Representation of Degenerate Affine Hecke Algebra and its Properties

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Abstract. Inspired by Vershik and Okounkov's approach to the representations of symmetry group, this paper reconstructs part of the results in their study, explores the irreducible representation of degenerate affine Hecke algebra and classifies weights of X_i's. This paper also describes the features of valid weights from an inflated $H(n)$-module and how to put weights in a nice "form".

Contents

1 Degenerate Affine Hecke Algebra

Definition 1.1. The degenerate affine Hecke algebra $H(n)$ is generated by elements $s_1, s_2, \ldots, s_{n-1}$ and $X_1, X_2, \ldots, X_n$ subject to the relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \ for \ |i-j| > 1,$$

$$s_i X_i s_i = X_{i+1} - s_i, \quad X_i X_j = X_j X_i.$$

Proposition 1.2. *We have a natural injective map*

$$\imath : H(m) \otimes H(n) \hookrightarrow H(m+n)$$

$$s_i \otimes 1 \mapsto s_i, 1 \otimes s'_j \mapsto s_{m+j},$$

$$X_i \otimes 1 \mapsto X_i, 1 \otimes X'_j \mapsto X_{m+j},$$

where $s_i, X_i \in H(m)$ *and* $s'_j, X'_j \in H(n)$.

We also know the linear basis of $\mathbb{C}[X_1, \ldots, X_n] =: \mathbb{C}[X]$ is monomials denoted by $\{X^\beta \mid \beta \in \mathbb{N}^n\}$ and the basis of $\mathbb{C}[S_n]$ is $\{w \in S_n\}$.

1.1 Representations of $H(n)$

Definition 1.3. There exists an algebra homomorphism

$$ev : H(n) \twoheadrightarrow \mathbb{C}[S_n]$$

$$s_i \mapsto s_i$$

$$X_1 \mapsto 0$$

such that $ev(X_2) = s_1 ev(X_1) s_1 + s_1 = s_1$ and by induction

$$ev(X_{i+1}) = s_i ev(X_i) s_i + s_i.$$

Therefore, we have $ev(X_i) = \sum_{k=1}^{i-1} (k\ i)$ for $i > 1$. This is a surjective algebra homomorphism with $\mathrm{Ker}(ev) = H(n) X_1 H(n)$.

Via ev one can turn any S_n-module M into an $H(n)$-module, we call $\mathrm{Infl}_{S_n}^{H(n)}$. So any $H(n)$-module M such that $X_1 M = 0$ is isomorphic to an inflated S_n-module.

Let V^λ be an irreducible representation of S_{l+k} and V^μ be an irreducible representation of S_l. We have S_l embeding in S_{l+k} on its first l elements. Define $V^{\lambda/\mu}$=$\mathrm{Hom}_{\mathbb{C}[S_l]}(V^\mu, V^\lambda)$. Since $\mathrm{Infl}_{S_l}^{H(l)}$ is a $H(l)$ representation, we can get

$$V^{\lambda/\mu} = \mathrm{Hom}_{\mathbb{C}[S_l]}(\mathrm{Infl}_{S_l}^{H(l)} V^\mu, \mathrm{Res}_{H(l)}^{H(l+k)} \mathrm{Infl}_{S_{l+k}}^{H(l+k)} V^\lambda)$$

which is a $H(k)$-module.

Example 1.4. Suppose V^μ is an irreducible representation of S_1, and V^λ is an irreducible representation of S_3. Suppose $\mu = \square$ and $\lambda = \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ and then $V^{\lambda/\mu}$ is a representation of $H(2)$. Let μ be the trivial representation of S_1, and λ be the standard representation of S_3. We choose basis $\{v_1, v_2\}$ of V^λ such that

$s_1 : v_1 \leftrightarrow v_2$,
$s_2 : v_1 \to v_1,\ v_2 \to -(v_1 + v_2)$.
Then we have

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} s_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

By *evaluation* homomorphism, we have

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

acting on $\mathrm{Infl}\, V^\lambda$. Thus, by Proposition 1.2 we get a new representation of $H(2)$.

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} s_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Since $X_i's$ commute, they can be upper triangulated simultaneously. In this representation of $H(2)$, if we choose basis $\{v_1 + v_2, v_1 - v_2\}$, we will get

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Remark: This is an irreducible representation since $v_1 + v_2$ and $v_1 - v_2$ are not the eigenvector of s_1.

Example 1.5. Suppose M is some $H(n)$ module. Let $v \in M$ be an X-eigenvector with spectrum (a_i). In other words $span\{v\}$ is an X-submodule. We also have $span\{v, s_i v\}$ is an X-submodule.

Case 1: v is also an eigenvector of s_i with eigenvalue b_i. Then we have $b_i = \pm 1$ and $a_{i+1} = a_i \pm 1$ with $\dim(span\{v, s_i v\}) = 1$.

Case 2: $a_i = a_{i+1} = a$. Then $\dim(span\{v, s_i v\}) = 2$. X_i and X_{i+1} can not be diagonalized. With respect to the $\{v, s_i v\}$ basis,

$$X_i = \begin{pmatrix} a & -1 \\ 0 & a \end{pmatrix} X_{i+1} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

Case 3: $a_i \neq a_{i+1} \pm 1$ and $a_i \neq a_{i+1}$. $X_i' s$ can be diagonalized when we choose the basis of this submodule to be $\{v, v + (a_i - a_{i+1})s_i v\}$. Then the spectrum $s_i \cdot \underline{a} = (a_1, a_2 ..., a_{i+1}, a_i, a_{i+2}, ...)$ for the eigenvector $v + (a_i - a_{i+1})s_i v$.

Example 1.6. Let M be an irreducible representation of $H(n)$ such that $X_1 M = 0$ $(\star)$. Then M can be written as $\mathrm{Infl}(N)$ for some N as an S_n-module. We have the correspondence:

$$\text{irreducible representation of } \mathbb{C}[X] \longleftrightarrow (a_1, a_2, ..., a_n)$$

For $v \in M$, we have following restrictions on its weight: $a_1 = 0$. For a_2, we know that $a_2 = \pm 1$. If $a_2 = 0$, by Example 1.5 Case 2, we have

$$X_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $a_2 \neq 1$, then by Case 3, we can get the weight vector of $v - a_2 s_i v$ to be $(a_2, 0, ...)$. In both situations, $X_1 \neq 0$.

1.2 Induced Representations

Let B be a $\mathbb{C}$-algebra with a subalgebra A and suppose B_A is free as a right A-module of rank r. Let M be an A-module, with $\dim_{\mathbb{C}} M = m$. Then we define a (left) B-module:

$$B \otimes_A M$$

which is $\mathrm{Ind}_A^B M$. Observe $\dim_{\mathbb{C}}(B \otimes_A M) = r \cdot m$.

Theorem 1.7 (Frobenius Reciprocity)**.**

$$Hom_B(\mathrm{Ind}_A^B M, N) = Hom_B(B \otimes_A M, N) \cong_{\mathbb{C}-vector\ space} Hom_A(M, \mathrm{Res}_A^B N)$$

We apply to $B = H(n)$ and $A = \mathbb{C}[X_1, X_2, ..., X_n]$. Then rank of $B_A = n! = |S_n|$ because the basis of $H(n)$ is $\{wX^\alpha | w \in S_n, \alpha \in \mathbb{N}^n\}$ and the basis for $H(n)_{\mathbb{C}[\underline{X}]}$ is $\{w | w \in S_n\}$. Now we apply to $B = H(n_1) \otimes H(n_2)$ and $A = \mathbb{C}[X_1, ... X_n]$. Then the rank of B_A is $n_1! n_2!$.

Proposition 1.8. *In Proposition 1.2, we define an injective map* $\iota : H(n_1) \otimes H(n_2) \to H(n)$ *where* $n_1 + n_2 = n$*. Thus* $H(n)_{H(n_1) \otimes H(n_2)}$ *is free of rank* $\binom{n}{n_1}$.

