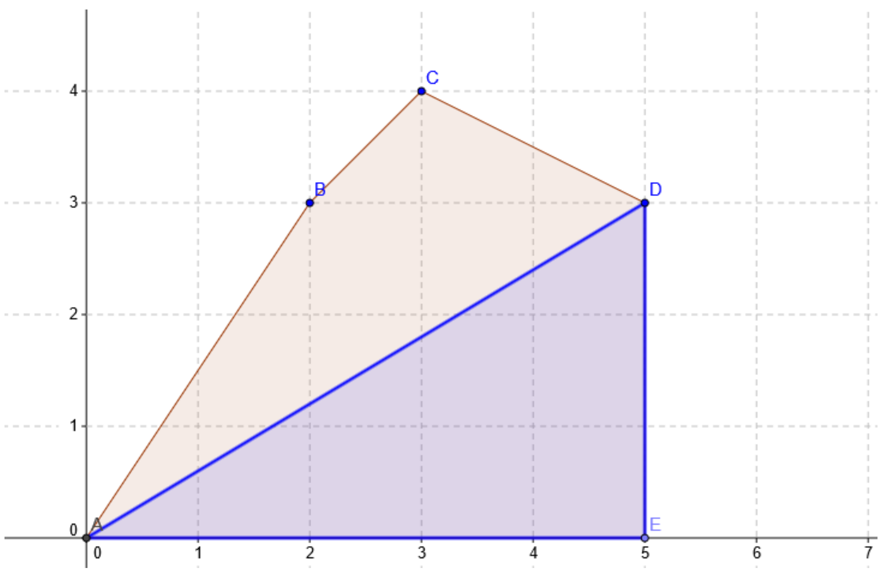


FIGURE 5. Lemma 2.2.3

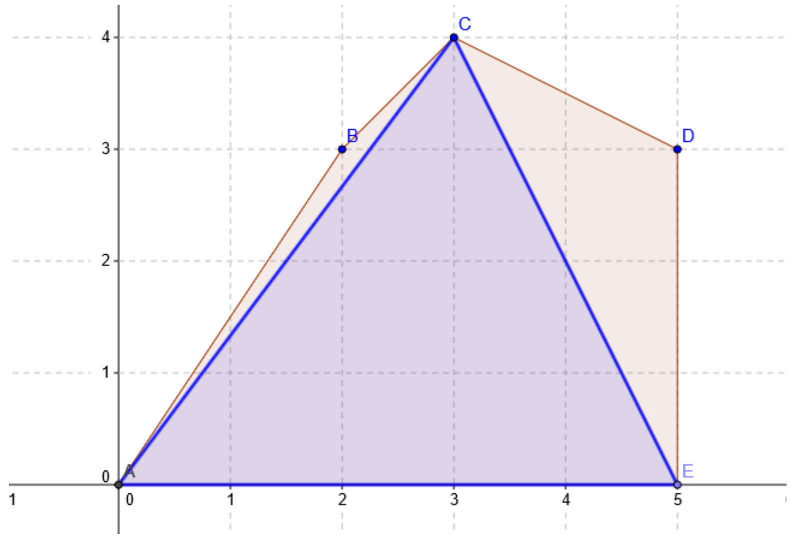


FIGURE 6. Lemma 2.2.3

CHAPTER 3

Prisms

The methodology for prisms is the same as for pyramids and the lemmas are proven in much the same manner. The only difference is that instead of looking at the T_1 slice of the prism, we can analyze the base directly as we do not have to worry about dilations as we do for pyramids. We will state and prove the dimensions of the bounding box as we did for the pyramids. Notice how much smaller they are then the bounds for pyramids.

THEOREM 3.0.4. *Every* 1*-lattice maximal prism is lattice equivalent to one found in the bounding box* $(11 + (12\sqrt{2})) \times 12 \times 6$.

The proof for this theorem is in cases as the pyramid theorem was.

3.1. Triangular Prisms

LEMMA 3.1.1. *In a triangular prism the the length of the prism base is always less than or equal to four.*

Proof. As the triangular base is a lattice polygon and maximal, it must contain an interior lattice point. Therefore, the height of the triangle cannot be one, as no integer point would be inside the triangle if it was. This means that the height is always greater than two. The smallest possible height for a triangle is two. Let $\triangle(A, B, C)$ be a triangular base with base length five and height two. Then a parallelogram with height two and base length four will always be able to fit inside. Thus, the base length of the triangular base is always less than four. □

This now gives different cases for the possible configuration of the triangular base depending on the base length. Looking at these various cases and taking the largest bounds provides the bounding box that the computer will use to search for triangular prisms. Case

1: The length of the base of the triangle is equal to one. In this case $h \leq 6$. This is because otherwise a triangle of the form $\triangle(1, k, h)$ with $h \geq 6$, $k \in \{0, \ldots h-1\}$ would fit inside and that was proven in lemma 3.12 to contain two lattice points.

