Characterizing Singular Polynomials for Dunkl Operators

Christian Smith
Thesis Advisor: Dr. Eugene Gorsky

May 25, 2016

Contents

Acknowledgments I would like to thank my thesis advisor, Dr. Eugene Gorsky, for introducing me to this research topic and for the tremendous guidance he has offered during the course of my work.

1 Introduction

Let f be a polynomial in n variables with coefficients in $\mathbb{R}$. Define $s_{ij}(f)$ as an operator which transposes x_i and x_j in f, e.g. if $f = ax_1 + bx_2 + cx_3$, then $s_{1,3}(f) = ax_3 + bx_2 + cx_1$.

Definition 1.1. *Now define the Dunkl Operator as:*

$$D_i(f) = \frac{\partial f}{\partial x_i} - c \sum_{i \neq j} \frac{f - s_{ij}}{x_i - x_j},$$

where i, j span over all possible n and c is some constant.

Dunk Operators have many applications in representation theory of rational Cherednik algebras, algebraic geometry, physics, and many other fields. In particular, Dunkl Operators are used to study the Calogero-Moser system; a discrete system of n particles. Specifically, we have:

$$\sum D_i^2 = H = \sum \frac{\partial}{\partial x_i^2} + k \sum \frac{1}{(x_i - x_j)^2},$$

where k is some constant dependent on c and H is the quantum Calogero-Moser Hamiltonian. [3] [4]

Definition 1.2. *A singular polynomial for the Dunkl Operator is a polynomial $f \in \mathbb{R}[x_1 \dots x_n]$ such that $D_i(f) = 0$ for all $i \in \{1, \dots, n\}$. [1]*

Our objective is to characterize these singular polynomials and to determine their dependence on the constant c.

Main Results. *The following are the main results contained in this paper.*

1. *If f is a degree 1 polynomial, then f is singular if and only if its coefficients sum to 0 and $c = \frac{1}{n}$.*
2. *Vandermonde determinants are singular for $c = \frac{1}{2}$.*
3. *Symmetric polynomials of degree greater than 0 are never singular.*
4. *There always exists a quadratic singular polynomial in n variables for $c = \frac{2}{n}$.*
5. *Any Garnir polynomial corresponding to a Young diagram λ such that λ corresponds to the partition $(b-1, n-b+1)$ where $b \geq \frac{n}{2}$ is singular for $c = \frac{1}{b}$.*
6. *Conjecture: Let λ be a Young diagram of size n such that $n = (b-1)q + r$, for some $b > 0, q > 0, r < b-1$, where λ has q rows of $(b-1)$ boxes and one row of r boxes. Then, any Garnir polynomial associated with λ is singular if and only if $c = \frac{1}{b}$. Furthermore, any Young diagram not adhering to these requirements has no associated Garnir polynomials which are ever singular.*

Proposition 1.3. *[1] The following are important properties of Dunkl Operators:*

1. $D_iD_j = D_jD_i$

2. D_i *is linear.*

Theorem 1.4. *[1, 2] If singular polynomials of positive degree exist, then* $c = \frac{a}{b}$*, where* a, b *are integers and* $b \leq n$*.*

Proposition 1.5. *The space of singular polynomials forms a representation of* S_n*.*

2 Degree 1 Polynomials

Theorem 2.1. *Let* $f = a_1x_1 + ax_2 + \ldots + a_nx_n$ *be some degree* 1 *polynomial in* n *variables. Then* f *is singular if and only if* $a_1 + a_2 + \ldots + a_n = 0$ *and* $c = \frac{1}{n}$*.*

Proof. It is easily seen that $D_i(x_i) = 1 - c(n-1)$ and $D_i(x_j) = c$ for $i \neq j$. Now we have a system of equations involving each D_i. Take this system of linear equations and construct the n by n matrix **A** below:

$$A = \begin{pmatrix} 1-c(n-1) & c & \ldots & c \\ c & 1-c(n-1) & c & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ c & \ldots & & 1-c(n-1) \end{pmatrix}$$

Let $a = 1 - c(n-1)$. Then

$$\det(A) = \prod_{i=0}^{i=n-1} \frac{(a+ic)(a-c)}{a+(i-1)c}.$$

This formula was found by row reducing the matrix above to an upper triangular matrix and then multiplying along the diagonal. We use induction to prove that this formula holds.

Suppose B is a $k \times k$ matrix with $1 - c(n-1)$ on the main diagonal and c everywhere else. It is easy enough to check that $\det(B) = \prod_{i=0}^{i=k-1} \frac{(a+ic)(a-c)}{a+(i-1)c}$ for $k = 2$.

Let $k = 2$ and let:

$$B = \begin{pmatrix} a & c \\ c & a \end{pmatrix}$$

where a is defined as above.

Then, after row reducing B we have the matrix:

$$B^* = \begin{pmatrix} a & c \\ 0 & a - \frac{c^2}{a} \end{pmatrix}$$

and $\det(B^*) = \det(B) = a \cdot \frac{(a+c)(a-c)}{a}$ as desired.

For the inductive step, suppose that for some $k-1$ we have that $\det(B)$ is as desired. Now for k we can construct the $k \times k$ matrix:

$$B' = \begin{pmatrix} 1-c(n-1) & c & \dots & c \\ c & 1-c(n-1) & c & \dots \\ \vdots & \vdots & \ddots & \vdots \\ c & \dots & & 1-c(n-1) \end{pmatrix}$$

where the upper left $(k-1) \times (k-1)$ block matrix is B.

Now we wish to row reduce B'. To do this, note that the row reduction process for the first $k-1$ rows of B' will be almost exactly the same as the row reduction process for B. Thus, we need only focus on the reduction of the kth row of B'. At the end of the $(k-1)$st step we will have the matrix:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c} & \frac{-c^2+ac}{a+(k-2)c} \\ 0 & \dots & \frac{-c^2+ac}{a+(k-2)c} & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c} \end{pmatrix}$$

To show that the $(k-1,k)$ and $(k,k-1)$ positions are both equal to $\frac{-c^2+ac}{a+(k-2)c}$, we use induction. In general, we wish to show that after the $i+1$ row has been reduced, we have that the non-zero and non-diagonal elements in that row are equal to $\frac{-c^2+ac}{a+ic}$. This can be easily shown to hold for $i+1 = 2, 3$ so we can move on to the induction step.

