SLOW NEIGHBORHOOD GROWTH ON THE HAMMING PLANE

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Abstract

We consider neighborhood growth dynamics on the Hamming plane. Each vertex on the Hamming plane is assigned an initial state, occupied or empty. A vertex x becomes occupied if the pair consisting of the counts of occupied vertices along the horizontal and vertical lines through x lies outside a fixed Young diagram, $\mathcal{Z}$. An initial set of occupied vertices spans for $\mathcal{Z}$ if after some number of iterations of the growth dynamics the entire Hamming plane is occupied. We study sets that take a long time to span for a fixed $\mathcal{Z}$. We give an upper bound on the maximal spanning time that is quadratic in the side length of the bounding square of $\mathcal{Z}$. We also give a lower bound of $(s-1)^{1/2}$, where s is the side length of the largest square contained in $\mathcal{Z}$.

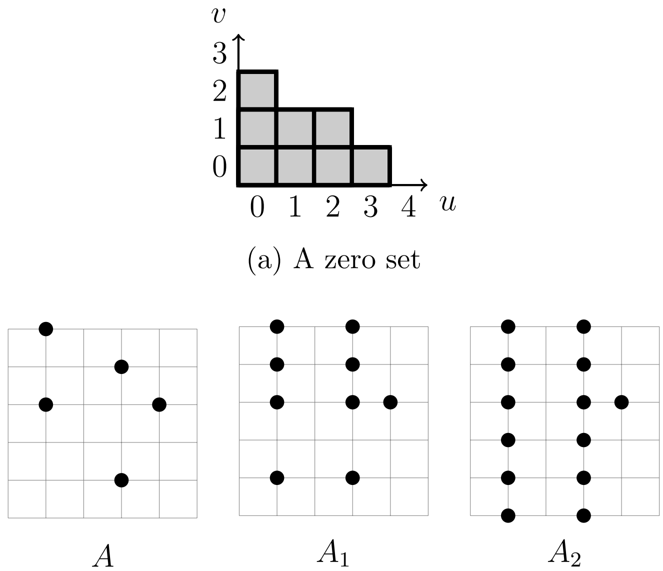
1 Introduction

Neighborhood growth dynamics [**GSS**] is a discrete time growth process that describes how an initial set of vertices on the Hamming plane grows over time. The Hamming plane is defined as the graph with vertex set $\mathbb{Z}^2$ and an edge between any two vertices that differ in a single coordinate. For simplicity we consider growth restricted to $\mathcal{Z}_+^2$. Percolation and growth processes on Hamming graphs are addressed in [**BBLN**, **GHPS**, **Siv**, **Sli**].

We use the notation and definitions introduced in [**GSS**]. For $a, b \in \mathbb{N}$, define the discrete $a \times b$ rectangle as $R_{a,b} = ([0, a-1] \times [0, b-1]) \cap \mathbb{Z}_+^2$. The set $\mathcal{Z} = \bigcup_{(a,b)\in\mathcal{I}} R_{a,b}$ consisting of the union of rectangles over a set $\mathcal{I} \subseteq \mathbb{N}^2$ is called a *zero set*. The cases when $\mathcal{Z} = \emptyset$ and $\mathcal{Z}$ is infinite are also allowed, however in this paper we consider only finite zero sets. Thus zero sets are equivalent to Young diagrams in the French notation.

Fix a zero set $\mathcal{Z}$. Let $A \subseteq \mathbb{Z}_+^2$ be a set of vertices and $x \in \mathbb{Z}_+^2$ a vertex. Let $L^h(x)$ and $L^v(x)$ denote the horizontal and vertical lines through x, respectively. We define the *neighborhood* of x by $N(x) = L^h(x) \cup L^v(x)$. The row and column counts of x are given by

$$\text{row}(x, A) = \left|L^h(x) \cap A\right| \text{ and } \text{col}(x, A) = |L^v(x) \cap A|\,.$$



(a) A zero set

(b) An initial set A and its growth behavior. Observe that the last configuration is inert since $\mathcal{T}^3(A) = \mathcal{T}^2(A)$. Thus A does not span.

Figure 1: Example of neighborhood growth dynamics.

Define the ordered pair of row and column counts of x by $(u, v) = (\text{row}(x, A), \text{col}(x, A))$. The *neighborhood growth transformation* $\mathcal{T} : 2^{\mathbb{Z}_+^2} \to 2^{\mathbb{Z}_+^2}$ is determined by a zero set $\mathcal{Z}$ and is described by the following *neighborhood growth rule*. Fix a zero set $\mathcal{Z}$ and an initial set of vertices $A \subseteq \mathcal{Z}_+^2$. If $x \in A$, then $x \in \mathcal{T}(A)$. If $x \notin A$, then $x \in \mathcal{T}(A)$ if and only if $(u, v) \notin \mathcal{Z}$.

Given an initial set of vertices $A \subseteq \mathcal{Z}_+^2$ and a neighborhood growth transformation $\mathcal{T}$, we can define a *discrete time trajectory*. The set of vertices at time $t \geq 0$ is defined by $A_t = \mathcal{T}^t(A)$. The vertices in A_t are called *occupied* and the vertices in A_t^c are called *empty*. The set $A \subseteq \mathbb{Z}_+^2$ is *inert* if $\mathcal{T}(A) = A$. Define the set of eventually occupied vertices by $A_\infty = \mathcal{T}^\infty(A) = \bigcup_{t \geq 0} A_t$. The set A *spans* if $A_\infty = \mathbb{Z}_+^2$. We also say that a subset $B \subseteq \mathcal{Z}_+^2$ is *spanned* if every vertex in B is occupied. Figure 1 shows an example of neighborhood growth dynamics.

We discuss the history and some previous results of neighborhood growth dynamics. There are several special cases of neighborhood growth. The simplest case is called *line growth*, introduced as *line percolation* by Bollobás et al. [**BBLN**], and is given by rectangular zero sets $\mathcal{Z} = R_{a,b}$ for $a, b \in \mathbb{N}$. Another special case is called L *growth*, given by L shaped zero sets consisting of the union of two rectangles $\mathcal{Z} = R_{a,b} \cup R_{c,d}$, where $a, b, c, d \in \mathbb{N}$ such that $a > c$ and $d > b$.