Case 2: $2 \leq b \leq 4$. In this case look at triangular base $\triangle(p, k, h)$ and let b be the length of base and (r, h) be the coordinate of the top point. We then look at a parallelogram inside the triangle. This parallelogram has height α and base length

$$y = (\alpha/h) \cdot r + (1 - \alpha/h) \cdot b. \tag{3.1}$$

Knowing that the base is less than or equal to four we now solve for h. We know that we want the parallelogram to have dimensions $1 + \epsilon \times 2 + \epsilon$, otherwise it is too big. So $\alpha \leq 2$ and $y \leq 2$. We can now plug in those values and solve for h.

$$\begin{aligned}(1 - \frac{\alpha}{h}) \cdot b < 2,\\ \alpha \leq h - \frac{h}{b},\\ 1 - \frac{1}{b} \geq \frac{2}{h},\\ h \geq \frac{2b}{b-1},\end{aligned}$$

Since this would make the base too big

$$h \leq \frac{2b}{b-1} \leq 4. \tag{3.2}$$

So $h \leq 4$ when $b \neq 2$. □

The bounding box for triangular prism can be calculated from these bounds and is $4 \times 12 \times 6$.

3.1.1. Prisms With Bases with More than Three Sides. Now we move on to prisms with bases that are not triangular. These, like the pyramids, are divided into four cases based on the position of the vertices, with differing bounds for each. The method of proof to show each bound is the same as it was for the pyramids. The numbers are different however because of the lower height bound for prisms.

LEMMA 3.1.2. *Given a base of a prism, P, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with v_0 equal to $(0,0)$. If v_1 is on the y-axis and $v_{2_y} = v_{1_y}$, then $x + b \leq 4$ and $h \leq 12$ where x is right-hand bound, b is the length of the base and h is the height of the base.*

Proof. This case is a trapezoid as $\overline{v_1 v_2}$ is parallel to the x-axis. Let ABCD be the parallelogram located with one side on the y axis and the top side the same as the top of the trapezoid. If this parallelogram is $2 + \epsilon \times 1 + \epsilon$, it will contain two lattice points inside it, making the base too big. This will happen if the point halfway on the line $\overline{CD}$ is greater than two as then the top will contain a rectangle. Or, in other words, if

$$\frac{x+b}{2} \geq 2. \tag{3.3}$$

Therefore $x + b$ must be less than or equal to four. □

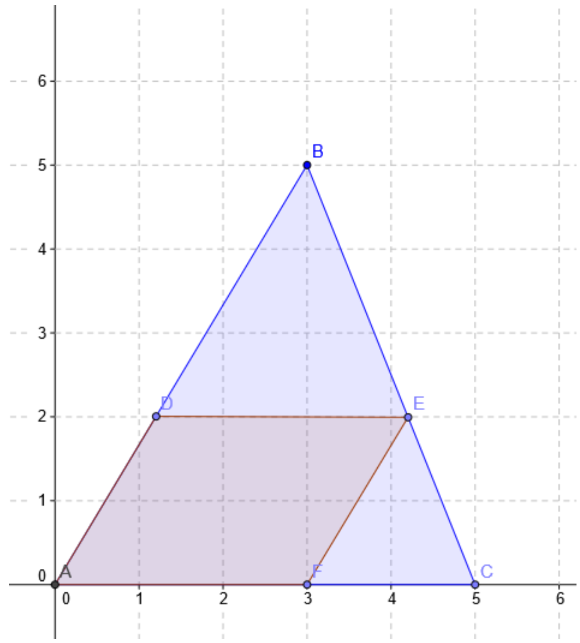


FIGURE 1. Triangular Prism Base

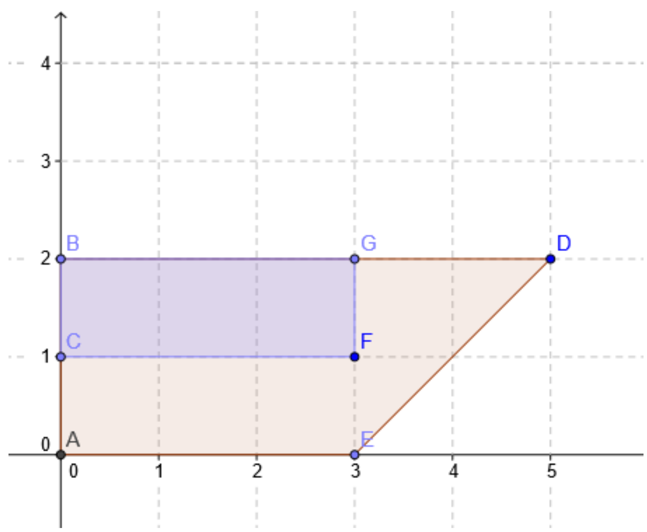


FIGURE 2. Lemma 3.1.2

LEMMA 3.1.3. *Given a quadrilateral base of a prism, P, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with $v_0 = (0,0)$. If v_1 is on the y-axis and*