Assume we have that this holds for some row i. Then the i and $i+1$ rows both have $\frac{-c^2+ac}{a+(i-1)c}$ in their non-zero, non-diagonal positions. Observe that we can further row reduce by multiplying row i by $\frac{-c}{a+ic}$ and adding it to row $i+1$. This gives:

$$\frac{-c^2+ac}{a+(i-1)c} \cdot \frac{-c}{a+ic} + \frac{-c^2+ac}{a+(i-1)c} = \frac{(-c^2+ac)(a+ic-c)}{(a+(i-1)c)(a+ic)} = \frac{-c^2+ac}{a+ic}$$

which is was we wished to show. Now, if we let $i = k-2$ (for the $k-1$ row) we get the entries above.

For the final part of the row reduction process, multiply the top row by $\frac{-c}{a+(k-1)c}$ and add it to the bottom row to form a new bottom row. The row reduced form, $B^{'*}$, of B' is:

$$B^{'*} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{(a+(k-1)c)(a-c)}{a+(k-2)c} & \frac{-c^2+ac}{a+(k-2)c} \\ 0 & \dots & 0 & \frac{(a+kc)(a-c)}{a+(k-1)c} \end{pmatrix}$$

which gives us $\det(B) = \prod_{i=0}^{i=k} \frac{(a+ic)(a-c)}{a+(i-1)c}$.

Now, if we let $k = n$, we get $\det(A) = \prod_{i=0}^{i=n-1} \frac{(a+ic)(a-c)}{a+(i-1)c}$ as desired.

For each i, the roots of the numerator in the product are $c = a, c = -\frac{a}{i}$. The $(a + ic)$ in the numerator of the ith term will cancel with the $(a + ic)$ in the denominator of the $i + 1$ term. Additionally, the $(a + ic)$ in the numerator of the $i = n - 1$ term becomes 1 when evaluated at $a = 1 - c(n - 1)$; thus, the $-\frac{a}{i}$ root can be discarded. Only the $c = a$ root is left. Substituting a for $1 - c(n - 1)$, gives $c = \frac{1}{n}$. Substituting $c = \frac{1}{n}$ into A gives a matrix where all entries are $\frac{1}{n}$. □

3 The Vandermonde Determinant and Symmetric Polynomials

Definition 3.1. *Let $W = \prod_{i<j}(x_i - x_j)$ where the product is taken over n variables. A polynomial of this form is called a Vandermonde Determinant.*

Proposition 3.2. *Let W be the Vandermonde Determinant. Then $s_{ij}(W) = -W$.*

Proof. Let W be the Vandermonde Determinant in n variables. First consider the action of s_{ij} on W. Note that $s_{ij}(W) = \prod_{i<j} s_{ij}[(x_i - x_j)]$. Without loss of generality, suppose $i < j$. We can ignore all factors $(x_k - x_t)$ such that $k, t \neq i, j$ since s_{ij} will have no effect on them. We now have the following four cases to consider.

$$(x_i - x_k)$$
$$(x_k - x_i)$$
$$(x_j - x_k)$$
$$(x_k - x_j)$$
$$(x_i - x_j)$$

Where $k \neq i, j$.

Note that, given any k, i and j will be in some factor of W together with k. Thus, when we apply s_{ij} to W we will get back all the factors of W multiplied by -1^{μ} for some μ. Thus, we only need to know the parity of μ to determine if $s_{ij}(W) = W$ or $-W$.

First consider the $(x_i - x_k)$ factors. If $k > j$ then $s_{ij}(x_i - x_k) = (x_j - x_k)$. If $k < j$ then

$$s_{ij}(x_i - x_k) = (x_j - x_k) = -(x_k - x_j)$$

and so we gain a factor of -1. There are $j - i - 1$ of these $(x_i - x_k)$ factors where $k < j$.

Now consider the $(x_k - x_i)$ and the $(x_j - x_k)$ factors. If $k < i$ then $k < j$ and if $k > j$ then $k > i$. Thus,

$$s_{ij}(x_k - x_i) = (x_k - x_j) \text{ and } s_{ij}(x_j - x_k) = (x_i - x_k)$$

and we pick up no factors of -1.

Next we look at the $(x_k - x_j)$ factors. If $i > k$ then s_{ij} gives back the original factor. If $i < k$, then

$$s_{ij}(x_k - x_j) = (x_k - x_i) = -(x_i - x_k)$$

and so we gain a factor of -1. There are $j - i - 1$ of these $(x_k - x_j)$ factors where $i < k$.

Finally, we have the solitary $(x_i - x_j)$ factors. Clearly we have, $s_{ij}(x_i - x_j) = -(x_i - x_j)$ and so we gain a single factor of -1.

We see that $\mu = 1 + 2(j - i - 1)$, which is odd. Therefore, $s_{ij}(W) = -W$. □

Corollary 3.3. *Let W be the Vandermonde Determinant and let $s \in S_n$. Then $s\,(W) = sgn(s)W$.*

Proof. By the Proposition above, $s_{ij}(W) = -W$. Note that $sgn(s_{ij}) = -1$. Since any $s \in S_n$ can be decomposed into a product of transpositions we have,

$$s(W) = (-1)^{\mu} W$$

where μ is number of transposition in the decomposition of s. In other words, $(-1)^{\mu} = sgn(s)$. □

Corollary 3.4. *The Vandermonde Determinant W spans a one dimensional representation of S_n. This representation is isomorphic to the sign representation.*

Theorem 3.5. *For any n, the Vandermonde Determinant W is singular if and only if $c = \frac{1}{2}$.*

Proof.

$$A = -c \sum_{i \neq j} \frac{W - s_{ij}(W)}{x_i - x_j} = -c \sum_{i} \frac{2W}{x_i - x_j} = -2c \sum_{i \neq j} \frac{W}{x_i - x_j} \tag{1}$$

Now let,

$$\Gamma_n = \prod_{k<p} (x_k - x_p)$$

where $k, p \neq i$. Then,

$$A = -2c\Gamma_n \sum_{i \neq j} \left[(-1)^{\mu} \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right]$$

where $\mu = 0$ if $i < j, 1$ if $i > j$.

We want to show that

$$\frac{\partial W}{\partial x_i} = \frac{A}{-2c} \text{ for all } i, \tag{2}$$

we do this by induction.

It is easy enough to show that (2) holds for $n = 1, 2$ and so we have base cases completed.