The most well studied type of neighborhood growth is called *threshold growth* [**GG**]. It was first introduced as *bootstrap percolation* by Chalupa, Leath, and Reich. Growth is defined by an integer $\theta \geq 1$ called the *threshold*, such that a vertex x becomes occupied when its neighborhood count is at least θ. Threshold growth on the Hamming plane is given by triangular zero sets $\mathcal{Z} = \{(u, v) : u + v \leq \theta - 1\}$.

Bootstrap percolation is usually studied in a random setting, where the vertices of the initial set A are chosen independently at random with probability p. The main question is to determine the value of p such that spanning occurs with probability at least $1/2$. This is called the *critical probability* and is denoted by p_c. Aizenman and Lebowitz [**AL**] initiated the study of the *classic bootstrap percolation*, where the growth process occurs on the $n \times n$ grid, denoted by $[n]^2$, with threshold $\theta = 2$. The first main result was proven by Holroyd [**Hol**], who showed that $p_c = \pi^2/(18 \log n) + o(1/\log n)$. Gravner, Holroyd, and Morris [**GHM**] obtained bounds on the second term, and Balogh, Bollobás, Duminil-Copin, and Morris [**BBDM**] extended these results and determined p_c for all $[n]^d$ and θ. Other variants have also been studied. Balogh and Bollobás [**BB**] addressed bootstrap percolation on the hypercube and Bollobás, Duminil-Copin, Morris, and Smith [**BDMS**] studied bootstrap percolation with drift.

Extremal problems are also of great interest. It is a well known result that the smallest spanning sets have exactly size n for the classic bootstrap percolation. For bootstrap percolation on $[n]^d$ with $\theta = 2$, Balogh, Bollobás, and Morris [**BBM**] found that the smallest spanning sets have size $\lceil d(n-1)/2 \rceil + 1$. Smallest spanning sets have also been studied by Riedl [**Rie2**] for bootstrap percolation on trees and by Balogh, Bollobás, Morris, and Riordan [**BBMR**] on hypergraphs. The largest size of inclusion-minimal spanning sets is addressed by Morris [**Mor**] for the classic bootstrap percolation and by Riedl [**Rie1**, **Rie2**] on hypercubes and on trees.

In this paper we study the extremal quantity $M(\mathcal{Z})$, called the *maximal spanning time* for $\mathcal{Z}$, defined by

$$M(\mathcal{Z}) = \max\{\min\{t \in \mathbb{N} : \mathcal{T}^t(A) = \mathbb{Z}_+^2\} : A \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of all spanning sets of $\mathcal{Z}$.

Benevides and Przykucki [**BP**] showed that the maximal spanning time is equal to $13n^2/18 + O(n)$ for the classical bootstrap percolation model. For neighborhood growth on the Hamming plane, Gravner, Sivakoff, and Slivken [**GSS**] found that the upper bound for $M(\mathcal{Z})$ is factorial in the length of the longest row or column of $\mathcal{Z}$.

We now state our main results. The first theorem shows that the upper bound for $M(\mathcal{Z})$ depends on the size of the bounding square $R_{m,n}$ of $\mathcal{Z}$, while the second theorem shows that the lower bound depends on the size of the largest square contained in $\mathcal{Z}$.

Theorem 1.1. *For any zero set $\mathcal{Z}$ such that $\mathcal{Z} \subseteq R_{n,n}$,*

$$M(\mathcal{Z}) \leq 4n^2 + 2n + 2.$$

Theorem 1.2. *Let $R_{s,s}$ be the largest square such that $R_{s,s} \subseteq \mathcal{Z}$. Then*

$$M(\mathcal{Z}) \geq (s-1)^{1/2}.$$

The paper is organized as follows. In Section 2 we introduce some notation, definitions, and lemmas which we use throughout the paper. In Section 3 we prove Theorem 1.1. Next we prove results on $M(\mathcal{Z})$ for special cases of neighborhood growth in Section 4. Finally in Section 5 we prove Theorem 1.2.

2 Preliminaries

In this section we prove some lemmas that we use throughout the paper. We first introduce some notation and definitions.

Let $\mathcal{Z}$ be a Young diagram and $k \in \mathbb{N}$. We define the Young diagrams obtained by removing the k leftmost columns or k bottommost rows of $\mathcal{Z}$, respectively, by

$$\mathcal{Z}^{\downarrow k} = \{(u, v-k) : (u,v) \in \mathcal{Z}, v \geq k\},$$
$$\mathcal{Z}^{\leftarrow k} = \{(u, v-k) : (u,v) \in \mathcal{Z}, u \geq k\}.$$

The Young diagram obtained by shifting $\mathcal{Z}$ diagonally by k is defined by

$$\mathcal{Z}^{\swarrow k} = \left(\mathcal{Z}^{\leftarrow k}\right)^{\downarrow k}.$$

We also define the *outside boundary* of a Young diagram $\mathcal{Z}$ by

$$\partial_o(\mathcal{Z}) = \{(x,y) \in \mathbb{Z}_+^2 \setminus \mathcal{Z} : (x-1,y) \in \mathcal{Z} \text{ or } (x,y-1) \in \mathcal{Z}\}.$$

Given two Young diagrams Y_1, Y_2, we define the *grid sum* of Y_1 and Y_2 by

$$Y_1 \boxplus Y_2 = \mathbb{Z}_+^2 \setminus [(\mathbb{Z}_+^2 \setminus Y_1) + (\mathbb{Z}_+^2 \setminus Y_2)].$$

Observe that $Y_1 \boxplus Y_2$ is also a Young diagram. Moreover the grid sum is commutative. See Figure 2 for an example of the grid sum.

Let $A \subseteq \mathbb{Z}_+^2$ be a finite set. We denote the *projections* of A onto the x-axis and y-axis by $\pi_x(A)$ and $\pi_y(A)$, respectively.