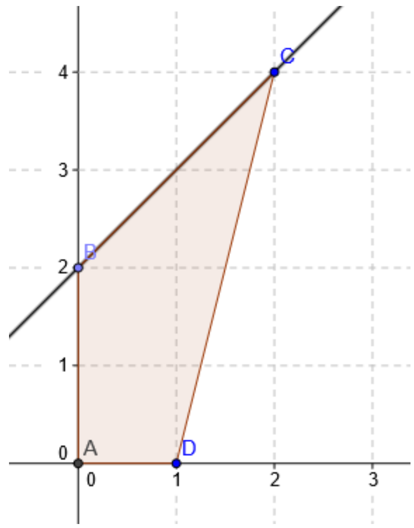


FIGURE 3. Lemma 3.1.3

$v_{2_y} \neq v_{1_y}$ then the slope of the line BC is greater than or equal to 1 *and $x \leq g$ where g is the intersection of the slope and the line y=12.*

Proof. If v_1 is on the y axis it must be on an integer point. The slope of the line connecting v_1 and v_2 will then also be an integer because of the properties of maximality. Thus, the y value and slope will both be greater than or equal to one and eventually the slope will intersect the height bound of y=12 for the bases. The x-value of this intersection will be the bound for the x-value of the vertices. The least steep the slope can then be is one. All other slopes are steeper so intersect y=12 at a smaller value. Using the formula for slope and known points (0,y) and (x,12) yields

$$1 = \frac{12-y}{x},$$
$$x = 12 - y,$$

$y \leq 12$ so $x \leq 12$.

□

LEMMA 3.1.4. *Given a quadrilateral base of a prism, P, let $v_0, v_1, \ldots, v_{s-1}$ be the vertices of P, labeled in a clockwise order starting with $v_0 = (0,0)$. If v_1 is not on the y axis then $x \leq 11 + (12\sqrt{2})$.*

Proof. Let polygon P be the base of the pyramid with v_0 on the origin. Let AD be the line segment from A to the rightmost vertex. Consider the triangle ABC. Let d

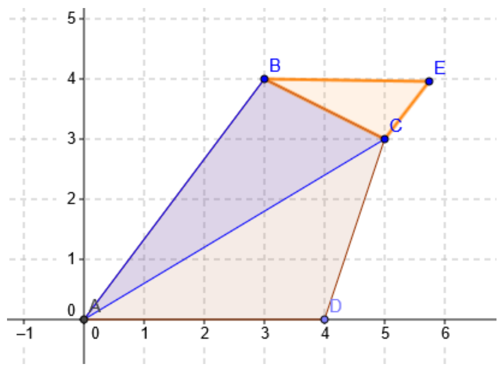


FIGURE 4. Lemma 3.1.4

be the distance from C to AB. By the x-axis lemma, there exists a series of unimodular transformations such that AB is on the x-axis and C is the apex of the triangle. This new triangle, $\triangle(A'B'C')$, will be lattice equivalent to $\triangle(ABC)$ and d will become d', the height of the new triangle. Since AB is getting mapped to the x-axis, the length will be the same or shrink as unimodular transformations preserve integer length. Since they also preserve area, d will either stay the same or grow larger, to preserve the area of $\triangle(ABC)$. This yields the inequality

$$d \leq d' \leq 11. \tag{3.4}$$

The bound for the x distance between C and B can be derived by looking at $\triangle(BCE)$ which is the triangle. C and B will be farthest apart when $\angle CBE$ is at a 45 degrees which would give a $45, 45, 90$ triangle as $\overline{CE}$ is perpendicular to $\overline{BC}$. The inequality is now

$$\frac{C_x}{\sqrt{2}} \leq d \leq d' \leq 11, \tag{3.5}$$

$C_x \leq 11\sqrt{2}$. This is combined with the fact that the furthest x-value of B is 12 because of the nature of canonical position. Thus, the rightmost bound for the base is $11 + (12\sqrt{2})$.
□

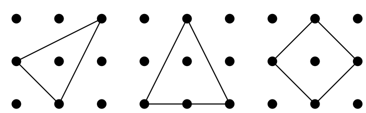
CHAPTER 4

Bounding the Height of Pyramids and Prisms

4.1. Introduction

Now that we have established the bounds for the length and height of the base of our polytopes we need to find bounds for the height of the polytope itself. We do this by looking at different cases based on. We can do these proofs for pyramids and prisms at the same time.

Consider the following three polygons:



(T_a) (T_b) (T_c)

Note that T_c is not actually a triangle. Each of these three polygons is a possible base for our pyramid our prism. Moreover, for bases that are larger and have more side, one of these three can fit inside the larger base so the bounds derived from looking at these shapes will still hold.