Now for the inductive step. Let W_n and W_{n+1} be the Vandermonde Determinants over n and $n+1$ variables respectively. Then,

$$W_{n+1} = (x_1 - x_{n+1}) \dots (x_n - x_{n+1}) W_n$$

Thus, we have

$$\frac{\partial W_{n+1}}{\partial x_i} = \frac{\partial}{\partial x_i}[(x_1 - x_{n+1}) \dots (x_n - x_{n+1}) W_n]$$

If $i \neq n+1$,

$$\begin{aligned}\frac{\partial W_{n+1}}{\partial x_i} =& (x_1 - x_{n+1}) \dots (x_n - x_{n+1}) \frac{\partial W_n}{\partial x_i} + \frac{\partial}{\partial x_i}[(x_1 - x_{n+1}) \dots (x_n - x_{n+1})] W_n \\ =& (x_1 - x_{n+1}) \dots (x_n - x_{n+1}) \Gamma_n \times \\ & \times \sum_{i,j \in [n], i \neq j} \left[(-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right] \\ & + \frac{(x_1 - x_{n+1}) \dots (x_n - x_{n+1})}{(x_i - x_{n+1})} W_n \\ =& \Gamma_{n+1} \sum_{i,j \in [n+1], i \neq j} \left[(-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right]\end{aligned}$$

If $i = n+1$, then

$$\begin{aligned}\frac{\partial W_{n+1}}{\partial x_i} &= 0 + W_n \left[\sum_{1 \leq j \leq n} \left(- \prod_{i=1, i \neq j}^{n} (x_i - x_{n+1}) \right) \right] \\ &= \Gamma_{n+1} \sum_{i,j \in [n+1], i \neq j} \left[(-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right]\end{aligned}$$

Thus, we have (2) and,

$$D_i(W) = \Gamma_n \sum_{i,j \in [n+1], i \neq j} \left[(-1)^\mu \prod_{i<k, k \neq j} (x_i - x_k) \prod_{p<i, p \neq j} (x_p - x_i) \right] (1 - 2c)$$

This implies that for any n, W is singular if and only if $c = \frac{1}{2}$. □

Theorem 3.6. *Let g be a symmetric polynomial in n variables of degree at least 1. Then g is never singular.*

Proof. Let g be a symmetric polynomials in n variables such that $\deg(g) \geq 1$. Then $s_{ij}(g) = g$. Thus,

$$\sum_{j \neq i} \frac{g - s_{i,j}(g)}{x_i - x_j} = 0$$

for all i. Since g is not constant, we have $\frac{\partial g}{\partial x_i} \neq 0$. Therefore,

$$D_i(g) = \frac{\partial g}{\partial x_i} \neq 0 \text{ for all } i.$$

□

4 Quadratic Polynomials

4.1 SAGE Calculations

Next we characterize quadratic singular polynomials for $n = 3$ and $n = 4$. Below are bases for the space of quadratic singular polynomials for certain values of c and n. To find these bases, we use the following procedure:

1. Take the polynomial

   $$f = a_1 x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2 + b_{1,2} x_1 x_2 + b_{1,3} x_1 x_3 + \ldots + b_{n-1,n} x_{n-1} x_n$$

   where the $a_i, b_{i,j}$ are unknown coefficients.

2. Find $D_i(f)$ for all i.

3. Combine like terms in the D_i and consider the coefficient of each term. These coefficients will be a combination of the a_i's.

4. We require that these coefficients equal 0 for each term to vanish (note that since like terms have already been collected, different terms cannot cancel with one another). Thus, we can create a system of linear equations, using the a_i as our unknowns and setting each of these coefficients equal to 0.

5. Solve this system of equation and use it to determine a basis for all singular quadratic polynomials in n variables for the given c.

Proposition 4.1. *Let f be a quadratic polynomial in 3 variables with $c = \frac{2}{3}$. Then f is singular if and only if it is in the span of*

$$(x_2 - x_3)(-2x_1 + x_2 + x_3)$$
$$(x_1 - x_3)(x_1 + x_3 - 2x_2)$$

Proposition 4.2. *Let f be a quadratic polynomial in 4 variables with $c = \frac{1}{2}$. Then f is singular if and only if it is in the span of*

$$(x_1 - x_4)^2 - (x_2 - x_3)^2$$
$$(x_1 - x_3)^2 - (x_2 - x_4)^2$$
$$(x_1 - x_2)^2 - (x_3 - x_4)^2$$

Proposition 4.3. *Let f be a quadratic polynomial in 4 variables with $c = \frac{1}{3}$. Then f is singular if it is in the span of*

$$(x_1 - x_2)(x_3 - x_4)$$
$$(x_1 - x_3)(x_2 - x_4)$$

4.2 Decomposition of $P_{n,2}$

To further characterize quadratic singular polynomials, we investigate the decomposition of the space of quadratic polynomials into irreducible representations of S_n. Let $P_{n,2}$ be the space of quadratic polynomials in n variables. We can decompose $P_{n,2}$ into irreducible representations of S_n in the way shown in Figure 1.

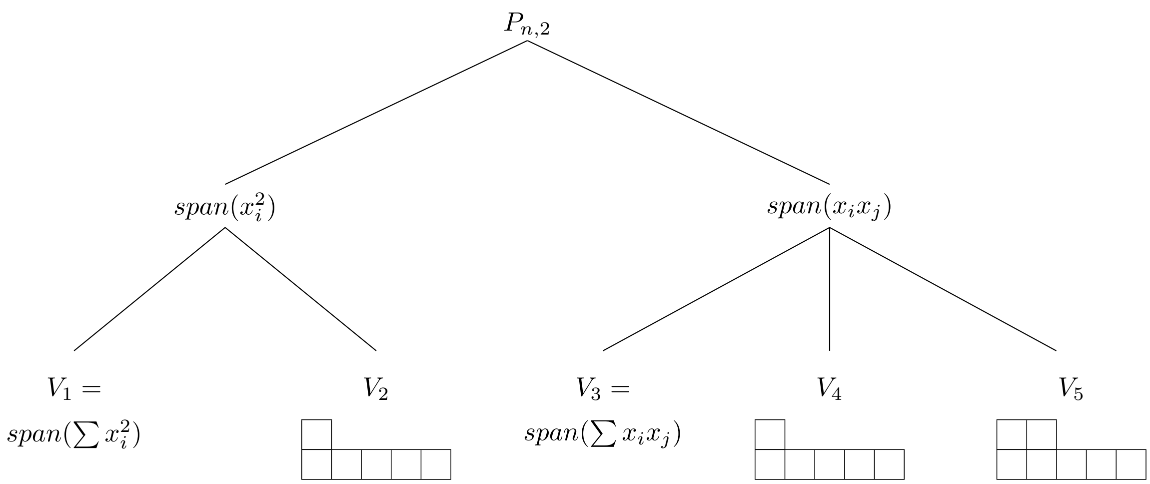
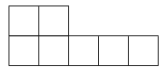


Figure 1: Decomposition of $P_{n,2}$

Here $\dim V_1 = \dim V_3 = 1, \dim V_2 = \dim V_4 = n - 1, \dim V_5 = \frac{n(n-3)}{2}$ and V_1, V_3 are the trivial representations of S_n.