In order to prove Theorem 1.2, we introduce a generalized version of neighborhood growth dynamics. The *enhancements* $\vec{r} = (r_0, r_1, \ldots) \in \mathbb{Z}_+^\infty$ and $\vec{c} = (c_0, c_1, \ldots) \in \mathbb{Z}_+^\infty$

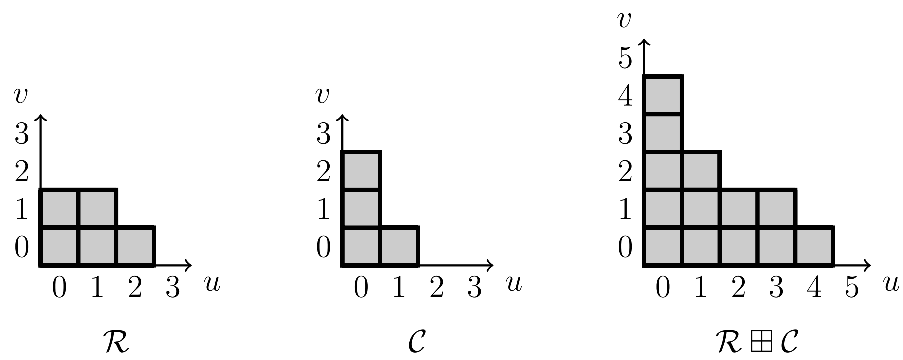


Figure 2: Example of grid sum.

are weakly decreasing sequences of non-negative integers which increase row and column counts, respectively, by fixed values. We call $\vec{r}$ the row enhancements and $\vec{c}$ the column enhancements. Then *enhanced neighborhood growth dynamics* is given by the triple $(\mathcal{Z}, \vec{r}, \vec{c})$, which defines a growth transformation $\mathcal{T}_{enhanced}$ as follows:

$$\mathcal{T}_{enhanced}(A) = A \cup \{(i,j) \in \mathbb{Z}_+^2 : (\mathrm{row}((i,j), A) + r_j, \mathrm{col}((i,j), A) + c_i) \notin \mathcal{Z}\}.$$

Note that enhanced growth given by $(\mathcal{Z}, \vec{0}, \vec{0})$ is equivalent to standard neighborhood growth given by $\mathcal{Z}$, but we do not differentiate between the two.

Given a pair of enhancements $(\vec{r}, \vec{c})$, we can form a pair of Young diagrams $(\mathcal{R}, \mathcal{C})$ where the row counts of $\mathcal{R}$ is given by $\vec{r}$ and the column counts of $\mathcal{C}$ is given by $\vec{c}$. Furthermore given a pair of Young diagrams $(\mathcal{R}, \mathcal{C})$, we can also extract a pair of enhancements $(\vec{r}, \vec{c})$ by setting r_j equal to the jth row of $\mathcal{R}$ and c_i equal to the ith column of $\mathcal{C}$. Thus for simplicity we will use $\mathcal{R}$ to denote the row enhancements $\vec{r}$, and similarly we use $\mathcal{C}$ to denote the column enhancements $\vec{c}$.

In this paper we consider enhanced growth starting from enhancements $\mathcal{R}$ and $\mathcal{C}$ only. That is, the initial set of occupied vertices is empty. We say that the pair of enhancements $(\mathcal{R}, \mathcal{C})$ span for a zero set $\mathcal{Z}$ if $\emptyset$ spans for $(\mathcal{Z}, \mathcal{R}, \mathcal{C})$. We let $\mathcal{A}_{enhanced}$ denote the set of all pairs of enhancements $(\mathcal{R}, \mathcal{C})$ that span for $\mathcal{Z}$.

Fix a zero set $\mathcal{Z}$. Given $A \in \mathcal{A}$, we define the spanning time of A under the growth dynamics $\mathcal{T}$ as

$$T(\mathcal{Z}, A) = \min\{t \in \mathbb{N} : \mathcal{T}^t(A) = \mathbb{Z}_+^2\}.$$

Similarly given enhancements $(\mathcal{R}, \mathcal{C}) \in \mathcal{A}_{enhanced}$, we define the spanning time of $(\mathcal{R}, \mathcal{C})$ under the enhanced growth dynamics $\mathcal{T}_{enhanced}$ as

$$T_{enhanced}(\mathcal{Z}, \mathcal{R}, \mathcal{C}) = \min\{t \in \mathbb{N} : \mathcal{T}_{enhanced}^t(\emptyset) = \mathbb{Z}_+^2\}.$$

By the definition above we can rewrite $M(\mathcal{Z})$ as

$$M(\mathcal{Z}) = \max\{T(\mathcal{Z}, A) : A \in \mathcal{A}\}.$$

We define an analogous extremal quantity for enhanced growth. The maximal spanning time for enhanced growth starting from enhancements $(\mathcal{R}, \mathcal{C})$ is given by

$$M_{enhanced}(\mathcal{Z}) = \max\{T_{enhanced}(\mathcal{Z}, \mathcal{R}, \mathcal{C}) : (\mathcal{R}, \mathcal{C}) \in \mathcal{A}_{enhanced}\}.$$

We now state and prove several useful lemmas. The first two lemmas give us inequalities that relate the maximal spanning time for $\mathcal{Z}$ to the maximal spanning time for small perturbations of $\mathcal{Z}$.

Lemma 2.1. *Let $\tau(A)$ denote the earliest time that a row or column of A is spanned. Define $\alpha = \alpha(\mathcal{Z}) = \max_{A \in \mathcal{A}}\{\tau(A)\}$. For any zero set $\mathcal{Z}$,*

$$M(\mathcal{Z}) \leq \alpha + \max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}.$$

Proof. Let $A \in \mathcal{A}$. Then A_α contains at least one spanned row, R, or at least one spanned column, C. Since A is a spanning set for $\mathcal{Z}$, $A_\alpha \setminus R$ is a spanning set for $\mathcal{Z}^{\downarrow 1}$ or $A_\alpha \setminus C$ is a spanning set for $\mathcal{Z}^{\leftarrow 1}$. Furthermore, A_α spans in at most $\max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}$ time steps. Therefore $M(\mathcal{Z}) \leq \alpha + \max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}$. □

Lemma 2.2. *For any zero set $\mathcal{Z}$,*

$$M(\mathcal{Z}) \geq \max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}, \tag{1}$$

$$M_{enhanced}(\mathcal{Z}) \geq \max\left\{M_{enhanced}\left(\mathcal{Z}^{\downarrow 1}\right), M_{enhanced}\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}. \tag{2}$$

Proof. Suppose that $\max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\} = M\left(\mathcal{Z}^{\leftarrow 1}\right)$. Let A be a spanning set for $\mathcal{Z}^{\leftarrow 1}$. Consider the set $A' = A \cup C \in \mathcal{A}$, where C is a fully spanned column such that $A \cap C = \emptyset$. Then A' is a spanning set for $\mathcal{Z}$ that spans in at most $M(\mathcal{Z})$ steps. Since A' contains a fully spanned column, $T(\mathcal{Z}, A') = T(\mathcal{Z}^{\leftarrow 1}, A)$. Therefore $M(\mathcal{Z}) \geq M\left(\mathcal{Z}^{\leftarrow 1}\right)$. If instead we have that $\max\left\{M\left(\mathcal{Z}^{\downarrow 1}\right), M\left(\mathcal{Z}^{\leftarrow 1}\right)\right\} = M\left(\mathcal{Z}^{\downarrow 1}\right)$, then by symmetry $M(\mathcal{Z}) \geq M\left(\mathcal{Z}^{\downarrow 1}\right)$. In any case this proves (1).