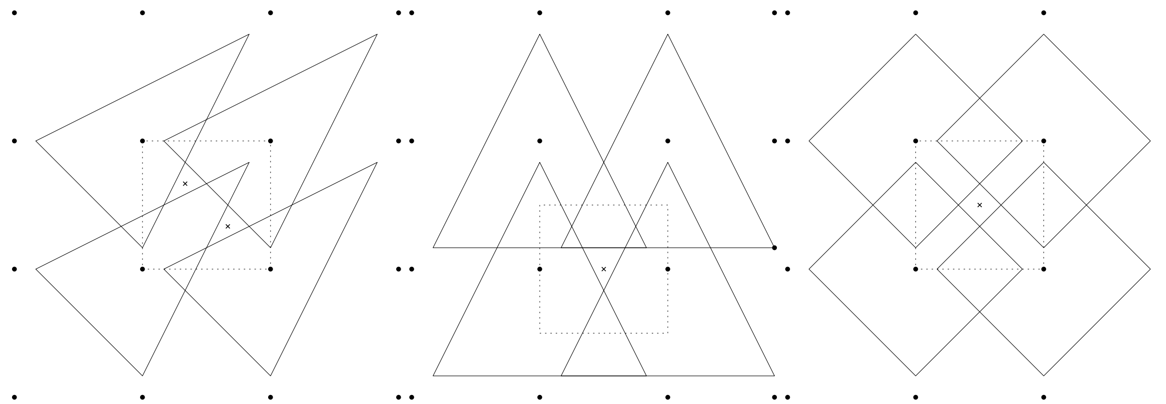
LEMMA 4.1.1. *For* $\alpha \in [0,1]$,

$$(1-\alpha)T_a^\circ = \mathbb{R}^2 \setminus \left(-\alpha T_a + \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \right\}\right) \mod \mathbb{Z}^2, \tag{4.1}$$

$$(1-\alpha)T_b^\circ = \mathbb{R}^2 \setminus \left(-\alpha T_b + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right) \mod \mathbb{Z}^2, \tag{4.2}$$

$$(1-\alpha)T_c^\circ = \mathbb{R}^2 \setminus \left(\alpha T_c + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right) \mod \mathbb{Z}^2. \tag{4.3}$$

Here are pictures that illustrate this lemma:



LEMMA 4.1.2. *Suppose T is a pyramid with a base of T_a, T_b, or T_c and height $h > 18$. Then T contains at least two interior lattice points.*

Proof. Let T be such a pyramid; let $\begin{pmatrix} x \\ y \\ h \end{pmatrix}$ be the apex of T. Note that $h > 12$. There are three cases:

(a) In this case, the base of T is T_a. Note that for any $k \in [0, h]$,

$$T \cap \{z = k\} = \left(1 - \frac{k}{h}\right) T_a + \frac{k}{h} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Either T contains an interior lattice point at height one or it does not.

(1) Suppose that $T \cap \{z = 1\}$ contains an interior lattice point. If $T \cap \{z = 2\}$ contains an interior lattice point, then we are done. Thus, suppose $T \cap \{z = 2\}$ does not contain an interior lattice point. That is,

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &\notin \left(1 - \frac{2}{h}\right) T_a^\circ + \frac{2}{h} \begin{pmatrix} x \\ y \end{pmatrix} \mod \mathbb{Z}^2, \\
-\frac{2}{h} \begin{pmatrix} x \\ y \end{pmatrix} &\notin \left(1 - \frac{2}{h}\right) T_a^\circ \mod \mathbb{Z}^2 \\
-\frac{2}{h} \begin{pmatrix} x \\ y \end{pmatrix} &\in -\frac{2}{h} T_a + \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \right\} \mod \mathbb{Z}^2, \quad \text{(by lemma 4.1.1)} \\
-\frac{6}{h} \begin{pmatrix} x \\ y \end{pmatrix} &\in -\frac{6}{h} T_a + \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \mod \mathbb{Z}^2, \\
-\frac{6}{h} \begin{pmatrix} x \\ y \end{pmatrix} &\in -\frac{6}{h} T_a \mod \mathbb{Z}^2.
\end{aligned}$$

Because $h > 18$, $-\frac{6}{h}T_a$ does not intersect $-\frac{6}{h}T_a + \left\{ \binom{\frac{1}{3}}{\frac{2}{3}}, \binom{\frac{2}{3}}{\frac{1}{3}} \right\}$. Thus $-\frac{6}{h}T_a \subseteq \left(1 - \frac{6}{h}\right) T_a^\circ$. Hence

$$-\frac{6}{h}\begin{pmatrix} x \\ y \end{pmatrix} \in \left(1 - \frac{6}{h}\right) T_a^\circ \mod \mathbb{Z}^2,$$
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \left(1 - \frac{6}{h}\right) T_a^\circ + \frac{6}{h}\begin{pmatrix} x \\ y \end{pmatrix} \mod \mathbb{Z}^2,$$
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in (T \cap \{z = 6\})^\circ \mod \mathbb{Z}^2.$$

So T contains an interior lattice point at height six.