Proof. Since $\mathbb{C}[\underline{X}], H(n_1) \otimes H(n_2)$ and $H(n)$ are all $\mathbb{C}$-vector spaces, we have

$$[H(n) : \mathbb{C}[\underline{X}]] = [H(n_1) \otimes H(n_2) : \mathbb{C}[\underline{X}]] \times [H(n) : H(n_1) \otimes H(n_2)]$$

Therefore, $H(n)_{H(n_1) \otimes H(n_2)}$ is free of rank $\frac{n!}{n_1! n_2!} = \binom{n}{n_1}$. □

Example 1.9. $H(3)_{H(2)\otimes H(1)}$ is free with basis $\{s_2, s_2 s_1, 1\}$ rank $= 3$.

Example 1.10. Pick 1-dimensional module of $\mathbb{C}[X_1, X_2]$ with a weight vector given by its weight (a, b). Denote this module by $(a, b) = span\{v_0\}$. Let $V = H(2) \otimes_{\mathbb{C}[X_1,X_2]} (a, b)$. $\dim V = 2 \times \dim(a, b) = 2$. Let $B = H(n)$ and ${}_BN$ be any *B-module*. By Frobenius Reciprocity,

$$\operatorname{Hom}_B(V, N) \cong \operatorname{Hom}_B(\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(a, b), N) \cong \operatorname{Hom}_{\mathbb{C}[\underline{X}]}((a, b), \operatorname{Res}_{\mathbb{C}[\underline{X}]} N).$$

Let N be the representation $V^{\lambda/\mu}$ in *Example* 1.4. $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N \cong (1, -1) \oplus (-1, 1)$; therefore, we have

$$\dim \operatorname{Hom}_{\mathbb{C}[\underline{X}]}((-1, 1), \operatorname{Res} N) = 1$$
$$\dim \operatorname{Hom}_{\mathbb{C}[\underline{X}]}((1, -1), \operatorname{Res} N) = 1$$
$$\dim \operatorname{Hom}_{H(2)}(\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(-1, 1), N) = 1$$

which means there exists a surjective nonzero $H(2)$-*homomorphism* such that $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(-1, 1) \twoheadrightarrow N$ since N is simple.

Example 1.11. Consider $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(1, -1) = H(2) \otimes_{\mathbb{C}[\underline{X}]} (1, -1)$ and denote $(1, -1) = span\{v_0\}$. Choose the basis $s_1 \otimes v_0$, $1 \otimes v_0$ for this left-module. By calculation, we get

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_1 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} X_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

which is irreducible. The eigenvectors of s_1 are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ but they are not preserved by X_1 and X_2. Therefore, $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(1, -1)$ is a simple $H(2)$-module.

Example 1.12. Now, consider induced representation $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(3, 4)$. Is this simple? We still choose the basis $s_1 \otimes v_0$, $1 \otimes v_0$. By calculation, we get

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_1 = \begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix} X_2 = \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$$

which is not irreducible because $\operatorname{span}((s_1 - 1) \otimes v_0)$ forms a $H(2)$-invariant subspace. $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(3, 4)$ is not a simple $H(2)$-module. If we choose new basis $\{(s_1 - 1) \otimes v_0\}, \{(s_1 + 1) \otimes v_0\}$ then we get matrices

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X_1 = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} X_2 = \begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix}$$

Thus, we have $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(3, 4) / \operatorname{span}\langle (s_1 - 1) \otimes v_0 \rangle \cong \operatorname{span}\langle 1 \otimes v_0 \rangle$.

Example 1.13. We consider induced representation $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(2)}(0, 0)$. Similar to previous calculations, we have

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which is irreducible.

Example 1.14. Let ${}_{H(3)}N$ be finite dimensional over $\mathbb{C}$ and suppose $N = \operatorname{Infl}_{S_3} M$ be a simple $H(3)$-module for some S_3-module M. Can we have X-weight of N which is $\underline{a} = (0, 1, 0)$? Since N is finite dimensional, we can choose $v \in N$ $v \neq \underline{0}$ to be a weight vector such that $X_i v = a_i v$. Therefore, we have $\operatorname{Hom}_{\mathbb{C}[\underline{X}]}((0, 1, 0), \operatorname{Res}_{\mathbb{C}[\underline{X}]} N) \neq 0$ and by Frobenius Reciprocity and previous statement, we have $\exists$ surjective nonzero $H(3)$-homomorphism f such that $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(3)}(0, 1, 0) \twoheadrightarrow N$. We choose $w \in \operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(3)}(0, 1, 0)$, such that $f(w) = v$. We have $f((X_3 - X_2)w) = (X_3 - X_2)f(w) = (X_3 - X_2)v = (s_2 s_1 s_2 + s_2 - s_1)v = -v$ by evaluation function. Similarly we have $f((X_2 - X_1)w) = (X_2 - X_1)v = s_1 v = v$, so $(s_2 s_1 s_2 + s_2)v = s_1 v - v = \underline{0}$. Then $s_1 s_2 s_2 s_1 s_2 v = -s_1 s_2 s_2 v = -v$, and we get $s_2 v = -v$. We have contradiction that $v = s_2 s_1 s_2 v = s_1 s_2 s_1 v = -v$.

1.3 General Approach

Suppose we have a ${}_{H(n)}N$ which is simple. Then there exists $\mathbb{C}[\underline{X}]$-module $(a_1, a_2, ..., a_n)$ and N is a quotient of $\operatorname{Ind}_{\mathbb{C}[\underline{X}]}^{H(n)}(a_1, ..., a_n)$.

Theorem 1.15. *If* $N = \operatorname{Infl}_{S_n} M$ *is irreducible and* M *is irreducible* $\mathbb{C}[S_n]$*-module, then* $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N$ *is semisimple.*

Proof. See Section 4. □

Example 1.16. Let $(a_1, a_2, ...a_n)$ denote 1-dimension $\mathbb{C}[X_1, ..., X_n]$-*module* and $[a_1, a_2, ...a_n]$ denote 1-dimension $H(n)$-*module* if $a_i = a_{i+1} + 1$ or $a_i = a_{i+1} - 1$ for all i. In module $[a_1, a_2, ...a_n]$, s_i act as 1 when $a_{i+1} = a_i + 1$ and s_i act as -1 when $a_{i+1} = a_i - 1$. Refer to example 1.12, let n=2 and consider $\operatorname{Ind}^{H(2)}(3, 4)$ which is not simple. We have exact sequence

$$0 \to [4, 3] \to \operatorname{Ind}(3, 4) \to [3, 4] \to 0$$

Does it split?
We know $\operatorname{Res}_{\mathbb{C}[X]} \operatorname{Ind}(3, 4) \cong (4, 3) \oplus (3, 4)$. Suppose $\operatorname{Ind}(3, 4) \cong A \oplus B$ and consider $\operatorname{Res} A$, $\operatorname{Res} B$. We have

$$\operatorname{Ind}(3, 4) \twoheadrightarrow A \text{ and } \operatorname{Ind}(3, 4) \twoheadrightarrow B$$

implies

$$(3, 4) \to \operatorname{Res} A \text{ and } (3, 4) \to \operatorname{Res} B.$$

Then we have $\mathbb{C}[\underline{X}]$-*module* isomorphisms,

$$(3, 4) \oplus (4, 3) \cong \operatorname{Res}_{\mathbb{C}[\underline{X}]} \operatorname{Ind}(3, 4) \cong \operatorname{Res}_{\mathbb{C}[\underline{X}]} A \oplus \operatorname{Res}_{\mathbb{C}[\underline{X}]} B$$

Contradiction! So $\operatorname{Ind}(3, 4)$ is not semi-simple.

Theorem 1.17. *Suppose $a_i \neq a_j$ whenever $i \neq j$. As $\mathbb{C}[\underline{X}]$-module we have*

$$\mathrm{Res}_{\mathbb{C}[\underline{X}]} \mathrm{Ind}_{\mathbb{C}[\underline{X}]}^{H(n)}(a_1, a_2, ..., a_n) = \oplus_{\sigma \in S(n)}(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)})$$

We need a lemma first.

Lemma 1.18. *Let B be a commutative finite dimensional $\mathbb{C}$-algebra.*

(1) Any simple B-module V is 1-dimensional.