We can show that V_1, V_2 are irreducible by noting that $span(x_i^2)$ is isomorphic to $\mathbb{R}^n$. Under the permutation representation over $\mathbb{R}^n$, it is well known that $\mathbb{R}^n$ is the direct sum of two irreducible representation. One is the trivial representation, to which V_1 is clearly isomorphic, which leaves V_2 as the other irreducible representation. Also, V_5 is isomorphic to the irreducible representation of S_n formed by the Garnir polynomials (discussed in section 5) and so V_5 is irreducible. Finally, $V_3 \oplus V_4 = span(\sum_{j \neq i, i \text{ fixed}} x_i x_j) \cong \mathbb{R}^n$. Since V_3 is isomorphic to the trivial representation (which is irreducible), we automatically have that V_4 is irreducible.

The Young diagrams below V_2, V_4, and V_5 correspond to these spaces in a way explained in section 5. For the moment, consider the Young diagram:

$\lambda_{V_5} =$ 

which represents V_5. As will be discussed in section 5, the number of SYT on λ_{V_5} equals the dimension of V_5. Thus, using the hook length formula, also discussed in section 5, we have:

$$\dim V_5 = \frac{n!}{2 \cdot (n-1) \cdot (n-2) \cdot (n-4)!} = \frac{n(n-3)}{2}.$$

Now we can refine our investigation into quadratic singular polynomials by using these V_i. Notice that all elements of V_1 and V_3 are symmetric polynomials. We can conclude, by Theorem 3.6, that there exist no non-trivial singular polynomials in either of these spaces.

Next we look at V_5. Let S be the space of singular polynomials. As mentioned before, S is a representation of S_n. Consider $Q = S \cap V_5$. The intersection of two representations of S_n is also be a representation of S_n. Since V_n is irreducible, this implies that either $Q = \{0\}$ or $Q = V_5$. Thus, if we show that any element of V_5 is singular, then we have that all elements in V_5 are singular. From this, we have the following.

Theorem 4.4. *Let $f \in V_5$, then f is singular, for all n, if and only if $c = \frac{1}{(n-1)}$.*

Proof. Note that $f = (x_1 - x_2)(x_3 - x_4) \in V_5$, this follows from Theorem 5.7. Thus, by the above argument, it suffices to show that f is singular. We have five cases to consider: $D_1(f), D_2(f), D_3(f), D_4(f), D_j(f)$ where $j \in [n]$ and $j \neq 1, 2, 3, 4$. First:

$$\begin{aligned} D_1(f) =& (1-3c)(x_3 - x_4) - c(n-4)(x_3 - x_4) = (1 - c(n-1))(x_3 - x_4) \\ &\Rightarrow D_1(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}. \end{aligned}$$

Since x_1 and x_3 behave in the same way, we also have:

$$D_3(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.$$

Now:

$$\begin{aligned} D_2(f) =& (3c-1)(x_3 - x_4) + c(n-4)(x_3 - x_4) = (c(n-1) - 1)(x_3 - x_4) \\ &\Rightarrow D_2(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}. \end{aligned}$$

Similarly, we have that x_2 and x_4 behave in the same way, so:

$$D_4(f) = 0 \text{ if and only if } c = \frac{1}{(n-1)}.$$

Finally:

$$D_j(f) = -(x_3 - x_4) + (x_3 - x_4) - (x_1 - x_2) + (x_1 - x_2) = 0.$$

Thus, we have that f is singular and therefore all elements in V_5 are singular. Furthermore, from Theorem 5.7 we have that $V_5 = span\{G_{T_{\lambda_{V_5}}}\}$, where $\{G_{T_{\lambda_{V_5}}}\}$ is the set of all Garnir polynomials corresponding to λ_{V_5} (Garnir polynomials will be discussed in the next section). Due to the shape of λ_{V_5}, all elements in $\{G_{T_{\lambda_{V_5}}}\}$ will behave the same way as f did when acted upon by the Dunkl Operators. Thus, each element in $\{G_{T_{\lambda_{V_5}}}\}$ will be singular if and only if $c = \frac{1}{(n-1)}$; and since any element of V_5 can be expressed as a linear combination of these polynomials, we have that any element of V_5 is also singular if and only if $c = \frac{1}{(n-1)}$.

□

Finally, consider $W = V_2 \oplus V_4$. As with V_5 we look at $H = S \cap W$. As before, H must be a representation of S_n; however, W is not irreducible so we cannot come to the same conclusion we did with V_5. Instead, we see that $H = AV_2 + BV_4$ for some $A, B \in \mathbb{C}$. This leads us to the following theorem.

Theorem 4.5. *Let,*

$$f_q = \frac{1}{4}(n-1)(n-2)x_q^2 - \frac{n-2}{4}\sum_{i \neq q} x_i^2 - \frac{n-2}{2}\sum_{i \neq q} x_q x_i + \sum_{i,j \neq q} x_i x_j$$

where $f \in \mathbb{R}[x_1 \dots x_n]$*; then, when* $c = \frac{2}{n}$*,* f *is singular for all* n*.*

Before we begin the proof, note that the first two terms in f_q are elements of V_2 and the last two terms of f_q are elements of V_4.

Proof. Since q is arbitrary, we can prove the theorem holds for $q = 1$ and we will have proven it true for all q. There are two cases to consider. The first is the action of D_1 on f and the second is the action of D_t, where $t \neq 1$, on f.

To find $D_1(f)$, we apply the Dunkl Operator to each term of f separately. This gives,

$$\begin{aligned} D_1(f) =& \frac{2(n-1)(n-2)}{4n}x_1 - \frac{2(n-1)(n-2)}{4n}\sum_{i \neq 1} x_i - \frac{n-2}{2n}\sum_{i \neq 1}(x_1 + x_i) \\ &+ \frac{(n-2)(n-4)}{2n}\sum_{i \neq 1} x_i + \frac{2}{n}\sum_{i,j \neq 1}(x_i + x_j) \end{aligned}$$

Summing the coefficients for the x_1 terms we have,

$$\frac{2(n-1)(n-1)}{4n} - \frac{(n-2)(n-1)}{2n} = 0$$

Similarly, for the x_i terms we have,

$$-\frac{2(n-1)(n-2)}{4n} - \frac{(n-2)}{2n} + \frac{(n-2)(n-4)}{2n} + \frac{2(n-2)}{n} = 0$$

Now, for the D_t case,

$$\begin{aligned} D_t(f) = & \frac{(n-1)(n-2)}{2n}(x_1 + x_t) \\ & + \left[-\frac{(n-2)}{2n} \sum_{i \neq t,1} (x_i + x_t) - \frac{(n-2)}{2n} x_t + \frac{(n-2)}{2n} \sum_{i \neq t} x_i \right] \\ & + \left[-\frac{(n-2)}{n} \sum_{i \neq t,1} (x_1 + x_i) + \frac{(n-2)(n-4)}{2n} x_1 \right] \\ & + \left[\frac{2}{n} \sum_{i,j \neq t,1} (x_i + x_j) - \frac{(n-4)}{n} \sum_{i \neq 1,t} x_i \right] \end{aligned}$$