(2) Suppose that $T \cap \{z = 1\}$ does not contain an interior lattice point. Using a similar argument, it can be shown that T contains interior lattice points at heights three and six

(b, c) In this case, the base of T is T_b or T_c. Either T contains an interior lattice point at height one or it does not. If it does, a similar argument to part (a) shows that T contains an interior lattice point at height four. If it does not, a similar argument to part (a) shows that T contains an interior lattice point at height two and an interior lattice point at height four.

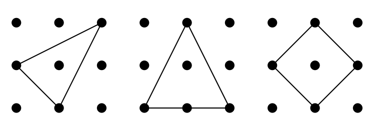
In all cases, T contains at least two interior lattice points. □

THEOREM 4.1.3. *Let P be a maximal integral 1-lattice polytope in canonical position (i.e., the xy-plane contains lattice-point-maximala face of P). Then the height of P (on the z-axis) is no greater than 12.*

Proof. Let P be such a polytope. Suppose for a contradiction that the height of P is greater than 12. Consider the base of P, $P \cap \{z = 0\}$.

Because P is maximal, $P \cap \{z = 0\}$ contains at least one interior lattice point. Consequently, there is an integral 1-lattice polygon contained in $P \cap \{z = 0\}$.

By [**1**], any integral 1-lattice polygon may be unimodularly transformed such that it contains at least one of the following integral 1-lattice polygons, each containing the origin as an interior point:



(T_a) (T_b) (T_c)

Apply such a transformation to P. Note that the transformation does not change the height of P. Let a be the apex of P; note that the z-coordinate of a is greater than 12.

Observe that some polygon T_0 is contained in $P \cap \{z = 0\}$, where T_0 is either T_a, T_b, or T_c. Note that conv(T_0, a) is a pyramid with a base of T_a, T_b, or T_c and a height greater than 12. By lemma 4.1.2, $\mathrm{conv}(T_0, a)$ contains at least two interior lattice points. Since $\mathrm{conv}(T_0, a) \subseteq P$, P contains at least two interior lattice points. This is a contradiction.

Therefore P has height no greater than 12. □

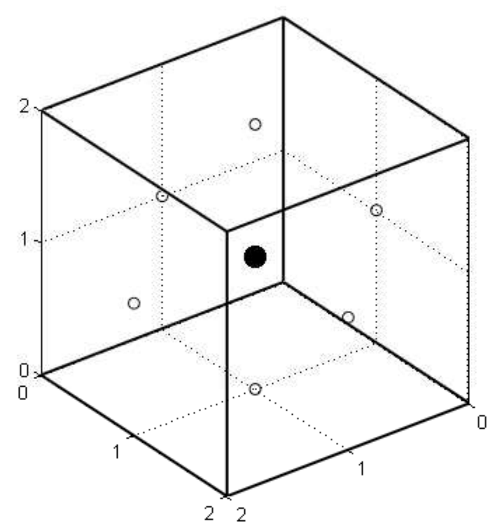
CHAPTER 5

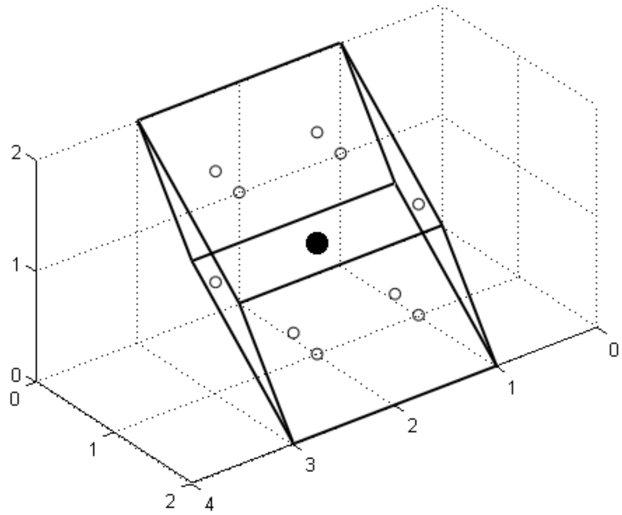
Results

After running the code, produced with help from graduate student Reuben La Haye, we were able to obtain the following results regarding 1-lattice maximal pyramids and prisms.There are

- 14 1-lattice maximal quadrilateral pyramids.
- 5 1-lattice maximal triangular prisms.
- 3 1-lattice maximal quadrilateral prisms.

Here are the pictures of the shapes, not including the simplices which were already classified. The clear dots are the lattice points in the facets and the bold dot is the interior lattice point.