For (2) suppose that $\max\left\{M_{enhanced}\left(\mathcal{Z}^{\downarrow 1}\right), M_{enhanced}\left(\mathcal{Z}^{\leftarrow 1}\right)\right\} = M_{enhanced}\left(\mathcal{Z}^{\leftarrow 1}\right)$. Let $\mathcal{R}' = (r_0, \ldots, r_n, 0, 0, \ldots)$ and $\mathcal{C}' = (c_0, \ldots, c_m, 0, 0, \ldots)$ be enhancements that span for $\mathcal{Z}^{\leftarrow 1}$. Define $\mathcal{R} = (r_0 + 1, r_1 + 1, \ldots, r_n + 1, 1, 1, \ldots)$ and $\mathcal{C} = \mathcal{C}'$. Let A'_t and A_t denote the set of occupied vertices at time t under the dynamics $(\mathcal{Z}^{\leftarrow 1}, \mathcal{R}', \mathcal{C}')$ and $(\mathcal{Z}, \mathcal{R}, \mathcal{C})$, respectively.

We use induction to show that $A_t = A'_t$ for all $t \geq 0$. The base case $A_0 = A'_0 = \emptyset$ holds. Suppose $A_t = A'_t$ for some $t \geq 0$. Let $x = (i, j)$ be a vertex that is not in A_t, and hence not in A'_t. Then $(\mathrm{row}(x, A'_t) + r_j, \mathrm{col}(x, A'_t) + c_i) \in \mathcal{Z}^{\leftarrow 1}$ if and only if $(\mathrm{row}(x, A_t) + r_j + 1, \mathrm{col}(x, A_t) + c_i) \in \mathcal{Z}$. This shows that $x \in A_{t+1}$ if and only if $x \in A'_{t+1}$. Therefore $T_{enhanced}(\mathcal{Z}^{\leftarrow 1}, \mathcal{R}', \mathcal{C}') = T_{enhanced}(\mathcal{Z}, \mathcal{R}, \mathcal{C})$, and we have $M_{enhanced}(\mathcal{Z}) \geq M_{enhanced}\left(\mathcal{Z}^{\leftarrow 1}\right)$.

If instead we have $\max\left\{M_{enhanced}\left(\mathcal{Z}^{\downarrow 1}\right), M_{enhanced}\left(\mathcal{Z}^{\leftarrow 1}\right)\right\} = M_{enhanced}\left(\mathcal{Z}^{\downarrow 1}\right)$, then by symmetry $M_{enhanced}(\mathcal{Z}) \geq M_{enhanced}\left(\mathcal{Z}^{\downarrow 1}\right)$. In either case (2) holds. □

Several proofs in this paper require us to construct enhancements that span for a given zero set and so we need an easy method to check if spanning occurs. The next lemma gives a useful criterion to determine whether a pair of enhancements $(\mathcal{R}, \mathcal{C})$ span for a zero set $\mathcal{Z}$.

Lemma 2.3. *Fix a zero set $\mathcal{Z}$. The enhancements $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$ if and only if*

$$\mathcal{Z} \subseteq \mathcal{R} \boxplus \mathcal{C}.$$

Proof. Suppose $\mathcal{Z} \subseteq \mathcal{R} \boxplus \mathcal{C}$. Let $m = |\pi_x(\mathcal{C})|$ and $n = |\pi_y(\mathcal{R})|$. Let $u_{i,j} = (r_j + i, c_i + j)$. Then

$$\{u_{i,j} : 0 \leq i \leq m, 0 \leq j \leq n\} \subseteq \mathbb{Z}_+^2 \setminus (\mathcal{R} \boxplus \mathcal{C}).$$

Suppose the vertex $x = (i, j)$ is occupied at some time $t > 0$. Every vertex to the left of and below x must also be occupied by time t. Let $x_v = (i, j+1)$ and $x_h = (i+1, j)$. Then the sites $(\mathrm{col}(x_v, A_t) + c_i, \mathrm{row}(x_v, A_t) + r_{j+1}) = u_{i,j+1}$ and $(\mathrm{col}(x_h, A_t) + c_{i+1}, \mathrm{row}(x_h, A_t) + r_j) = u_{i,j+1}$ lie outside of $\mathcal{R} \boxplus \mathcal{C}$. This implies that the vertices x_v and x_h must be occupied by time $t + 1$. Since $(0, 0)$ is occupied at $t = 1$, every vertex in $\mathcal{Z}_+^2$ becomes occupied. Thus $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$.

If $\mathcal{Z} \not\subseteq \mathcal{R} \boxplus \mathcal{C}$, there exists $k \in [0, m]$ and $\ell \in [0, n]$ such that $u_{k,\ell} \in \mathcal{Z}$. Suppose vertex $x = (k, \ell) \notin A_t$ for some time $t > 0$. Let (u, v) denote the row and column counts of x. Then $u \leq r_\ell + k$ and $v \leq c_k + \ell$, which implies that (u, v) lies in $\mathcal{Z}$. Thus $x \notin A_{t+1}$ and so x never becomes occupied. Moreover every vertex $x' = (k', \ell')$ with $k < k'$ and $\ell < \ell'$ never becomes occupied. Therefore spanning does not occur. □

We introduce the notion of thin sets as a way to connect enhanced neighborhood growth to standard neighborhood growth. A set $A \subseteq \mathbb{Z}_+^2$ is *thin* if for every vertex $x \in A$, either $\mathrm{row}(x, A) = 1$ or $\mathrm{col}(x, A) = 1$. That is, no other vertices of A lie either on the horizontal line through x or the vertical line through x.