In this case we must check the coefficients for $x_1, x_t,$ and x_i.
For x_1 we have,

$$\frac{(n-1)(n-2)}{2n} + \frac{(-2n+4)(n-2)}{2n} + \frac{(n-2)(n-4)}{2n} + \frac{(n-2)}{2n} = 0$$

For x_t,

$$\frac{(n-1)(n-2)}{2n} - \frac{(n-2)(n-2)}{2n} - \frac{(n-2)}{2n} = 0$$

For x_i,

$$-\frac{(n-1)}{2n} + \frac{(n-2)}{2n} - \frac{(n-2)}{n} + \frac{2(n-3)}{n} - \frac{(n-4)}{n} = 0$$

And so we have the desired result. □

Proposition 4.6. $F = \sum_{q=1}^{n} f_q = 0.$

Proof. Consider x_j^2 where j is fixed. Then for f_j we have $\frac{(n-1)(n-2)}{4}$ as the coefficient of this term. Now consider all other f_q, there are $(n-1)$ such f_q and for each the coefficient of x_j is $-\frac{(n-2)}{4}$. Thus we have:

$$\frac{(n-1)(n-2)}{4} - \frac{(n-1)(n-2)}{4} = 0$$

and the coefficient of x_j^2 in the sum is 0. Since j is arbitrary, F has no terms of the form x_i^2.

Similarly, take $x_{j,p}$ such that j, p are fixed. Then for f_j there are $-\frac{(n-2)}{2}$ such terms and for f_p there are also $-\frac{(n-2)}{2}$ such terms. Now, looking at all other f_q, each has only one copy of $x_{j,p}$ and there are $(n-2)$ remaining f_q. So:

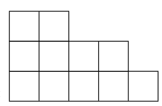
$$-\frac{(n-2)}{2} - \frac{(n-2)}{2} + (n-2) = 0.$$

Again, i and j are arbitrary so no such terms appear in F. Since these are all possible terms in F, we have $F = 0$. $\square$

5 Young Diagrams and Garnir Polynomials

5.1 Young Diagrams

In this section, we recall some facts from combinatorics and representation theory [5]. In particular, standard Young tableaux will be discussed. As a point of clarity, note that we will use the convention of writing Young diagrams such that the longest row of boxes will be at the bottom of the diagram and the shortest row will be at the top. For instance, if we had the partition $(5, 4, 2)$, the corresponding Young diagram would be:

$\lambda =$ 

Definition 5.1. Let λ be some Young diagram. A Standard Young Tableau (SYT), denoted T_λ, is a filling of the boxes of λ with the numbers $1, \ldots, n$, where n is the number of boxes in λ. These numbers must be placed in λ such that, when moving from the bottom of any column to the top, the numbers increase and similarly, when moving from left to right along a row, the numbers also increase.

For example, if λ is the Young diagram corresponding to the partition, $(4, 3, 2)$, we could have the SYT:

$$T_\lambda = \begin{array}{|c|c|cc} \hline 4 & 6 & & \\ \hline 2 & 5 & \multicolumn{1}{c|}{9} & \\ \hline 1 & 3 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

However, the following filling:

$$A = \begin{array}{|c|c|cc} \hline 2 & 6 & & \\ \hline 4 & 5 & \multicolumn{1}{c|}{9} & \\ \hline 1 & 3 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

would not be a SYT because in the first column, numbers do not increase from bottom to top, we move from 4 in the box of the middle row to 2 in the box of the top row.

Definition 5.2. Let λ be some Young diagram. Let a be some box in λ. Let u_i be the number of boxes, in the same column as a, above a and let r be the number of boxes, in the same row of a, to the right of a. Then $h_a = u + r + 1$ is the hook length of a.

The following is a well known fact in representation theory.

Hook Length Formula 5.3. Let λ be a Young diagram for some partition of n. Let f_λ be the number of standard Young tableaux that can be formed from λ, that is, the number of ways the boxes of λ can be filled with the numbers $1, \ldots, n$ such that the rules for SYT are obeyed. Then:

$$f_\lambda = \frac{n!}{\prod_{a \in \lambda} h_a}.$$

This is called the hook length formula.

For example, consider:

$$Y = \begin{array}{|c|c|c|c|} \cline{1-2} 2 & 1 & \multicolumn{2}{c}{} \\ \hline 5 & 4 & 2 & 1 \\ \hline \end{array}$$

The tableau Y is not a standard Young tableau. Rather, Y is a Young diagram where the boxes of Y have been filled with their respective hook length. The box in the bottom lefter corner has 3 boxes to the right of it and 1 above it, thus its hook length is 5. Then, from the hook length formula, we have that the number of possible SYT on this shape is:

$$f_\lambda = \frac{6!}{2 \cdot 1 \cdot 5 \cdot 4 \cdot 2 \cdot 1} = 9$$

Theorem 5.4. The irreducible representations of the symmetric group S_n are in bijection with the Young diagrams that give partitions of n.

Recall the figures below the V_i in the earlier decomposition of the space of quadratic polynomials in n variables. Since each V_i is an irreducible representation of S_n, these figures were the Young diagrams corresponding to those particular irreducible representations.

Theorem 5.5. Let V be some irreducible representation of S_n and let λ be the Young diagram associated with V. Then:

$$\dim V = f_\lambda$$

where the SYT are taken with respect to shape λ and f_λ is found using the hook length formula given above.

5.2 Garnir Polynomials

Definition 5.6. Let T_λ be a standard Young tableaux for some Young diagram λ. Denote each column of T_λ by a^i, so column one (the column farthest to the left) is a^1, column two is a^2, and so on. Now denote the number in a box of T_λ by a^i_j where j is the box's position (counting from bottom to top) in column i. Thus, the filling in the lower left box will be denoted by a^1_1, the filling of the box above it by a^1_2, the filling of the box to its immediate right by a^2_1, and so on. Associated with T_λ is the Garnir polynomial G_{T_λ} where:

$$G_{T_\lambda} = \prod_{i=1}^{q} \left[\prod_{j=1}^{\lambda_i} \left(\prod_{k>j} (x_{a^i_j - a^i_k}) \right) \right]$$

where q is the number of columns in T_λ, λ_i is the number of boxes in column i, and $k \leq \lambda_i$ [6].