(2) If the character of V is $\chi_1 + \chi_2 + \cdots + \chi_d$ with all $\chi_i \neq \chi_j$ where χ_i is the character of simple B-module V_i, then $V \cong \oplus V_i$.

(3) If we have a filtration $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ of B-submodules with each V_{j+1}/V_j simple and $V_{j+1}/V_j \neq V_{k+1}/V_k$ as B-module when $j \neq k$. Then $V \cong \oplus_j(V_{j+1}/V_j)$.

Proof of Theorem 1.17. We choose the $\mathbb{C}$ basis of $V = \mathrm{Res}_{\mathbb{C}[\underline{X}]} \mathrm{Ind}_{\mathbb{C}[\underline{X}]}^{H(n)}(a_1, a_2, ..., a_n)$ to be $\{\sigma \otimes v | \sigma \in S_n\}$. We can upper-triangularize all operators $X_i's$ with diagonal entries $\{(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, ...a_{\sigma^{-1}(n)}) | \sigma \in S_n\}$, if we order the basis by length of elements (length is defined in Definition 2.4) and denote it by $\{e_1, e_2, ...e_{n!}\}$. Let B be the subalgebra of $\mathbb{M}_{n!}(\mathbb{C})$ generated by matrices of $X_i's$ in this basis. Then we have the filtration

$$0 \subset span\{e_1\} \subset span\{e_1, e_2\} \cdots \subset V$$

Denote $span\{e_1, e_2, ...e_i\} = V_i$. We have $V_{j+1}/V_j \neq V_{k+1}/V_k$ when $j \neq k$ since $a_i's$ are distinct. Furthermore, V_i/V_{i-1} is a 1-dimensional submodule spanned by a weight vector with ith weight. Thus, by the previous lemma, we get

$$\mathrm{Res}_{\mathbb{C}[\underline{X}]} \mathrm{Ind}_{\mathbb{C}[\underline{X}]}^{H(n)}(a_1, a_2, ..., a_n) = \oplus_{\sigma \in S(n)}(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)})$$

□

Definition 1.19. We define an operator called "*inter twiner*" in $H(n)$ by $\varphi_i = s_i X_i - X_i s_i = s_i(X_i - X_{i+1}) + 1$

Example 1.20. If v is a weight vector of $H(2)$ with weight (a_1, a_2), then $\varphi_1 \otimes v = (s_1(X_1 - X_2) + 1) \otimes v = (a_1 - a_2)s_1 \otimes v + 1 \otimes v$ which potentially gives us another weight vector by Example 1.5, Case 3.

Proposition 1.21. *Inter twiner operator has following properties:*
(1) $\varphi_i \varphi_j = \varphi_j \varphi_i$ if $|i - j| > 1$
(2) $\varphi_i \varphi_j \varphi_i = \varphi_j \varphi_i \varphi_j$ if $i = j \pm 1$

(3) $\varphi_i^2 = (X_i - X_{i+1} + 1)(-X_i + X_{i+1} + 1) \in \mathbb{C}[X_i, X_{i+1}]$
(4) $X_k \varphi_i = \varphi_i X_{s_i(k)}$
(5) Let $w \in S_n$*. Because properties (1) and (2),* φ_w *makes sense.*
(6) $\varphi_w = w(\textit{polynomials of X's}) + \sum_{l(u)<l(w)} u(\textit{polynomials of X's})$

Consider algebras $C \supseteq B \supseteq A$ and a left *A-module* M. Then we have transitivity of induction:

$$C \otimes_A M \cong C \otimes_B B \otimes_A M = \operatorname{Ind}_B^C \operatorname{Ind}_A^B M \cong \operatorname{Ind}_A^C M$$

In our setting, B_A is free as *A-module*, C_B is free as *B-module* and C_A is free as *A-module*. Free modules are $flat$ which means functor $B_A \otimes_A -$ is exact. For example Ind_A^B and Ind_B^C are exact functors. Thus,

Lemma 1.22. *Given a short exact sequence of B-module*

$$0 \to L \xrightarrow{\iota} M \to N \to 0$$

then, the sequence

$$0 \to \operatorname{Ind}_B^C L \to \operatorname{Ind}_B^C M \to \operatorname{Ind}_B^C N \to 0$$

is exact.

Proof. We just need to show a $flat$ module is exact. Due to the freeness, we can write $C \cong \oplus_i B$. Thus, the second sequence becomes

$$0 \to \oplus_i (B \otimes L) \xrightarrow{\oplus \iota} \oplus_i (B \otimes M) \to \oplus_i (B \otimes N) \to 0$$

$$0 \to \oplus_i L \xrightarrow{\oplus \iota} \oplus_i M \to \oplus_i N \to 0$$

Clearly, $\oplus_i \iota$ is injective since ι is injective. Thus, we get the exactness. □

Theorem 1.23 (Jordan-Holder)**.** *Let* R *be a ring with unity and* M *be a left R-module. Suppose we have two filtrations* $0 = M_0 \subset M_1 \subset M_2 \ldots M_r = M$ *and* $0 = L_0 \subset L_1 \subset L_2 \ldots L_k = M$ *such that* M_{i+1}/M_i *and* L_{j+1}/L_j *are simple R-modules for all i, j. Then we have r=k and the list of quotients* $(M_1/M_0, M_2/M_1, M_3/M_2 \ldots M_r/M_{r-1})$ *is a rearrangement of the list* $(L_1/L_0, L_2/L_1, L_3/L_2 \ldots L_k/L_{k-1})$*.*

Corollary 1.24. *Suppose we have short exact sequence of R-module*

$$0 \to L \to M \to N \to 0$$

such that L, N are simple modules and $L \ncong N$*. Then the only possible quotient modules of M are L or N and the only possible submodules of M are L or N.*

Proof. The proof follows from Jordan-Holder Theorem. □

Suppose $n = \sum_{i=1}^{l} a_i$ and ${}^j x_i \in \mathbb{Z}$ for all i, j. We define the notation

$$\mathrm{Ind}^n_{a_1,a_2,\dots,a_l}[{}^1x_1, {}^1x_2, \dots {}^1x_{a_1}] \boxtimes [{}^2x_1, \dots, {}^2x_{a_2}] \boxtimes \cdots \boxtimes [{}^lx_1, \dots, {}^lx_{a_l}]$$

representing the induced representation from the representation of $H(a_1) \otimes H(a_2) \otimes \cdots \otimes H(a_l)$ to $H(n)$.

Example 1.25. Given

$$0 \to [4,3] \to \mathrm{Ind}(3,4) \to [3,4] \to 0$$

We have

$$0 \to \mathrm{Ind}^3_{1,2}(2) \boxtimes [4,3] \to \mathrm{Ind}^3_{1,1,1}(2) \boxtimes (3,4) \to \mathrm{Ind}^3_{1,2}(2) \boxtimes [3,4] \to 0$$

where subscripts of Ind represent the tensor product of $H(i)$ and
$\mathrm{Ind}^3_{1,1,1}(2) \boxtimes (3,4) = \mathrm{Ind}^3_{1,2}(2) \boxtimes (\mathrm{Ind}^2_{1,1}(3,4)) = \mathrm{Ind}^3_{1,1,1}(2,3,4)$ by transitivity.