Given enhancements $(\mathcal{R}, \mathcal{C})$, we define the *canonical thin set* with respect to $(\mathcal{R}, \mathcal{C})$ as the thin set A constructed by populating column i with c_i occupied vertices and row j with r_j occupied vertices such that $A \cap R_{m,n} = \emptyset$, where m is the index of the first zero row enhancement and n is the index of the first zero column enhancement. See Figure 3 for an example.

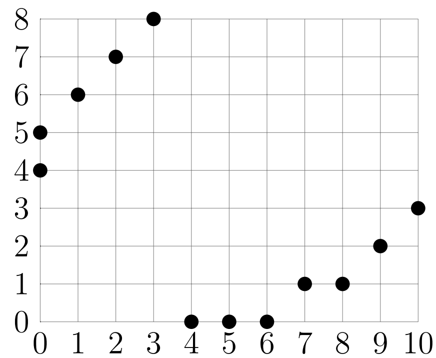


Figure 3: The canonical thin set with respect to enhancements
$\mathcal{R} = (3, 2, 1, 1, 0, 0, \ldots)$ and $\mathcal{C} = (2, 1, 1, 1, 0, 0, \ldots)$.

In order to relate lower bounds on $M_{enhanced}(\mathcal{Z})$ to $M(\mathcal{Z})$ we consider spanning by thin sets. For any pair of finite row and column enhancements, $(\mathcal{R}, \mathcal{C})$, that span under the dynamics $\mathcal{T}_{enhanced}$, the canonical thin set with respect to $(\mathcal{R}, \mathcal{C})$, $B_{thin} = B_{thin}(\mathcal{R}, \mathcal{C})$, will span under the dynamics $\mathcal{T}$. However, the spanning time for B_{thin} under $\mathcal{T}$ might be faster than the spanning time for $(\mathcal{R}, \mathcal{C})$ under $\mathcal{T}_{enhanced}$. In the following lemma, we use thin sets to show that lower bounds on $M_{enhanced}(\mathcal{Z}^{\swarrow 1})$ are also lower bounds on $M(\mathcal{Z})$.

Lemma 2.4. *For any zero set* $\mathcal{Z}$*,*

$$M_{enhanced}\left(\mathcal{Z}^{\swarrow 1}\right) \leq M(\mathcal{Z}).$$

Proof. Let $(\mathcal{R}, \mathcal{C})$ denote a pair of enhancements with row counts $(r_0, \cdots, r_{n-1})$ for $\mathcal{R}$ and column counts $(c_0, \cdots, c_{m-1})$ for $\mathcal{C}$. Suppose further that $r_{n-1} = c_{m-1} = 1$. If $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$ then the shifted enhancements $(\mathcal{R}', \mathcal{C}')$ where $\mathcal{R}' = \mathcal{R}^{\leftarrow 1}$, $\mathcal{C}' = \mathcal{C}^{\downarrow 1}$ will span for $\mathcal{Z}^{\swarrow 1}$. Let $\mathcal{R}^1$ and $\mathcal{C}^1$ denote the Young diagrams obtained by modifying $\vec{r}$ and $\vec{c}$ so that $r_0 = c_0 = \infty$. We show that the spanning times $T(\mathcal{Z}, \mathcal{R}^1, \mathcal{C}^1)$ and $T_{enhanced}(\mathcal{Z}^{\swarrow 1}, \mathcal{R}', \mathcal{C}')$ are the same.

Let A'_t and A^1_t be the sets of occupied vertices at time t for $(\mathcal{Z}^{\swarrow 1}, \mathcal{R}', \mathcal{C}')$ and $(\mathcal{Z}, \mathcal{R}^1, \mathcal{C}^1)$, respectively. By induction on t, we show that $A'_t = A^1_t$ for all $t \geq 0$. Therefore $(\mathcal{R}', \mathcal{C}')$ span for $\mathcal{Z}^{\swarrow 1}$ if and only if $(\mathcal{R}^1, \mathcal{C}^1)$ span for $\mathcal{Z}$ and the spanning times are the same.

The base case $A'_0 = A_0 = \emptyset$ is satisfied. Suppose $A'_t = A^1_t$ for some $t \geq 0$. Let $x = (i, j)$ be a vertex not in A'_t, and hence not in A^1_t. Define $k = \text{col}(x, A^1_t) = \text{col}(x, A'_t)$ and $\ell = \text{row}(x, A^1_t) = \text{row}(x, A'_t)$. Then $(r_j + \ell, c_i + k) \notin \mathcal{Z}$ if and only if $((r_j - 1) + \ell, (c_i - 1) + k) \notin \mathcal{Z}^{\swarrow 1}$. This implies that $x \in A'_{t+1}$ if and only if $x \in A^1_{t+1}$ and the spanning times are the same.

Let B_{thin} denote the canonical thin set for the pair $(\mathcal{R}, \mathcal{C})$. The proof will follow if we can show

$$T(\mathcal{Z}, B_{thin}) \geq T_{enhanced}(\mathcal{Z}, \mathcal{R}^1, \mathcal{C}^1). \tag{3}$$

Let $B_t = \mathcal{T}^t(B_{thin})$ and $B_t^* = B_t \cap R_{m,n}$. Similarly let $A_t^* = A_t \cap R_{m,n}$. Since $r_{n-1} = c_{m-1} = 1$, if $(m-1, n-1) \in A_t^*$ then $t \geq T_{enhanced}(\mathcal{Z}, \mathcal{R}^1, \mathcal{C}^1)$. We claim that for all $t > 0$,

$$A_t^* = B_t^*. \tag{4}$$

We proceed by induction on t. The base case is satisfied as $B_t^* = A_t^* = \emptyset$. Suppose for some $t > 0$, $B_t^* = A_t^*$. Let $x = (i,j) \in R_{m,n} \backslash B_t^*$. If

$$\text{row}(x, A_t) + r_j = \text{row}(x, B_t) \tag{5}$$

$$\text{col}(x, A_t) + c_i = \text{col}(x, B_t) \tag{6}$$

then $x \in B_{t+1}^*$ implies $x \in A_{t+1}^*$. By the induction hypothesis $\text{row}(x, A_t^*) = \text{row}(x, B_t^*)$. Therefore if $\text{row}(x, B_t \backslash B_t^*) = r_j$ then (5) holds. By the construction of B_{thin} we know that $\text{row}(x, B_t \backslash B_t^*) \geq r_j$. For each row, sites in that row outside of $R_{m,n} \cup B_0$ will not become occupied before a site in same row that is contained in $R_{m,n}$ as the column counts for sites outside of $R_{m,n}$ will be less than or equal to the column counts of sites in the same row inside $R_{m,n}$. Therefore if $x \in R_{m,n}$ is not occupied then no site in that row outside of $B_0 \cup R_{m,n}$ will be occupied and $\text{row}(x, B_t \backslash B_t^*) = r_j$. Similarly $\text{col}(x, B_t \backslash B_t^*) = c_i$ so (6) also holds. Equations (5) and (6) combined imply (4) which in turn implies (3). □