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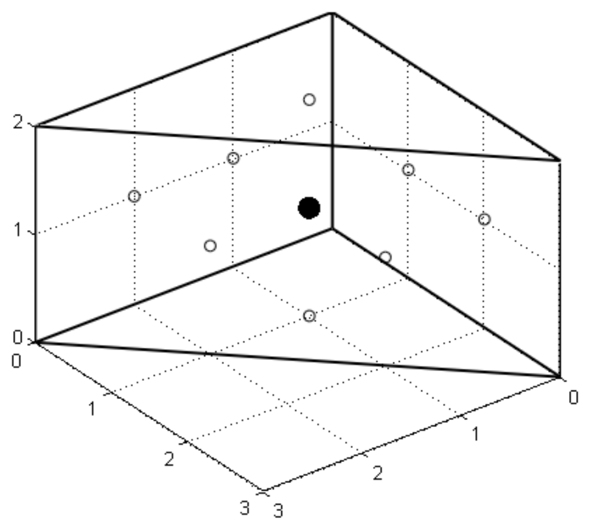
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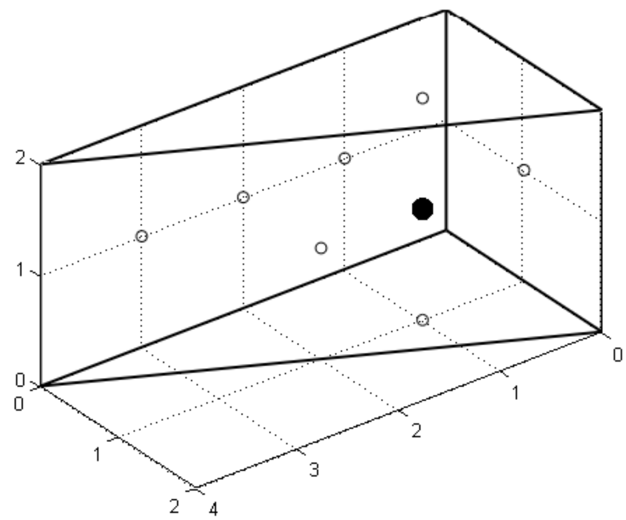
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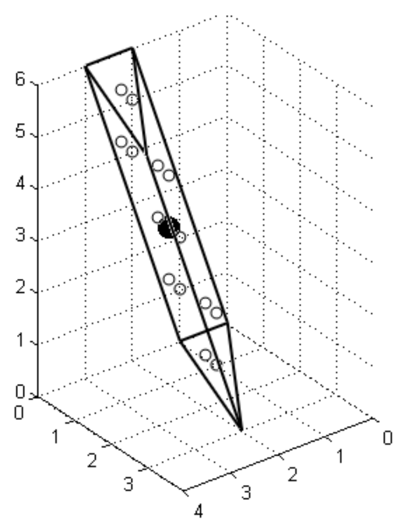
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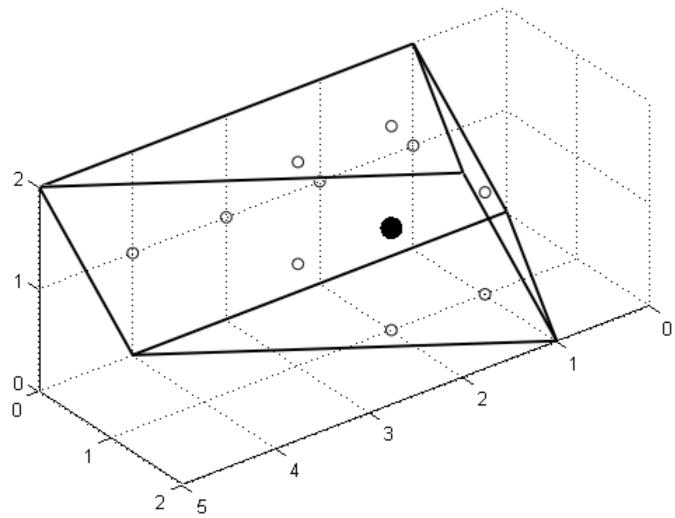

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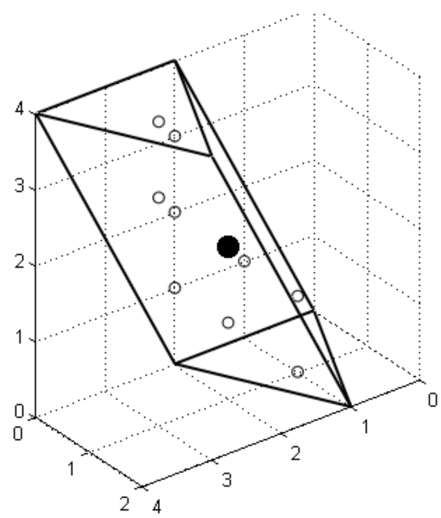