Example 1.26. Let's think about Ind(0, 1, 0)

$$0 \to [1,0] \to \mathrm{Ind}(0,1) \to [0,1] \to 0$$

$$0 \to \mathrm{Ind}^3_{2,1}[1,0] \boxtimes (0) \to \mathrm{Ind}^3_{1,1,1}[0,1] \boxtimes (0) \to \mathrm{Ind}^3_{2,1}[0,1] \boxtimes (0) \to 0$$

Choose $\mathbb{C}$-basis of $H(3)$ over $H(2) \otimes H(1)$ to be $\{1, s_2, s_1s_2\}$ In this basis, we get a 3 dimensional representation of $H(3)$ on module $\mathrm{Ind}^3_{2,1}[0,1] \boxtimes (0)$

$$X_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} X_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} X_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} s_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, X-weight is (0,1,0) and a generalized weight (0,0,1).
Similarly, we also get a 3 dimensional representation of $H(3)$ on module $\mathrm{Ind}^3_{2,1}[1,0] \boxtimes (0)$

$$X_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} X_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$s_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} s_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $L = \mathrm{Ind}^3_{2,1}[1,0] \boxtimes (0)$ and $N = \mathrm{Ind}^3_{2,1}[0,1] \boxtimes (0)$. L or N are not inflations. If $S \subsetneq N$ is an $H(3)$-*submodule*, in particular $\mathrm{Res}_{\mathbb{C}[\underline{X}]} S \subseteq \mathrm{Res}_{\mathbb{C}[\underline{X}]} N$ is a $\mathbb{C}[\underline{X}]$-*submodule*. We know N has a basis of generalized X-weight vectors and S does too.
Analyze $\mathrm{Res}_{\mathbb{C}[\underline{X}]} S$:

Case 1: S has (0,1,0) weight vector which must be $1 \otimes v$, but $H(3) \cdot (1 \otimes v) = N$.
Case 2: S must have (0,0,1) weight vector w and $w \in \text{span}\{1 \otimes v, s_2 \otimes v\}$ with nonzero coefficient of $s_2 \otimes v$, but not s_1 invariant. We know $w = 1 \otimes v + s_2 \otimes v$ which is a weight vector with weight $(0,0,1)$. Therefore, $s_1 w = 1 \otimes v - s_1 s_2 \otimes v$ and $X_1 s_1 w = (s_1 s_2 X_3 - s_1 - s_2) \otimes v = -1 \otimes v - s_2 \otimes v$. Then we have $1 \otimes v \in S$ so N is irreducible and not inflated.
Similarly, Let $T \subsetneq L$.
Analyze $\text{Res}_{\mathbb{C}[\underline{X}]} T$:
Case 1: T has (0,1,0) weight vector which must be $w = (1 \otimes v - s_2 \otimes v - s_1 s_2 \otimes v)$, but $w - s_1 w = 1 \otimes v - s_1 \otimes v = 2 \otimes v$. So $H(3) \cdot (w) = L$.
Case 2: T must have (1,0,0) weight vector or generalized weight vector $w \in \text{span}\{1 \otimes v\}$, but $H(3) \cdot (1 \otimes v) = L$. Thus, L is also an irreducible representation and not an inflation.

Suppose Q is an irreducible $H(3)$-*module* such that $(0,1,0) \subseteq \text{Res}\, Q$, then $\text{Ind}^3_{1,1,1}(0,1,0) \twoheadrightarrow Q$ by Frobenius Reciprocity. Thus $Q \cong L$ or $Q \cong N$ and $Q \neq \text{Infl}^{H(3)}_{S_3}$ (any module) .

Example 1.27. Recall that if M is simple $H(5)$-*module* such that $\text{Res}_{\mathbb{C}[\underline{X}]} M \supseteq (0,3,1,4,6) = v$, could $M = \text{Infl}_{S_5}$ (any module)? No! Because $\varphi_1 v$ is weight vector of weight $(3,0,1,4,6)$ and $\varphi_1 v \neq 0$, $X_1 \varphi_1 v = 3\varphi_1 v \neq 0$. If $\varphi_1 v = 0$ then $\varphi_1^2 v = (X_1 - X_2 + 1)(-X_1 + X_2 + 1)v = -8v = 0$.

So what if $\text{Res}_{\mathbb{C}[\underline{X}]} M \supseteq (0,1,0,2,1,4,6)$ where M is a $H(7)$ simple module? First consider $N = \text{Res}^{H(7)}_{H(3)\otimes H(4)} M$ and $\text{Res}_{\mathbb{C}[\underline{X}]} N \supseteq (0,1,0) \boxtimes (2,1,4,6)$. Thus, by previous example, N will contain either $[1,0,0] \boxtimes [\dots]$ or $[0,0,1] \boxtimes [\dots]$. Neither of them can appear in an inflated module. Then M cannot be an inflation.

Proposition 1.28. *By the analogue of Example 1.26 and Corollary 1.24, we can conclude that whenever we have an irreducible H(3)-module* $\mathbb{M}$ *such that* $(a, a+1, a) \in \text{Res}\, \mathbb{M}$*, we have either* $(a,a,a+1)$ *or* $(a+1,a,a)$ *as a weight in* $\mathbb{M}$*. Additionally, whenever we have an irreducible H(3)-module* $\mathbb{L}$ *such that* $(a,a,a+1) \in \text{Res}\, \mathbb{L}$*, we have a map* $\text{Ind}(a,a,a+1) \twoheadrightarrow \mathbb{L}$ *and short exact sequence,*

$$0 \to \text{Ind}^3_{1,2}(a) \boxtimes [a+1,a] \to \text{Ind}(a) \boxtimes (a,a+1) \to \text{Ind}^3_{1,2}(a) \boxtimes [a,a+1] \to 0$$

the left term has generalized weights $(a,a+1,a)$*,* $(a+1,a,a)$ *and* $(a+1,a,a)$ *which is definitely not* $\mathbb{L}$*. The right term which is* $\mathbb{L}$ *has weight* $(a,a,a+1)$*,* $(a,a,a+1)$ *and* $(a,a+1,a)$*. Therefore,* $\mathbb{L}$ *contains weight* $(a,a+1,a)$*. Thus we have diagram to illustrate:*

$$(a,a,a+1) \longrightarrow (a,a+1,a) \longleftarrow (a+1,a,a),$$

$$(a,a+1,a) \longrightarrow (a,a,a+1) \text{ or } (a,a+1,a) \longrightarrow (a+1,a,a).$$

1.4 Theory of Weights

Definition 1.29. Denote by $a_k : H(n) \to H(n)$ the automorphism of $H(n)$ such that:

$$s_i \mapsto s_i$$
$$X_i \mapsto X_i + k$$

for all possible i.

Definition 1.30. Denote by $\epsilon : H(n) \to H(n)$ the automorphism of $H(n)$ such that:

$$X_i \mapsto -X_i$$
$$s_i \mapsto -s_i$$

for all possible i. Thus, $\epsilon(w) = (-1)^{l(w)} w$

Definition 1.31. Denote by rev : $H(n) \to H(n)$ the automorphism of $H(n)$ such that:

$$\begin{aligned} X_1 &\mapsto X_n \\ X_2 &\mapsto X_{n-1} \\ &\vdots \\ X_n &\mapsto X_1 \\ s_1 &\mapsto -s_{n-1} \\ s_2 &\mapsto -s_{n-2} \\ &\vdots \\ s_{n-1} &\mapsto -s_1 \end{aligned}$$

If ${}_{H(n)}M$ is a left $H(n)$-module, then we can define $H(n)$-module M^{a_k} by $\forall h \in H(n), m \in M$, then $h \cdot m = a_k(h)m$. We can define M^ϵ, M^{rev} in the same manner.

With these automorphisms we have

$$\begin{aligned} (\mathrm{Ind}(a,b,c))^{rev} &\cong \mathrm{Ind}(c,b,a) \\ (\mathrm{Ind}(a,b,c))^{\epsilon} &\cong \mathrm{Ind}(-a,-b-c) \\ (\mathrm{Ind}(a,b,c))^{a_k} &\cong \mathrm{Ind}(a+k,b+k,c+k) \end{aligned}$$

Proposition 1.32. *Same set up as Proposition 1.28, we can switch elements in the weight tuple as follows:*

① $(a, a \pm 1, a) \to (a, a, a \pm 1)$ *or* $(a \pm 1, a, a)$.

② $(a, a, a \pm 1) \to (a, a \pm 1, a)$.

③ $(a \pm 1, a, a) \to (a, a \pm 1, a)$.

④ $(a, a \pm 1)$ *are stuck in the absence of structure* ①, ②, ③.

⑤ $(a, b) \rightleftharpoons (b, a)$ *if* $a \neq b \pm 1$.