3 Upper Bound on $M(\mathcal{Z})$

The previous upper bound for $M(\mathcal{Z})$ is factorial in the size of the longest row or column of $\mathcal{Z}$ [**GSS**]. We improve this result and show that if $\mathcal{Z}$ is bounded by an $n \times n$ square, then the upper bound for $M(\mathcal{Z})$ is quadratic in n.

Proof of Theorem 1.1. Fix a zero set $\mathcal{Z}$ contained in an $n \times n$ square $R_{n,n}$. Let A be a spanning set for $\mathcal{Z}$ and let A_t denote the set of occupied vertices at time t, with $A_0 = A$. At each time $t > 0$, at least one new vertex is occupied until the entire Hamming plane is spanned. Let $\tau = T(\mathcal{Z}, A)$ denote the spanning time for A. For $0 < t \leq \tau$, let $B_t = A_t \setminus A_{t-1}$.

After n vertices become occupied in a fixed row or column, the row or column must become spanned by the next step. For every t and every $u \in B_t$, there are at most $2n$ other times $t' \neq t$ when a vertex in the neighborhood of u may become occupied. Therefore if $\tau \geq 2n(2n+1)$, there exists at least $2n$ vertices $\{u_i\}_{i=1}^{2n}$ such that no two u_i and u_j are in the same row or column and each u_i becomes occupied by time $t_n = 2n(2n+1)$.

Let (r_i, c_i) denote the row and column counts for vertex u_i at time $t_n - 1$. Since u_i becomes occupied either at or before time t_n, $(r_i, c_i) \notin \mathcal{Z}$.

For $1 \leq i \leq 2n$, let (x_i, y_i) denote the coordinates of the site u_i. For $1 \leq i, j \leq 2n$ let $v_{ij} = (x_i, y_j)$ and $w_{ij} = (x_j, y_i)$. At time t_n at least one of the pairs (r_j, c_i) or (r_i, c_j) lies outside of $\mathcal{Z}$, so at least one, or both, of v_{ij} or w_{ij} must be occupied by time t_n. For $1 \leq i \leq 2n$, let

$$V_i = \left(\bigcup_{j=1}^{2n} \{v_{ij}\} \right) \cap A_{t_n} \text{ and } W_i = \left(\bigcup_{j=1}^{2n} \{w_{ij}\} \right) \cap A_{t_n}.$$

All vertices in V_i lie in the same column and all vertices in W_i lie in the same row. Let M_i denote the larger of V_i and W_i, and let L_i denote the row or column that contains M_i. Each M_i must contain at least n vertices, and hence by time $t_n + 1$ each L_i must be spanned. Of the $2n$ different L_i, at least n must be rows or at least n must be columns. In either case, the entire Hamming plane is spanned by time $t_n + 2$. Therefore we conclude that $M(\mathcal{Z}) \leq 4n^2 + 2n + 2$. □

4 Special Cases

In this section we prove results on $M(\mathcal{Z})$ for special cases of neighborhood growth. In particular, we find exact results on $M(\mathcal{Z})$ for line growth and we obtain upper and lower bounds on $M(\mathcal{Z})$ that are of the same order for L growth.

We start by addressing line growth. First we make a preliminary observation. Let $\mathcal{Z} = R_{m,n}$ be a rectangular zero set. If a vertex x becomes occupied then its row count is at least m or its column count is at least n, which imply that $L^h(x)$ or $L^v(x)$, or both, are spanned. That is, spanning occurs rows or columns at a time.

We first prove a lemma that describes the spanning behavior in line growth. In particular it shows that either some number of rows and columns get spanned at the same time, or spanning alternates between some number of rows at time t followed by some number of columns at $t + 1$.

Lemma 4.1. *Let $\mathcal{Z} = R_{m,n}$ and let $A \subseteq \mathbb{Z}_+^2$. If $L^h(x) \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$, then there exists at least one vertex $y \in L^h(x)$ such that $L^v(y) \in \mathcal{T}(A) \setminus A$. Similarly, if $L^v(x) \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$, then there exists at least one vertex $y \in L^v(x)$ such that $L^h(y) \in \mathcal{T}(A) \setminus A$.*

Proof. Suppose $L^h(x) \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$. Then $\text{row}(x, \mathcal{T}(A)) \geq m$ and $\text{row}(x, A) < m$. There exists at least one vertex $y \in L^h(x)$ such that $y \in \mathcal{T}(A) \setminus A$. Since $\text{row}(y, A) < m$, we must have that $\text{col}(y, A) \geq n$. This implies $L^v(y) \in \mathcal{T}(A) \setminus A$. By symmetry the same argument holds if we first suppose $L^v(x) \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$. □

It is interesting to note that we obtain different values for $M(\mathcal{Z})$ depending on whether $\mathcal{Z}$ is a rectangular zero set or a square zero set. However in either case we show that $M(\mathcal{Z})$ is linear in the side length of $\mathcal{Z}$.

Proposition 4.2. *If $\mathcal{Z} = R_{m,n}$, with $m \neq n$, then $M(\mathcal{Z}) = 2\min\{m, n\}$.*

Proof. Without loss of generality, assume $m > n$. Let A be a spanning set for $\mathcal{Z}$. At least one row or one column is spanned at every time step. By Lemma 4.1, there are at least $n-1$ spanned rows and columns at $t = 2n-2$. At $t = 2n-1$, every column containing at least one vertex that lies outside of the $n-1$ spanned rows and $n-1$ spanned columns becomes spanned. There must be at least m spanned columns at $t = 2n-1$. If not then spanning does not occur, contradicting the fact that A is a spanning set. Therefore spanning occurs at $t = 2n$. This gives us the upper bound $M(\mathcal{Z}) \leq 2n$.