For example, let:

$$T_\lambda = \begin{array}{|c|c|c|c|} \hline 3 & 6 \\ \cline{1-2} 2 & 5 & 8 \\ \cline{1-3} 1 & 4 & 7 & 9 \\ \hline \end{array}$$

Then:

$$G_{T_\lambda} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)(x_4 - x_6)(x_5 - x_6)(x_7 - x_8)$$

Notice that x_9 is not part of any factor since there are no boxes above the box containing the number 9.

Now let:

$$T_\lambda = \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

Then:

$$G_{T_\lambda} = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

Notice that in this case, G_{T_λ} is the Vandermonde determinant for $n = 4$.

Finally:

$$T_\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

Then:

$$G_{T_\lambda} = 1$$

G_{T_λ} is a constant because T_λ consists of only one row, so no factors with any x_i can be formed.

Theorem 5.7. [6] Let λ be some Young diagram. Let $\{G_{T_\lambda}\}$ be the set of all possible Garnir polynomials on λ. Then $\{G_{T_\lambda}\}$ is an irreducible representation of S_n and $\{G_{T_\lambda}\}$ is isomorphic to the irreducible representation of S_n corresponding to λ.

Recall V_5 in the decomposition of $P_{n,2}$. The Young diagram below V_5 in Figure 1 corresponds to V_5 itself. By the above theorem, this Young diagram, and by implication V_5, is isomorphic to the irreducible representation of S_n formed from the Garnir polynomials of the Young diagram. Thus, V_5, as stated earlier, is irreducible.

Next we consider a specific kind of Young diagram. Let, λ be the Young diagram corresponding to the partition $(b-1, n-b+1)$, where $b \geq \frac{n}{2}$. Let T_λ be a standard Young Tableau over λ such that, if k is the filling of some box, then the number occupying the box directly above it in the same column is $k+1$. Now letG_{T_λ} be the Garnir polynomial corresponding to T_λ. Thus, if $n = 5, b = 4$, then:

$$T_\lambda = \begin{array}{|c|c|c} \hline 2 & 4 & \\ \hline 1 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

and

$$G_{T_\lambda} = (x_1 - x_2)(x_3 - x_4)$$

An interesting interpretation of T_λ is that it can be constructed from any b satisfying:

$$n = (b-1)1 + (n-b+1)$$

where $(b-1)$ gives the number of boxes in the bottom row and $(n-b+1)$ gives the number of boxes in the top row. In other words, it is a way of depicting division with remained of n by $b-1$.

From this construction, we have the following theorem.

Theorem 5.8. Let,

$$G_{T_\lambda} = (x_1 - x_2)(x_3 - x_4)\ldots(x_{2(n-b+1)-1} - x_{2(n-b+1)})$$

where G_{T_λ} is the Garnir polynomial described above and $b \geq \frac{n}{2}$. Then G_{T_λ} is singular if and only if $c = \frac{1}{b}$.

Proof. There are three cases to consider, $D_t(G_{T_\lambda})$ when

1. t is even, and present in G_{T_λ}
2. t is odd, and present in G_{T_λ}
3. t is not in G_{T_λ}.

Case 1: Let t be even and in G_{T_λ}. Then,

$$
\begin{aligned}
D_t(G_{T_\lambda}) =& \frac{G_{T_\lambda}}{(x_t - x_{t+1})} - c\left[\frac{2G_{T_\lambda}}{(x_t - x_{t+1})}\right.\\
&+ \sum_{i \text{ even}, i \neq t+1} \frac{G_{T_\lambda}(x_{t+1} - x_{i+1})}{(x_t - x_t + 1)(x_i - x_{i+1})}\\
&+ \sum_{i+1 \text{ odd}, i \neq t+1} \frac{-G_{T_\lambda}(x + t + 1 - x_i)}{(x_t - x_{t+1})(x_i - x_{i+1})}\\
&\left. + (-n + 2b - 2)\frac{G_{T_\lambda}}{(x_t - x_{t+1})}\right]\\
=& \frac{G_{T_\lambda}}{(x_t - x_{t+1})}[1 - 2c - c(n-b) - c(-n + 2b - 2)]
\end{aligned}
$$

If we want $D_t(G_{T_\lambda}) = 0$, then we need $1 - 2c - c(n-b) - c(-n+2b-2) = 0$ and so

$$
1 - 2c - c(n-b) - c(-n+2b-2) = 1 - cb = 0 \Rightarrow c = \frac{1}{b}
$$

Thus, for t even, the theorem holds.

Case 2: Notice that $D_t(-G_{T_\lambda}) = -D_t(G_{T_\lambda})$. Also, if t is odd we make apply the Dunkl Operators to $-G_{T_\lambda}$ and we can treat t as if it were even. Thus, by Case 1 we have that the theorem holds.

Case 3: Now suppose t is not present in G_{T_λ}. Then:

$$
D_t(G_{T_\lambda}) = 0 - c\left[\sum_{i \text{ odd}} \frac{-G_{T_\lambda}}{(x_i - x_{i+1})} + \sum_{i+1 \text{ even}} \frac{G_{T_\lambda}}{(x_i - x_{i+1})}\right] = 0
$$

Thus, in the third case, the value of c is irrelevant, we will always have $D_t(G_{T_\lambda}) = 0$.

Since all three cases hold, we have that the theorem is true.

□

Corollary 5.9. Let λ be a Young diagram corresponding to G_{T_λ} from theorem 5.8. If G_{T_μ} is a different Garnir polynomial found from some filling of λ other than the one used to construct G_{T_λ}, then G_{T_μ} is also singular.

Proof. From theorem 5.7 we have that the set of Garnir polynomials corresponding to λ, $\{G_{T_\lambda}\}$, forms an irreducible representation of S_n and from proposition 1.5 we know that the space of singular polynomials, S, forms a representation of S_n.

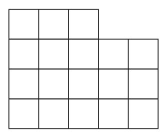
Let $Q = S \cap \{G_{T_\lambda}\}$. Then Q must also be a representation of S_n. Since $\{G_{T_\lambda}\}$ is irreducible, it follows that either $Q = \{0\}$ or $Q = \{G_{T_\lambda}\}$. Since we have already shown that G_{T_λ} is singular, we know $Q \neq \{0\}$. Thus, $Q = \{G_{T_\lambda}\} \Rightarrow \{G_{T_\lambda}\} \subseteq S_n$. Therefore, G_{T_μ} is singular. □

We end our results with a conjecture closely related to the theorem just proven.

Conjecture 5.10. Let $n = (b-1)q + r$ for some $n > 0, b > 0, q > 0, r < b - 1$. Let λ be the Young diagram with q rows of $b-1$ boxes and one row of r boxes. Now let T_λ be some standard Young Tableau over λ. The Garnir polynomial associated with T_λ, $G_{T_\lambda} \in \mathbb{R}[x_1, \ldots, x_n]$, is singular if and only if $c = \frac{1}{b}$.