Lemma 1.33. *If weight* $\alpha = (a_1, a_2, \dots, a_n)$ *with* $a_{i+1} = a_i$ *for some* i*, then* α *can not be a weight of inflated module.*

Proof. If $a_i = a_{i+1} = a$, then any module M with $(\dots, a, a, \dots) \in \operatorname{Res} M$ is not semisimple since X_i and X_{i+1} are not diagonalizable by Example 1.5 Case 2. Thus, by Theorem 1.15, α can not be a weight of an inflated module. □

Theorem 1.34. *Suppose* M *is an inflated* $H(n)$*-module with* n*-tuple* $\underline{\alpha}$ *as a weight of* $\operatorname{Res} M$*. Then* $\underline{\alpha} = (a_1, a_2, \dots, a_n)$ *must satisfy the following properties:*

i. : $a_1 = 0$,

ii. : $\{a_i - 1, a_i + 1\} \cap \{a_1, a_2, \dots, a_{i-1}\} \neq \emptyset \ \forall i$,

iii. : *if* $a_p = a_q = a$ *for some* $p < q$ *then*

$$\{a - 1, a + 1\} \subset \{a_{p+1}, \dots, a_{q-1}\}.$$

Proof. If M is an inflated module, then $X_1 M = 0$ and $a_0 = 0$. If $\{a_i - 1, a_i + 1\} \cap \{a_1, a_2, \dots, a_{i-1}\} = \emptyset$ for some i, then we can move a_i to the first place of the tuple by operatoion ⑤ in previous proposition which gives us that M is not an inflated module. If (iii) fails, then pick p, q with the minimal $q - p$ for which this happens. If $\{a - 1, a + 1\} \cap \{a_{p+1}, \dots, a_{q-1}\} = \emptyset$, then we can get weight $(\dots, a, a, \dots)$ by ⑤ which is not valid. Without loss of generality, if $\{a + 1\} \in \{a_{p+1}, \dots, a_{q-1}\}$ but $\{a - 1\} \notin \{a_{p+1}, \dots, a_{q-1}\}$, then there is only one $a + 1$ between two a's, otherwise there must be another a between two $a + 1$'s. This will contradict the minimality of $q - p$. Thus, we will get $(\dots, a, a + 1, a, \dots)$ by operation ⑤ and get $(\dots, a, a, \dots)$ by operation ① which is not valid. □

Definition 1.35. For an integer tuple which satisfies three conditions in Theorem 1.34, we call this tuple "valid weight".

Lemma 1.36. *Operation* ⑤ *in Proposition 1.32 does not change the validity of weight.*

Proof. We can only do operation ⑤ when $a_i \neq a_{i+1}$, $a_{i+1} \pm 1$. Thus, it will not interfere with validity since we only care about the position of $a_i \pm 1$ in the tuple. □

Definition 1.37. We say a weight is in a *nice form* if it has the form of $(r+1)$-tuple $(g_0, g_{-1}, ...g_{-r})$ where each g_{-k} represents a tuple $(-k, -k+1, -k+2\dots)$ and $|g_{-k}| \geq |g_{-k-1}|$. $|g_{-k}|$ means length of tuple.

Example 1.38. $(0, 1, 2, 3, 4, -1, 0, 1, 2, -2, -3)$ is an example of nice form. $(0, 1, 2, -1, 0, 1, 2)$ is not an example of nice form.

Lemma 1.39. *Let $\alpha = (a_0, a_1, \dots)$ be a valid weight. If $a > 0$, then the leftmost entry with value $a+1$ can not appear on the left of the leftmost entry with value a.*

Proof. Suppose the lemma is false. Then out of all such pairs $\{(a_i, a_j)|a_i = a_j + 1\}$ where $i < j$, $a_j > 0$ and $a_k \neq a_j$ for all $k < j$, pick i minimal. Let $a = a_j$. If $a_k \neq a+2$ for $k < i$, then we can move a_i to first place by operation ⑤. If $a_k = a+2$ for some $k < i$, then the pair $a_k = a_i + 1$, $k < i$ contradicts the minimality of i, unless there is an $a + 1 = a_r$ left to $a + 2$, i.e. $r < k$. But as $r < i$, this contradicts the minimality of i. Hence the lemma holds. □

If $a < 0$, we have a similar lemma.

Lemma 1.40. *Let $\alpha = (a_0, a_1, \dots)$ be a valid weight. If $a < 0$, then the leftmost entry with value $a-1$ can not appear on the left of the leftmost entry with value a.*

Theorem 1.41. *Every valid weight can be transformed to a nice form using Proposition 1.32 operations.*

Proof. Consider a valid weight $\alpha = (a_0, a_1, \dots)$. Observation: if $a > 0$, there cannot be two $a-1$ appear prior to the leftmost a in the weight, and if $a < 0$, there cannot be two $a+1$ appear prior to the leftmost a in the weight by criterion (iii). If $a_i = 1$ for some $i > 0$, then we can apply operation ⑤ to α repeatedly to the leftmost occurrence of 1 to create a new valid weight α' with $a'_0 = 0, a'_1 = 1$. We can do this as previous lemmas and observation ensure there are no $2's$ to the left of this 1 in α and there are no $0's$ between a_0 and first 1. We can apply similar reasoning to the leftmost 2 in α' to bring it to the third position and so on, until we reach the largest value $b_0 := \max_i\{a_i\}$. The new valid weight α'' has first $b_0 + 1$ positions determined $\alpha'' = (0, 1, 2, \dots, b, \dots, ...)$. Denote $g_0 = (0, 1, \dots, b)$ and observe $|g_0| = b + 1$. Now we claim $a''_{b+1} = -1$ if $|g_0| < |\alpha''|$. First if $a''_{b+1} \geq 0$, then $a''_{b+2} = a''_i$ for some $0 \leq i \leq b$ by maximality of b. But by (iii), $a''_{b+1} - 1$ occurs in j^{th} position for some $i < j < b+1$ which cannot happen since the sequence g_0 is increasing. If $\alpha''_{b+1} < -1$ then repeat ⑤ brings α''_{b+1} to the first position contradicting validity. Now we follow the same process until reach the largest number in $\alpha \setminus g_0$ and denote it as g_{-1}. We will have $|g_0| \geq |g_{-1}|$, otherwise $b \in g_{-1}$ which contradicts the maximality of b. Then we can group rest of elements in the same manner and have $|g_{-k}| \leq |g_{-k-1}|$. □

Definition 1.42. We define a second nice form of weight such that the n-tuple is in form of $(g_0, g_1, ...g_r)$ where each g_k represents a tuple $(k, k-1, k-2\dots)$ and $|g_k| \geq |g_{k+1}|$.

Example 1.43. Same example as Example 1.38 but in second nice form: $(0, -1, -2, -3, 1, 0, 2, 1, 3, 2, 4)$.

2 Center of $H(n)$

We can also regard degenerate affine Hecke algebra as a vector space by $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[S_n]$. Therefore, the basis of $H(n)$ is $\{X^\alpha w | \alpha \in \mathbb{N}^n, w \in S_n\}$.

Lemma 2.1.

$$s_i X_i^a = X_{i+1}^a s_i - \sum_{k=0}^{a-1} X_i^k X_{i+1}^{a-1-k}$$

and

$$s_i X_{i+1}^a = X_i^a s_i + \sum_{k=0}^{a-1} X_i^k X_{i+1}^{a-1-k}$$

for $a \geq 1$.

Proof. $s_i X_i^a = (s_i X_i) X_i^{a-1} = (X_{i+1} s_i - 1) X_i^{a-1} = X_{i+1}(s_i X_i) X_i^{a-2} - X_i^{a-1} = X_{i+1}(X_{i+1} s_i - 1) X_i^{a-2} - X_i^{a-1} = X_{i+1}^2 s_i X_i^{a-2} - X_{i+1} X_i^{a-2} - X_i^{a-1}$.
Continue the same process we will get $s_i X_i^a = X_{i+1}^a s_i - \sum_{k=0}^{a-1} X_i^k X_{i+1}^{a-1-k}$.
Similarly, we will also get $s_i X_{i+1}^a = X_i^a s_i + \sum_{k=0}^{a-1} X_i^k X_{i+1}^{a-1-k}$. □

Lemma 2.2. $s_i X_i^a X_{i+1}^a = X_i^a X_{i+1}^a s_i$, *where* $a, b \geq 1$.