For the lower bound consider the enhancements $\mathcal{R} = (m-1, m-2, \ldots, 1)$ and $\mathcal{C} = (n, n-1, \ldots, 1)$. By Lemma 2.3, $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$. Let A be the canonical thin set with respect to $(\mathcal{R}, \mathcal{C})$. Then A spans for $\mathcal{Z}$ and one can check that A spans in $2n$ steps. Thus $M(\mathcal{Z}) \geq 2n$. □

Proposition 4.3. *If $\mathcal{Z} = R_{n,n}$, then $M(\mathcal{Z}) = 2n-1$.*

Proof. Let A be a spanning set for $\mathcal{Z}$. At least one row or one column is spanned at each time step. By $t = 2n-3$, there are at least $n-1$ spanned columns or at least $n-1$ spanned rows. Without loss of generality, assume that there are at least $n-1$ spanned rows. At $t = 2n-2$, every column containing at least one occupied vertex outside of the $n-1$ spanned columns is spanned. There must be at least n spanned columns at $t = 2n-2$. If not then spanning does not occur, contradicting the fact that A is a spanning set. Therefore spanning occurs at $t = 2n-1$. This gives us the upper bound $M(\mathcal{Z}) \leq 2n-1$.

For the lower bound, consider the enhancements $\mathcal{R} = (n, n-1, \ldots, 1)$ and $\mathcal{C} = (n-1, n-1, \ldots, 1)$. By Lemma 2.3, $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$. Let A be the canonical thin set with respect to $(\mathcal{R}, \mathcal{C})$. Then A spans for $\mathcal{Z}$ and one can check that A spans in $2n-1$ steps. Thus $M(\mathcal{Z}) \geq 2n-1$. □

We now turn our attention to L growth. We first prove the following useful lemma that describes the spanning behavior of L growth.

Lemma 4.4. *Let $\mathcal{Z} = R_{a,b} \cup R_{c,d}$ and let A be a set of vertices in $\mathbb{Z}_+^2$. In every two time steps at least one column or row is spanned.*

Proof. Suppose $x \in \mathcal{T}^2(A) \backslash \mathcal{T}(A)$. There exists $y \in N(x)$ such that $y \in \mathcal{T}(A) \backslash A$. Suppose

$y \in L^h(x)$. Then either $\text{col}(y, A) \geq d$, or $b \leq \text{col}(y, A) < d$ and $c \leq \text{row}(y, A) < a$.

If $\text{col}(y, A) \geq d$, then column $L^v(y)$ becomes spanned in $\mathcal{T}(A)$ and we are done. Otherwise, $b \leq \text{col}(y, A) < d$ and $c \leq \text{row}(y, A) = \text{row}(x, A) < a$. We must have that $\text{col}(x, A) < b$ and $\text{col}(x, \mathcal{T}(A)) \geq b$. There exists $z \in L^v(x)$ such that $z \in \mathcal{T}(A) \setminus A$. Then $\text{col}(x, A) = \text{col}(z, A) < b$ which implies that $\text{row}(z, A) \geq a$. Thus row $L^h(z)$ becomes spanned in $\mathcal{T}(A)$.

In any case, at least one row or column becomes spanned in $\mathcal{T}(A)$. The same argument holds for the case when $y \in L^v(x)$. □

Although we are unable to obtain an exact result on $M(\mathcal{Z})$, we find upper and lower bounds that are of the same order. In particular given an L shaped zero set $\mathcal{Z} = R_{a,b} \cup R_{c,d}$, the upper and lower bounds are linear in b or c.

Proposition 4.5. *If $\mathcal{Z} = R_{a,b} \cup R_{c,d}$, then*

$$2 \min\{b, c\} \leq M(\mathcal{Z}) \leq 2(b + c)$$

Proof. Without loss of generality assume $b \leq c$. By Lemma 4.4 at least one row or column is spanned in every two steps, and Lemma 2.1 gives us

$$M(\mathcal{Z}) \leq 2 + \max\left\{M\left(\mathcal{Z}^{\leftarrow 1}\right), M\left(\mathcal{Z}^{\downarrow 1}\right)\right\}$$

We prove the upper bound by induction. If $\mathcal{Z}^{\leftarrow 1} = R_{a,b} \cup R_{0,d} = R_{a,b}$ or $\mathcal{Z}^{\downarrow 1} = R_{a,0} \cup R_{c,d} = R_{c,d}$, Proposition 4.2 implies that $M(R_{a,b}) \leq 2 \min\{a, b\} \leq 2b$ and $M(R_{c,d}) \leq 2 \min\{c, d\} \leq 2c$, and thus the base case holds.

Now assume that the upper bound holds for $M\left(\mathcal{Z}^{\leftarrow 1}\right)$ and $M\left(\mathcal{Z}^{\downarrow 1}\right)$. Observe that $\mathcal{Z}^{\leftarrow 1} = R_{a-1,b} \cup R_{c-1,d}$ and $\mathcal{Z}^{\downarrow 1} = R_{a,b-1} \cup R_{c,d-1}$. By the inductive hypothesis $M\left(\mathcal{Z}^{\leftarrow 1}\right) \leq 2(b + c - 1)$ and $M\left(\mathcal{Z}^{\downarrow 1}\right) \leq 2(b - 1 + c)$. Therefore

$$\begin{aligned} M(\mathcal{Z}) &\leq 2 + \max\left\{M\left(\mathcal{Z}^{\leftarrow 1}\right), M\left(\mathcal{Z}^{\downarrow 1}\right)\right\} \\ &\leq 2 + 2(b + c - 1) \\ &= 2(b + c) \end{aligned}$$

which proves the upper bound.

For the lower bound, we consider two cases. Without loss of generality assume $b \leq c$. First suppose $a - c > b$ and $d - b > c$. On the first step, span b rows. Then $\mathcal{Z}^{\downarrow b} = R_{c,d-b}$. By repeated applications of Lemma 2.2, $M(\mathcal{Z}) \geq M(R_{c,d-b}) = 2c$.

Otherwise suppose $a - c \geq b$ or $d - b \geq c$. On the first step, span $c - b$ columns. Let $\mathcal{Z}' = \mathcal{Z}^{\leftarrow c-b} = R_{a-c+b,b} \cup R_{b,d}$. We claim that $M(\mathcal{Z}') \geq 2b$.