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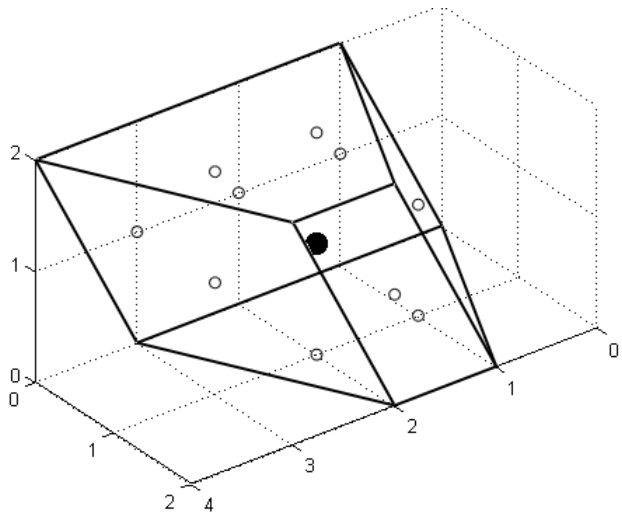
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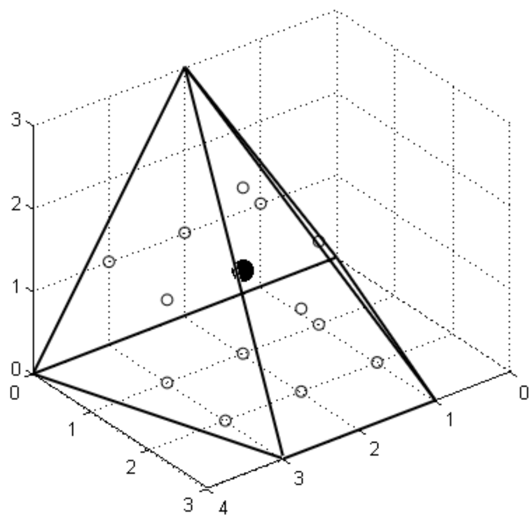
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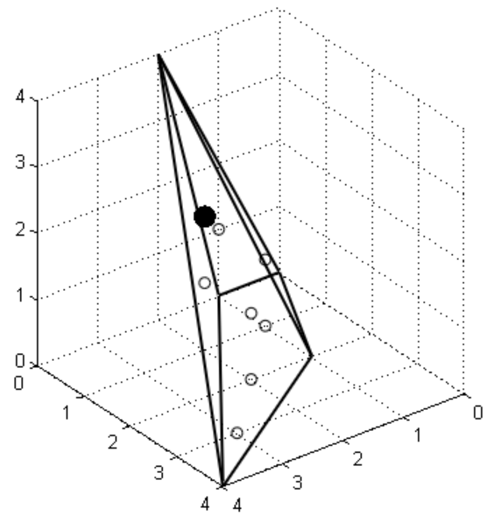
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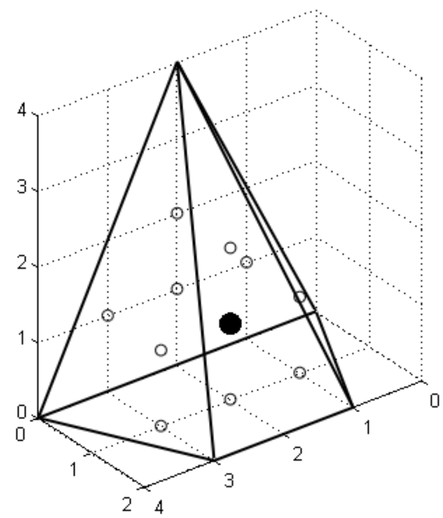
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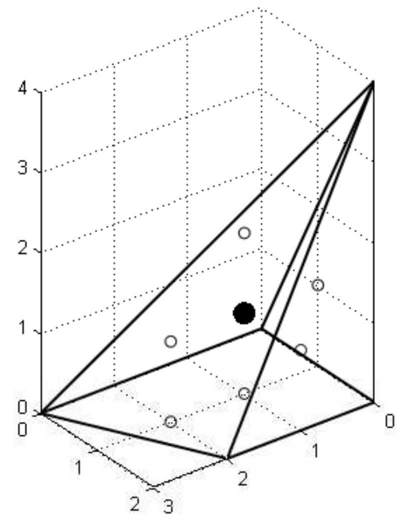
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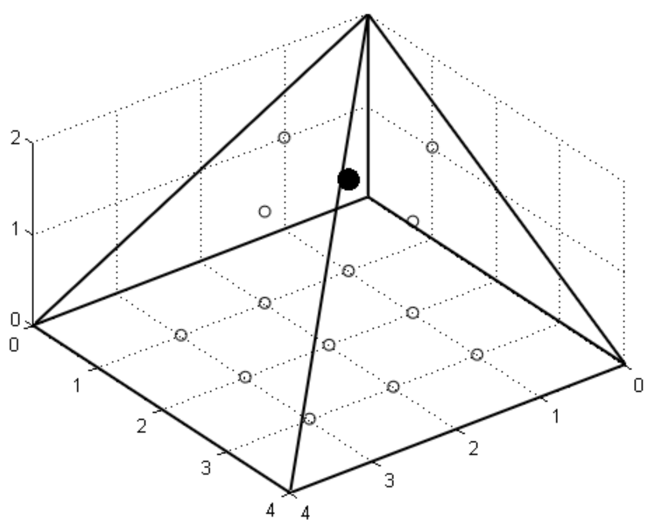