Also, let μ be a Young diagram which does not conform to the above requirements and which does not correspond to the constant Garnir polynomial equal to 1. Then there does not exist any standard Young tableau, T_μ, such that G_{T_μ} is singular.

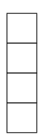
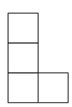
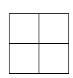
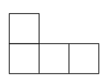
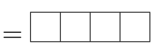
For instance, we could have $18 = (6-1)3 + 3$. Then we have $b = 6, q = 3, r = 3$, giving us the Young diagram:



Then G_{T_λ} is the Garnir polynomial associated to some T_λ. The previous theorem is a special case of this conjecture and just like the previous theorem, this conjecture has an interesting relationship to division with remainder.

The following are several instances where this conjecture holds. We do not mention the Garnir polynomial corresponding to a Young diagram that is a row of n boxes, though we do list this Young diagram, since the conjecture does not comment on this Garnir polynomial and since it equals 1, so it is singular for any c.

The possible Young diagrams for $n = 4$ are:

$\lambda_1 =$  $\lambda_2 =$  $\lambda_3 =$  $\lambda_4 =$  $\lambda_5 =$ 

Notice that $\lambda_1, \lambda_3, \lambda_4$ all obey the criteria set out in the conjecture, with $b_{\lambda_1} = 2, b_{\lambda_3} = 3, b_{\lambda_4} = 4$, while λ_2 does not. To determine if the conjecture holds, we use the SAGE program (see appendix).

This program first finds a Garnir polynomial corresponding to each λ_i and then applies the Dunkl operators to it to determine for what c, if any, it is singular. Note that the Garnir polynomial found for each λ_i is the one found from the SYT where, if a box contains the filling k, then the box immediately above it in the same column has the filling $k+1$. Using the same reasoning as in corollary 5.9, if this particular Garnir polynomial is singular, then all other Garnir polynomials corresponding to λ_i are also singular and, similarly, if this particular Garnir polynomial is not singular for any c, then all other Garnir polynomials corresponding to λ_i are also not singular for any c.

Since the Garnir polynomial corresponding to λ_1 is the Vandermonde determinant over $n = 4$, we already know from theorem 3.5 that it is singular if and only if $c = \frac{1}{2}$ as desired.

Using the program we find:

$$\begin{aligned}
D_1(G_{T_{\lambda_3}}) &= -(3c-1)(x_3 - x_4)\\
D_2(G_{T_{\lambda_3}}) &= -(3c-1)(x_3 - x_4)\\
D_3(G_{T_{\lambda_3}}) &= -(3c-1)(x_1 - x_2)\\
D_4(G_{T_{\lambda_3}}) &= (3c-1)(x_1 - x_2)
\end{aligned}$$

Thus, $G_{T_{\lambda_3}}$ is singular if and only if $c = \frac{1}{3}$ as desired.

And:

$$\begin{aligned}
D_1(G_{T_{\lambda_4}}) &= -4c + 1\\
D_2(G_{T_{\lambda_4}}) &= 4c - 1\\
D_3(G_{T_{\lambda_4}}) &= 0\\
D_4(G_{T_{\lambda_4}}) &= 0
\end{aligned}$$

So $G_{T_{\lambda_4}}$ is singular if and only if $c = \frac{1}{4}$ as desired.

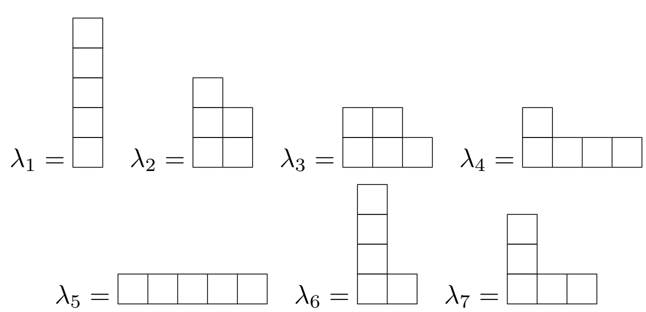
Finally:

$$\begin{aligned}
D_1(G_{T_{\lambda_2}}) &= -(5cx_1 - 3cx_2 - 3cx_3 + cx_4 - 2x_1 + x_2 + x_3)(x_2 - x_3)\\
D_2(G_{T_{\lambda_2}}) &= -(3cx_1 - 5cx_2 + 3cx_3 - cx_4 - x_1 + 2x_2 - x_3)(x_1 - x_3)\\
D_3(G_{T_{\lambda_2}}) &= (3cx_1 + 3cx_2 - 5cx_3 - cx_4 - x_1 - x_2 + 2x_3)(x_1 - x_2)\\
D_4(G_{T_{\lambda_2}}) &= 0
\end{aligned}$$

Therefore, $G_{T_{\lambda_2}}$ is not singular for any c and any other Garnir polynomial corresponding to λ_2 is not singular for any c.

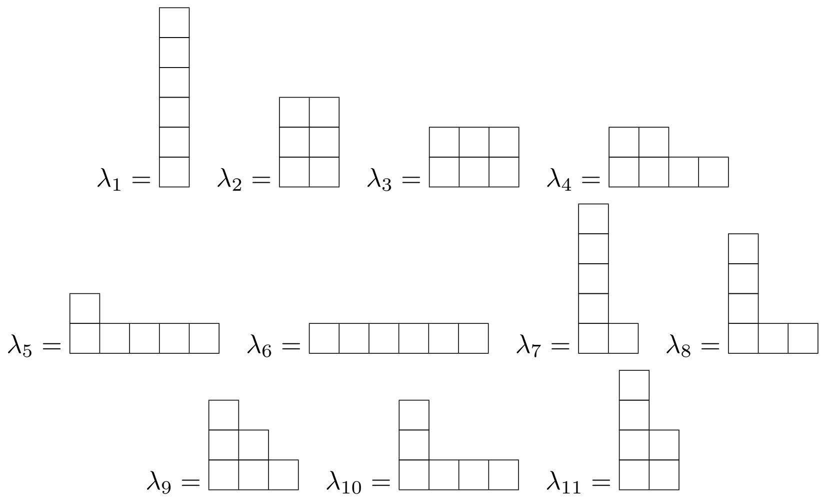
For the remaining two examples, we omit the SAGE program calculations.

The possible Young diagrams for $n = 5$ are:



We see that the λ_i, for $1 \leq i \leq 4$, adhere to the requirements in the conjecture while λ_6 and λ_7 do not. Furthermore, after using the SAGE program (see appendix) we see that those which do adhere to the conjecture's requirements have singular Garnir polynomials for $c = \frac{1}{b}$ while those that do not have no Garnir polynomials that are singular for any c.