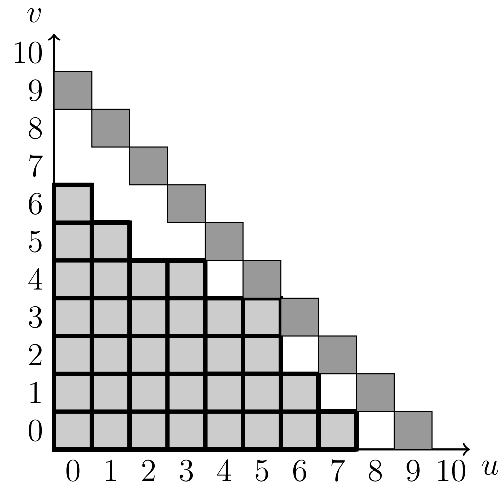


Figure 4: A zero set that lies below a discrete −1 slope line and intersects the line at the site $(5, 3)$.

To show this, consider the enhancements $\mathcal{R} = (n, n-1, \dots, 1)$ and $\mathcal{C} = (b, b-1, \dots, 1)$, where $n = \max\{a-c+b, d\}$. By Lemma 2.3, $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$. Let A be the canonical thin set with respect to $(\mathcal{R}, \mathcal{C})$. One can check that A spans in $2b$ steps. Hence $M(\mathcal{Z}') \geq 2b$.

Again by Lemma 2.2, $M(\mathcal{Z}) \geq M(\mathcal{Z}') \geq 2b$. In any case $M(\mathcal{Z}) \geq 2\min\{b, c\}$ which proves the lower bound. □

5 Lower Bound on $M(\mathcal{Z})$

Recall from Lemma 2.4 that a lower bound on $M_{enhanced}(\mathcal{Z}^{\swarrow 1})$ is also a lower bound on $M(\mathcal{Z})$. Thus in order to prove Theorem 1.2, we first find a lower bound on $M_{enhanced}(\mathcal{Z})$. We start by restricting our attention to zero sets that are contained below a discrete −1 slope line and intersect the line in at least one site. See Figure 4 for an example. The following lemma gives us a lower bound on $M_{enhanced}(\mathcal{Z})$ for this type of zero set.

Lemma 5.1. *Let L be the discrete line $x+y = c$ for some constant c. If for all $(i, j) \in \mathcal{Z}$, $i + j < c$ and there exists $R_{m,n} \subseteq \mathcal{Z}$ such that $m + n - 1 = c$, then*

$$M_{enhanced}(\mathcal{Z}) \geq 2\min\{m, n\}.$$

Proof. Assume without loss of generality that $m \geq n$. Consider the enhancements $\mathcal{R} = (m-1, m-2, \dots, 1)$ and $\mathcal{C} = (n, n-1, \dots, 1)$. Then $\mathcal{R} \boxplus \mathcal{C}$ is the Young diagram $\{(x, y) : x + y \leq m + n - 2\}$. For all sites (x, y) on L, the site $(x - 1, y - 1)$ is in $\mathcal{R} \boxplus \mathcal{C}$. Then $\mathcal{Z} \subseteq \mathcal{R} \boxplus \mathcal{C}$ and Lemma 2.3 implies that $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$.

One can check that $(\mathcal{R}, \mathcal{C})$ spans $R_{m,n}$ in $2\min\{m,n\}+1$ steps when $m \neq n$ and $2n$ steps when $m = n$. Thus $T_{enhanced}(R_{m,n}, \mathcal{R}, \mathcal{C}) \geq 2\min\{m,n\}$. Since $R_{m,n} \subseteq \mathcal{Z}$, $T_{enhanced}(\mathcal{Z}, \mathcal{R}, \mathcal{C}) \geq T_{enhanced}(R_{m,n}, \mathcal{R}, \mathcal{C})$ and the result follows. $\square$

For any zero set $\mathcal{Z}$ there exists a largest square, $R_{s,s}$, contained in $\mathcal{Z}$. The next proposition gives a lower bound on $M_{enhanced}(\mathcal{Z})$ that is dependent on the size of this square.

Proposition 5.2. *Let $R_{s,s}$ be the largest square such that $R_{s,s} \subseteq \mathcal{Z}$. Then*

$$M_{enhanced}(\mathcal{Z}) \geq s^{1/2}.$$

Proof. Let k_r and k_c denote the largest number of rows and columns, respectively, in $\mathcal{Z}$ that are all of the same length and least length s. If $k_r < s^{1/2}$, then there are at least $s^{1/2}$ rows of different lengths in $\mathcal{Z}$. Let $(\mathcal{R}, \mathcal{C})$ be enhancements such that $\mathcal{R} = \mathcal{Z}$ and $\mathcal{C} = \emptyset$. Then rows with different lengths are spanned at different times, which gives a spanning time of at least $s^{1/2}$. Similarly if $k_c < s^{1/2}$, there are at least $s^{1/2}$ columns of different lengths in $\mathcal{Z}$. Setting $\mathcal{R} = \emptyset$ and $\mathcal{C} = \mathcal{Z}$ gives a spanning time of at least $s^{1/2}$.

Otherwise suppose $k_r \geq s^{1/2}$ and $k_c \geq s^{1/2}$. Let a and b denote the number of columns longer than the k_c repeated columns and the number of rows longer than the k_r repeated rows, respectively. Define $\mathcal{Z}' = (\mathcal{Z}^{\leftarrow a})^{\downarrow b}$. Let L be a line with slope -1 that passes through the site $(m,n) \notin \mathcal{Z}'$ such that $(m-1, n-1) \in \mathcal{Z}'$ and $\mathcal{Z}'$ lies below L. Then $m \geq k_c$ and $n \geq k_r$. By Lemma 5.1, $M_{enhanced}(\mathcal{Z}') \geq 2\min\{k_r, k_c\} \geq s^{1/2}$. By Lemma 2.2, $M_{enhanced}(\mathcal{Z}') \leq M_{enhanced}(\mathcal{Z})$.

Therefore we may conclude that

$$M_{enhanced}(\mathcal{Z}) \geq s^{1/2}.$$

$\square$

Proof of Theorem 1.2. If $R_{s,s} \subseteq \mathcal{Z}$, then $R_{s-1,s-1} \subseteq \mathcal{Z}^{\swarrow 1}$. The theorem then follows from Lemma 2.4 and Proposition 5.2. $\square$

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