Proof. We compute a more general case. Suppose $A = X_i^a X_{i+1}^b s_i - s_i X_i^a X_{i+1}^b$, $X_{i+1}^b X_i^a s_i = X_{i+1}^b s_i X_{i+1}^a - C$ with $C = X_{i+1}^b X_i^{a-1} + X_{i+1}^{b+1} X_i^{a-2} + ... X_{i+1}^{a+b-1}$ and $s_i X_i^a X_{i+1}^b = X_{i+1}^a s_i X_{i+1}^b - C$. Therefore, $A = X_{i+1}^b s_i X_{i+1}^a - X_{i+1}^a s_i X_{i+1}^b$. We get $A = 0$ when $a = b$. □

Lemma 2.3. *Suppose* $f \in \mathbb{C}[X]$, *then* $s_i f = (s_i \circ f) s_i + \frac{f - s_i \circ f}{X_{i+1} - X_i}$.

Proof. Choose an element $v = X_1^{a_1} X_2^{a_2} ... X_n^{a_n}$ from the basis of $\mathbb{C}[X]$.

Suppose $a_i > a_{i+1}$, then

$$\begin{aligned} s_i v &= s_i X_1^{a_1} X_2^{a_2} ... X_i^{a_i - a_{i+1}} (X_i X_{i+1})^{a_{i+1}} ... X_n^{a_n} \\ &= X_1^{a_1} X_2^{a_2} ... (X_{i+1}^{a_i - a_{i+1}} s_i - \sum_0^{a_i - a_{i+1} - 1} X_i^k X_{i+1}^{a_i - a_{i+1} - 1 - k})(X_i X_{i+1})^{a_{i+1}} ... X_n^{a_n} \\ &= (... X_{i+1}^{a_i} X_i^{a_{i+1}} ...) s_i + \sum_0^{a_i - a_{i+1} - 1} (... X_i^{k + a_i} X_{i+1}^{a_i - k - 1} ...) \\ &= (s_i \circ f) s_i + \frac{f - s_i \circ f}{X_{i+1} - X_i} \end{aligned}$$

Similarly for $a_i < a_{i+1}$. □

Definition 2.4. *Length* of an element from S_n is

$$\begin{aligned} l : Sn &\to \mathbb{Z} \\ w &\mapsto l(w) \end{aligned}$$

defined by

$$\begin{aligned} l(w) &= \#\{(i, j) \in [n] \times [n] \mid i < j \text{ and } w(i) > w(j)\} \\ &= \#\{ \text{ inversion of } w\} \\ &= \#Inv(w) \end{aligned}$$

Theorem 2.5. *The center of $H(n)$ is $\mathbb{C}[X_1, X_2, ... X_n]^{S_n}$*

Proof. By Lemma 2.3, it is clear that all symmetric functions are in the center, since $f \in \mathbb{C}[X_1, X_2, ... X_n]^{S_n}$ commute with all generators of $H(n)$. Conversely, let $f = \sum_{w \in S_n} f_w w$ be an element in center of $H(n)$ such that exists nontrivial $w \in S_n$ with $f_w \neq 0$. We choose $u \in S_n$ with maxim $l(u)$ and $f_u \neq 0$. Also choose i such that $u(i) \neq i$. Then,

$$\begin{aligned} X_i f &= \sum_{l(w) \leq l(u)} X_i f_w w \\ &= X_i f_u u + \sum_{l(w) \leq l(u), w \neq u} X_i f_w w \end{aligned}$$

which has u coefficient $X_i f_u$. While,

$$
\begin{aligned}
fX_i &= \sum_{w \in S_n} f_w w X_i \\
&= f_u u X_i + \sum_{l(w) \le l(u), w \ne u} f_w w X_i \\
&= f_u X_{u(i)} u + \sum_{l(w) \le l(u), w \ne u} f'_w w
\end{aligned}
$$

Therefore, $X_i f_u = X_{u(i)} f_u$. Then $u(i) = i$, Contradiction. We have $f \in \mathbb{C}[X]$. By Lemma 2.3, we have $s_i f = f s_i = (s_i \circ f) s_i + \frac{f - s_i \circ f}{X_{i+1} - X_i}$. Therefore, we have $f = s_i \circ f$ for $\forall i$ □

Proposition 2.6. *We have homomorphism $\varphi : Z(H(n)) \to \mathbb{C}$ by $\varphi(f) = \mathrm{Tr}(\rho(f))/n$ where ρ is a n dimensional representation.*

Proof. The center of $\mathbb{M}_{k \times k}$ is $c\mathbb{I}$. □

3 Central Character

V^λ is a S_n-module and $\mathrm{Infl}_{S(n)}^{H(n)} V^\lambda$ is a $H(n)$-module. We want to know how $Z(H(n))$ acts on $H(n)$-module.
$n = 3$

Example 3.1. $V^{(3)} = V^{\square\square\square}$. This is a trivial representation; therefore, $X_1 - 0 = 0$, $X_2 - 1 = 0$, $X_3 - 2 = 0$. Then $f \in Z(H(3)) = \mathbb{C}[x]^{S_3}$ acts as scalar $f(0, 1, 2)$ on $V^{\square\square\square}$ or $f(X_1, X_2, X_3) - f(0, 1, 2) = 0$.

Example 3.2. $V^{(2\ 1)} = V^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}$, we have $X_1 + X_2 + X_3 = 0\mathbb{I}$, $X_1 X_2 X_3 = 0\mathbb{I}$, $X_1 X_2 + X_2 X_3 + X_1 X_3 = -\mathbb{I}$. Therefore, $f \in Z(H(3))$ acts on $V^{(2\ 1)}$ as $f(0, 1, -1)$

Definition 3.3. Given $\underline{a} \in \mathbb{C}^n$, we associate the *central character*
$\chi_{\underline{a}} : Z(H(n)) \to \mathbb{C}$, $f(X_1, X_2 .... X_n) \mapsto f(a_1, a_2, ... a_n)$.

Define an equivalence relation: $\underline{a} \sim \underline{b}$ if $\underline{a} = \sigma \underline{b}$ for $\sigma \in S_n$.

Because $X_i's$ commute, there exists common eigenvector v_0 such that $X_i v_0 = a_i v_0$. For example, in *Example* 1.4, $v_0 = v_1 + v_2$ we have the computation that

$$X_1(v_1 + v_2) = 0(v_1 + v_2)$$

$$X_2(v_1+v_2) = 1(v_1+v_2)$$
$$X_3(v_1+v_2) = -1(v_1+v_2).$$

Therefore, $\chi(f) = f(0,1,-1)$ for $f \in Z(H(n))$.

Theorem 3.4. *There exists unique $\underline{a} \in \mathbb{C}^n$ up to equivalence such that $\forall f \in Z(H(n))$ we have $\chi(f) = f(\underline{a})$ which are determined by $\chi(e_i) = e_i(\underline{a})$.*

Proof. Suppose that $g(t) = \sum_{k=0}^{n} b_k t^{n-k}$ where $b_0 = 1$ and $b_i = \chi(e_i)$. We can also write $g(t) = \prod_{k=1}^{n}(t - x_k) = t^n + e_1(\underline{x})t^{n-1}+, ..., +e_n(\underline{x})$ where $x_k's$ are the roots of this polynomial. Then we have $\underline{a} = (x_1, x_2, ..., x_n)$. □

Corollary 3.5. *If N is an irreducible $H(n)$-module, $v \in N$ such that $X_i v = a_i v$ for all i. If $w \in N$ with $X_i w = b_i w$ for all i, then $(\underline{a}) = \sigma(\underline{b})$ for some $\sigma \in S_n$.*

Proof. $e_k(\underline{X})v = e_k(\underline{a})v$. Since $e_k(\underline{X})$ is in the center of $H(n)$, we have $e_k(\underline{X})w = e_k(\underline{a})w = e_k(\underline{b})w$. Therefore, $e_k(\underline{a}) = e_k(\underline{b})$ for all $k = 1, 2, 3, ..., n$, and $(\underline{a}) = \sigma(\underline{b})$ for some $\sigma \in S_n$. □

4 Proof of Theorem 1.15

Statement of Theorem 1.15 If $N = \operatorname{Infl}_{S(n)} M$ is irreducible and M is irreducible S_n module, then $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N$ is semisimple.