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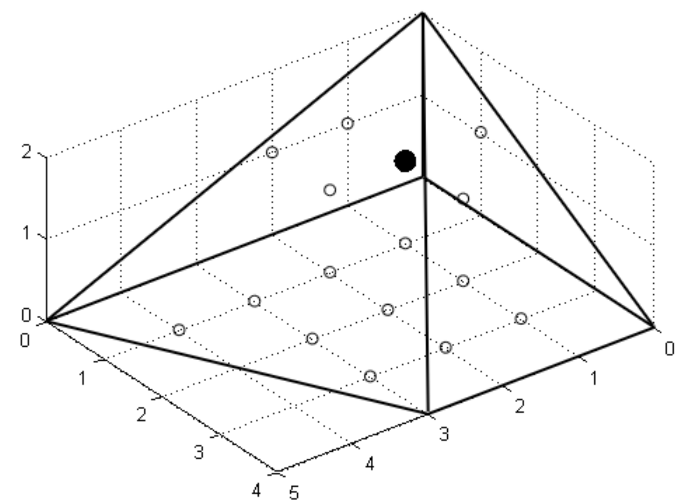
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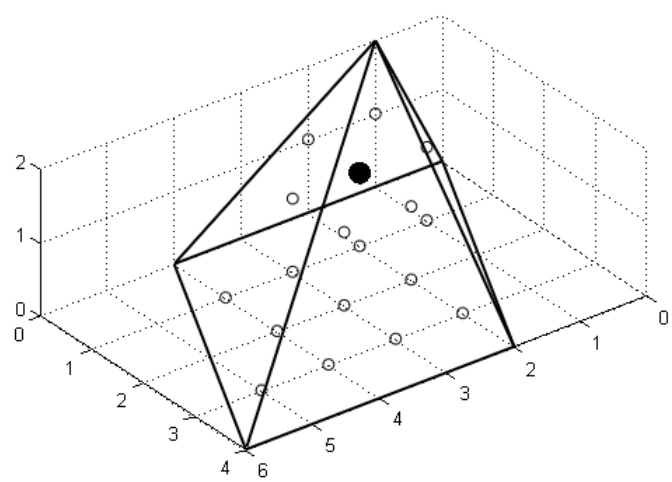
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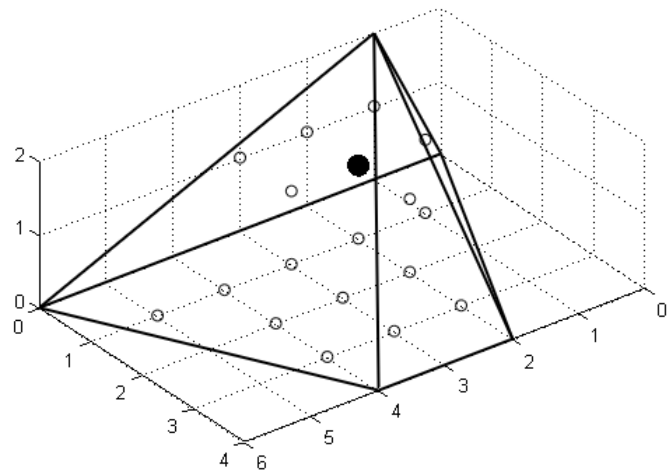
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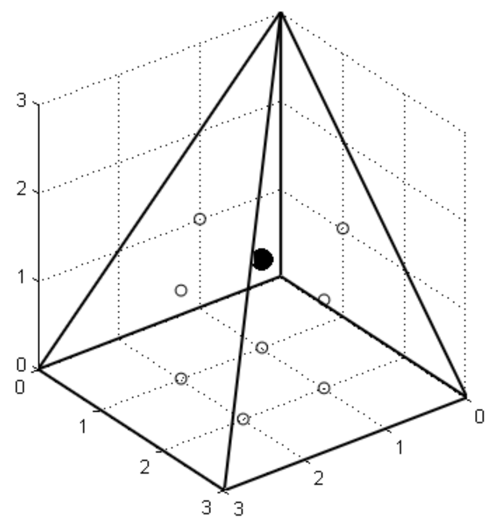
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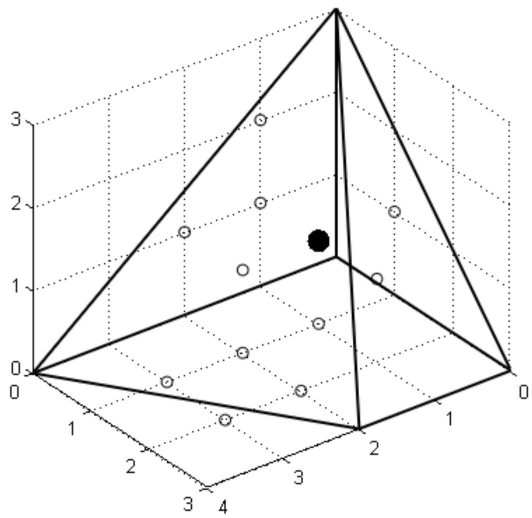
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