The possible Young diagrams for $n = 6$ are:



The λ_i, for $1 \leq i \leq 5$, meet the requirements set out in the conjecture while λ_i, for $7 \leq i \leq 11$, do not. Furthermore, after applying the SAGE code (see Appendix), we see that the Garnir polynomials associated with the first 6 Young diagrams are singular for $c = \frac{1}{b}$ while the Garnir polynomials associated to the last 4 Young diagrams are never singular for any c.

References

[1] Charles Dunkl. Singular Polynomials for the Symmetric Groups. Int. Math. Res. Not. 2004, no. 67, 3607-3635

[2] C. Dunkl, M. de Jeu, and E. Opdam. Singular Polynomials for Finite Reflection Groups. Trans. Amer. Math. Soc. 349 (1994), no. 1, 237-256.

[3] Pavel Etingof. Lectures on Calogero-Moser Systems. arXiv:math/0606233v4[math.QA]. 2009.

[4] Pavel Etingof and Xiaoguang Ma. Lecture Notes on Cherednik Algebras. arXiv:1001.0432v4[math.RT]. 2010.

[5] Stanley, Richard P. Algebraic Combinatorics: Walks, Trees, Tableaux, and More. 2013.

[6] M. Haiman. Combinatorics, symmetric functions and Hilbert schemes Current Developments in Mathematics 2002, no. 1 (2002), 39-111.

6 Appendix

The following are two programs, written using SAGE, which we used for our calculations. The first program takes an arbitrary polynomial in n variables and of degree k and first applied the Dunkl Operators to it and then creates a matrix from which we constructed the bases for the space of singular polynomials for the instances found in section 4. The second program calculated Garnir polynomials for various SYT and them applied the Dunkl Operators to them.

Program 1

```
#DUNKL CALCULATIONS SECTION
#creates a polynomial ring in n variables over a rational coefficient polynomial
#ring in m variables (the m variables ring will act as coefficients for variables of
#the n variable ring)
m = 28
A = PolynomialRing(QQ, m, 'a')
a = A.gens()

n = 7
R = PolynomialRing(A, n, 'x')
x = R.gens()

#determines the constant c and the function f to be acted on by the
#Dunkl Operators
c = 2/7
f = a[0]*x[0]^2 + a[1]*x[1]^2 + a[2]*x[2]^2 + a[3]*x[3]^2 + a[4]*x[4]^2
+ a[5]*x[5]^2 +a[6]*x[6]^2+ a[7]*x[0]*x[1] + a[8]*x[0]*x[2]
+ a[9]*x[0]*x[3] + a[10]*x[0]*x[4] + a[11]*x[0]*x[5] + a[12]*x[0]*x[6]
+ a[13]*x[1]*x[2] + a[14]*x[1]*x[3] + a[15]*x[1]*x[4] + a[16]*x[1]*x[5]
+ a[17]*x[1]*x[6] + a[18]*x[2]*x[3] + a[19]*x[2]*x[4] + a[20]*x[2]*x[5]
 + a[21]*x[2]*x[6] + a[22]*x[3]*x[4] + a[23]*x[3]*x[5] + a[24]*x[3]*x[6]
+ a[25]*x[4]*x[5] + a[26]*x[4]*x[6] + a[27]*x[5]*x[6]



#creates a list of values found after each D_i acts on f
D_found = []
for i in range(n):
    rem_ind = list(range(n))
    rem_ind.remove(i)
    sD_i = 0
    for j in rem_ind:
        sD_i = sD_i + (f-f.subs({x[i]:x[j],x[j]:x[i]}))/(x[i]-x[j])
    D_i = diff(f,x[i]) - c*sD_i
    D_found.append(D_i)
```

```
#displays the found values for D_i(f)
#for i in range(n):
#    D_found[i]

print('__________________________________________________________________
#MATRIX CREATION AND ROW REDUCTION SECTION
#Creates Good variables and assignes it to True if any D_i is 0
Good = False
for i in range(n):
    if D_found[i] == 0:
        Good = True

#Matrix is only found if all D_i are not equal to 0. The program fails otherwise.
if Good == False:
    #Creates T, a list of lists where the sublists are the
#collection of monomials in each D_i
    T = []
    for i in range(n):
        T.append(D_found[i].numerator().monomials())

    #Creates V, a list containing the coefficients of each monomial in each D_i
    V = []
    for i in range(n):
        for j in range(len(T[i])):
            V.append(D_found[i].numerator().monomial_coefficient(T[i][j]))

    #Creates W, a list of lists where each sublist is the
#collection of coefficients for the
#coefficient sum of each monomial in each D_i
    W = []
    for i in range(len(V)):
        U = []
        for j in range(m):
            U.append(V[i].monomial_coefficient(a[j]))
        W.append(U)

    #Creates a matrix Q which has, as its rows, the coefficients of the coefficient sum
# of each monomial in each D_i
    G1 = matrix(W[0])
    M = G1.transpose()
    for i in range(1, len(W)):
        G2 = vector(W[i])
        M = M.augment(G2)

    Q = M.transpose()
    QF = Q.echelon_form()
```

```
    rowcount = QF.nrows()
    colcount = QF.ncols()
    for i in range(rowcount):
        AllZero = True
        for j in range(colcount):
            if QF.row(i)[j] != 0:
                AllZero = False
        if AllZero == False:
            QF.row(i)
```

Program 2

```
n = 10
R = PolynomialRing(QQ, n, 'x')
x = R.gens()
#c = var('c')

# partition

lam = [2,2,2]

# Garnir polynomial for lambda
f = 1
l=len(lam)
tmp = 0
for k in range(l):
    w=prod([prod([x[i]-x[j] for j in range(i+1,tmp+lam[k])])
     for i in range(tmp,tmp+lam[k])])
    f=f*w
    tmp=tmp+lam[k]


#Dunkl operators

f = (x[0]-x[1])*(x[2]-x[3])*(x[4]-x[5])*(x[6]-x[7])*(x[8]-x[9])
c = 1/6
factor(f)
D_found = []
for i in range(n):
    rem_ind = list(range(n))
    rem_ind.remove(i)
    sD_i = 0
    for j in rem_ind:
        sD_i = sD_i + (f-f.subs({x[i]:x[j],x[j]:x[i]}))/(x[i]-x[j])
    D_i = diff(f,x[i]) - c*sD_i
    D_found.append(D_i)
```

```
for i in range(n):
    if D_found[i] != 0:
        factor(D_found[i])
    else:
        D_found[i]
```