Theorem 4.1 (Maschke)**.** *If char$(\mathbb{K})$ does not divide the order of G, then $\mathbb{K}[G]$ is semisimple. In particular, $\mathbb{C}[G]$ is semisimple.*

In particular,

$$\operatorname{Res}_{S_{n-1}}^{S_n} M = \bigoplus_{1}^{r} L_i,$$

where L_i are simple S_{n-1} modules.
Combining with inflation, we have

$$\operatorname{Infl}(\bigoplus L_i) = \operatorname{Infl}(\operatorname{Res}_{S_{n-1}}^{S_n} M) = \operatorname{Res}_{n-1}^{n} N = \operatorname{Res}_{n-1}^{n-1,1} \operatorname{Res}_{n-1,1}^{n} N$$

Proof of Theorem 1.15. We interpret $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N$ as $\operatorname{Res}_{1,...,1}^{n} N$ on which we can do the restriction recursively. By proposition 2.6, we know $c_n = \sum X_i \in Z(H(n))$ acts on N

by some scalar c. Consider $\operatorname{Res}^n_{n-1,1} N$, c_n still acts as $c\mathbb{I}$ and $c_n = c_{n-1} + X_n$ where $c_{n-1} \in Z(H(n-1)), X_n \in Z(H(1))$, therefore, c_{n-1} acts as scalars c_i on $\operatorname{Infl} L_i$ and X_n will act as scalar $c - c_i$ on $\operatorname{Infl} L_i$. If we keep doing the restriction, we will get $M = \bigoplus_T V_T$ where V_T is irreducible S_1-modules and sum is over all possible chains. We also know that all c_j will act diagonally on S_j-simples if basis is compatible with restriction and same for $X_j = c_j - c_{j-1}$. Thus, for each T, we will get a X_i-weight $(a_1, a_2, \ldots, a_n)_T$ on $\operatorname{Infl} V_T$. Therefore, $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N = \bigoplus_T (a_1, \ldots, a_n)_T$. $\operatorname{Res}_{\mathbb{C}[\underline{X}]} N$ is semisimple and, in particular, all X_i can be diagonalized simultaneously in the basis compatible with restriction. □

5 Young Tableaux

Definition 5.1. Let λ be a young diagram. Given a box $\square \in \lambda$, I define the content of $\square$, $c(\square)$ = column number of $\square$ − row number of $\square$, where column is counted from left to right and row is counted from top to bottom.

Example 5.2.

0	1	2	3
−1	0	1	
−2			

shows the content of cells of the partition $\lambda = (4, 3, 1)$.

Definition 5.3 (Equivalent definition of standard Young tableaux)**.** For each standard Young tableau of shape ν, we can represent it as a standard chain of Young diagram,

$$\emptyset = \nu_0 \nearrow \nu_1, \ldots, \nearrow \nu_n = \nu$$

where ν_i/ν_{i-1} is the cell with label i. By $\operatorname{Tab}(\nu)$, we denote the set of all possible paths from $\emptyset$ to ν. Then write

$$\operatorname{Tab}(n) = \bigcup_{|\nu|=n} \operatorname{Tab}(\nu).$$

Denote the set of all valid weight of length n by $\operatorname{VW}(n)$.

Proposition 5.4. *Suppose we are given a path,*

$$T = \nu_0 \nearrow \nu_1, \ldots, \nearrow \nu_n \in \operatorname{Tab}(n)$$

and a map from $\operatorname{Tab}(n)$ *to* $\mathbb{Z}^n$

$$\varphi(T) = (c(\nu_1/\nu_0), c(\nu_2/\nu_1), \ldots, c(\nu_n/\nu_{n-1})).$$

This map gives a bijection between $\operatorname{Tab}(n)$ *and* $\operatorname{VW}(n)$

Proof. Let $\varphi(T) = (a_1, \ldots, a_n)$. First show that $\mathrm{im}(\varphi) \subset \mathrm{VW}(n)$. Clearly, $a_1 = c(\nu_1/\nu_0) = 0$.
If $q \in \{2, \ldots, n\}$ is placed in the position (i, j) such that $a_q = j - i$, we have either $i > 1$ or $j > 1$. Therefore, there is a box either on the left or above the box ν_q/ν_{q-1}. There exists $p < q$ such that $a_q = a_p + 1$ or $a_q = a_p - 1$.
Now suppose $a_p = a_q$ for some $q < p$ which means if q is placed at position (i, j), then p will be place at position $(i + k, j + k)$. Denote p' and p'' are placed at position $(i+k-1, j+k)$ and $(i+k, j+k-1)$ respectively so we have $a_{p'} = a_p+1$, $a_{p''} = a_p-1$. We will also have $\{p', p''\} \subset \{q + 1, \ldots, p - 1\}$ since T is a standard young tableau. Thus $\mathrm{im}(\varphi) \subset \mathrm{VW}(n)$.
Now we claim that φ is injective. If $T_1 \neq T_2 \in \mathrm{Tab}(n)$ and denote $T_1 = \nu_0 \nearrow \nu_1, \ldots, \nearrow \nu_n$, $T_2 = \nu'_0 \nearrow \nu'_1, \ldots, \nearrow \nu'_n$, pick minimum p such that $\nu_p \neq \nu'_p$. We will have $c(\nu_p/\nu_{p-1}) \neq c(\nu'_p/\nu'_{p-1})$; hence $\varphi(T_1) \neq \varphi(T_2)$.
To show that $\mathrm{VW}(n) \subset \mathrm{im}(\varphi)$, we prove by induction on n. When $n = 1$ and $n = 2$, the cases are trivial. Suppose $\mathrm{Tab}(n - 1) \to \mathrm{VW}(n - 1)$ is surjective. Pick $\alpha = (a_1, a_2, \ldots, a_{n-1}, a_n) \in VW(n)$, we can restrict to $\alpha' = (a_1, a_2, \ldots, a_{n-1}) \in VW(n - 1)$. So by hypothesis, we have $\varphi(T') = \alpha'$ for some $T' \in \mathrm{Tab}(n - 1)$. If $a_n \notin \{a_1, \ldots, a_{n-1}\}$, then $a_n + 1 = \min\{a_1, \ldots, a_{n-1}\}$ or $a_n - 1 = \max\{a_1, \ldots, a_{n-1}\}$. We will put a new box ν_n/ν_{n-1} at the end of first row or the end of first column of T' respectively. If $a_n \in \{a_1, \ldots, a_{n-1}\}$, pick largest p such that $a_p = a_n$ where the box with label p at position (i, j). Claim that we will put the new box ν_n/ν_{n-1} at position $(i + 1, j + 1)$. Since p is the largest integer with such property, then there exists unique r and s such that $a_r = a_n-1$ and $a_s = a_n+1$ where $r, s \in \{p+1, \ldots, n\}$.
Thus, this new box is addable and illustrated by $\begin{array}{|c|c|}\hline p & s \\ \hline r & n \\ \hline\end{array}$.

□

Thus, there is a bijective correspondence between $\mathrm{VW}(n)$ and $\mathrm{Tab}(n)$ and it is clear that operation ⑤ in Proposition 1.32 for the weight corresponds to the exchange of the blocks ν_{i+1}/ν_i and ν_{i+2}/ν_{i+1} if they are not in the same column or row.

6 Classification of weights of inflated H(n)-modules

Let M be an irreducible $\mathbb{C}[S_n]$-*module* and $N = \mathrm{Infl}\, M$. Then N is X-semisimple by Theorem 1.15. By Theorem 1.34 and Proposition 5.4, weights of N can be mapped to standard Young tableaux. Thus, given an inflated $H(n)$-*module* N, we can recover some $\lambda \vdash n$ by previous map. We then denote $N = V^\lambda$.

Theorem 6.1. *The above correspondence is well defined which means the following:*

1. *If v and w are weight vectors of N, then their respective weights will recover the same partition of n.*
2. *Let M and M' be irreducible representations of $\mathbb{C}[S_n]$ and λ_M and $\lambda_{M'}$ be the corresponding partitions respectively. Then we have $M \cong M' \iff \lambda_M = \lambda_{M'}$.*

Proof. Proof of (1) of Theorem 6.1: By Corollary 3.5, we know that if $X_i v = a_i v$ and $X_i w = b_i w$, then $(\underline{a}) = \sigma(\underline{b})$, $\sigma \in S_n$. Thus, $(\underline{a}) = (\underline{b})$ as multisets. Then their corresponding standard Young tableaux have the same shape because the numbers of cells on each diagonal are same by Proposition 5.4. We will postpone the proof of part (2). □

Let's look at a lemma first.

Lemma 6.2. *$ev : H(n) \to \mathbb{C}[S_n]$ induces $ev' : Z(H(n)) \to Z(\mathbb{C}[S_n])$ and ev' is surjective.*

Proof. Let $X'_i = ev(X_i) = (1\ i) + (2\ i) + \ldots (i-1\ i)$ and recall $X'_1 = 0$. We prove that the elements $p(X'_2, X'_3, \ldots, X'_n)$, with p symmetric polynomial, span $Z(\mathbb{C}[S_n])$. Since the dimension of $Z(\mathbb{C}[S_n])$ equals to the number of conjugacy classes in S_n which is number of partitions of n, it suffices to construct a set of linearly independent symmetric polynomials indexed by partitions of n. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \vdash n$, $\mu_t > 0$ and $\mu' = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_t - 1)$ and $X_\mu = m_{\mu'}(0, X'_2, \ldots, X'_n)$ is defined as the monomial symmetric polynomial. We set $X_{(1^n)} = 1$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \vdash n$, where $\mu_s > 1$ and $\mu_{s+1} = \mu_{s+2} = \ldots \mu_t = 1$. By abuse of notation in the following, we write X_i for $ev(X_i)$

Claim 1: Among all $\sigma \in S_n$ in the summands of expansion of X_μ in the basis $\{\sigma | \sigma \in S_n\}$, those permutations with smallest number of fixed element can be written as a product of disjoint cycle as
$(i_1\ a_1\ a_2\ \ldots a_{\mu_1-1})(i_2\ b_1\ b_2\ \ldots b_{\mu_2-1}) \ldots (i_s\ c_1\ c_2\ \ldots c_{\mu_s-1})$, where
$i_1, \ldots, i_s, a_1 \ldots a_{\mu_1-1}, b_1, \ldots b_{\mu_2-1}, c_1, \ldots, c_{\mu_s-1}$ are all distinct. In addition all such σ have coefficient 1 in X_μ.

Proof of Claim 1. $X_{i_1}^{\mu_1-1} X_{i_2}^{\mu_2-1} \ldots X_{i_s}^{\mu_s-1} = \prod_{j=1}^{s} X_{i_j}^{\mu_j-1}$. For each term, we have $X_{i_j}^{\mu_j-1} = ((1\ i_j) + (2\ i_j) +, \ldots, (i_j - 1\ i_j))^{\mu_j-1} = (i_j\ d_{\mu_j-1})(i_j\ d_{\mu_j-2}) \ldots (i_j\ d_1) +$ other terms, where $d_1, d_2, \ldots, d_{\mu_j-1} \subset \{1, 2, \ldots, i_j - 1\}$. So we deduce that

$\sigma = (i_1\ a_{\mu_1-1})(i_1\ a_{\mu_1-2})\dots(i_1\ a_1)(i_2\ b_{\mu_2-1})(i_2\ b_{\mu_2-2})\dots(i_2\ b_1)\dots(i_s\ c_{\mu_s-1})(i_s\ c_{\mu_s-2})\dots(i_s\ c_1)$. Thus, σ at most permutes $\mu_1+\mu_2+\dots+\mu_s$ elements and the maximum is obtained when all numbers in previous expression are all distinct. In such case, we have $\sigma = (i_1\ a_1\ a_2\dots a_{\mu_1-1})(i_2\ b_1\ b_2\dots b_{\mu_2-1})\dots(i_s\ c_1\ c_2\dots c_{\mu_s-1})$ and it only appears once in the summand. In particular the cycle type of σ is μ. □

For a partition $\mu = (\mu_1, \cdots \mu_t) \vdash n$ with $\mu_s > 1$ and $\mu_{s+1} = \cdots = \mu_t = 1$, we define $\tilde{\mu_i} = \mu_1 + \mu_2 + \cdots + \mu_i$. We also define $\tilde{\mu_o} = 0$

Claim 2: Consider the type of element in the previous claim, $\sigma_\mu = \prod_{i=1}^{s}(n-\tilde{\mu}_{i-1}-1\ n-\tilde{\mu}_{i-1}\dots n-\tilde{\mu}_i+1)$. Then σ_μ is a summand of X_μ with coefficient 1.

Proof of Claim 2. I claim that σ_μ comes from the monomial $(X_n)^{\mu_1-1}(X_{n-\tilde{\mu_1}})^{\mu_2-1}(X_{n-\tilde{\mu_2}})^{\mu_3-1}\dots(X_{\mu_t})^{\mu_t-1}$. For $i = 1, 2, \dots, s$, we have $(X_{n-\tilde{\mu}_{i-1}})^{\mu_i-1} = ((1\ n-\tilde{\mu}_{i-1})+(2\ n-\tilde{\mu}_{i-1})+\dots(n-\tilde{\mu_i}\ n-\tilde{\mu_{i-1}})+\dots+(n-\tilde{\mu}_{i-1}-1\ n-\tilde{\mu}_{i-1}))^{\mu_i-1} = (n-\tilde{\mu}_{i-1}\ n-\tilde{\mu}_i+1)(n-\tilde{\mu}_{i-1}\ n-\tilde{\mu}_i+2)\dots(n-\tilde{\mu}_{i-1}\ n-\tilde{\mu}_{i-1}-1) + \text{other terms} = (n-\tilde{\mu}_{i-1}\ n-\tilde{\mu}_{i-1}-1\dots n-\tilde{\mu}_{i-1}+1) + \text{other terms}$. Note that all numbers in the decomposition of σ_μ are distinct; thus, we get a cycle decomposition of σ_μ and from Claim 1 we have the coefficient equals to 1 in X_μ. □

Now I give an example to illustrate what we have done so far. Let $n = 10$ and $\mu = (3, 3, 2, 1, 1)$ and then $\mu' = (2, 2, 1)$. We have $\tilde{\mu} = (3, 6, 8, 9, 10)$ as a 5-tuple. $X_{\mu'} = m_{\mu'}(0, X'_2, \dots, X'_{10}) = m_{221}(0, X'_2, \dots, X'_{10})$. Thus, ${X'_{10}}^2 {X'_7}^2 X_4$ is a summand of the symmetric function and in particular $(10\ 9)(9\ 8)(7\ 6)(6\ 5)(4\ 3) = (10\ 9\ 8)(7\ 6\ 5)(4\ 3)$ appears in the m_{221} as expected by Claim 2.

We give a total order of partitions of n denoted by $\preccurlyeq$. $\mu \preccurlyeq \lambda$ if $l(\mu) < l(\lambda)$ or $l(\mu) = l(\lambda)$ and $\mu \le \lambda$ in lexicographic order, where l is the length of partition.

Claim 3: Let $\lambda, \mu \vdash n$ and suppose σ_λ appears in X_μ. Then $\mu \preccurlyeq \lambda$.

Proof of Claim 3. I use the fact that : Let $\sigma \in S_n$. Suppose that σ has cycle structure $\mu = (\mu_1, \mu_2, \dots, \mu_r) \vdash n$ which means $\sigma = \omega_1\omega_2\dots\omega_r$ with ω_j of length μ_j and $\omega'_i s$ are disjoint cycles. Let $\sigma = t_1 t_2 \dots t_m$ where $t'_i s$ are transpositions. Then $m > n - l(\mu)$.

Moreover, if $m = n - l(\mu)$, we may rearrange $t_i's$ such that

$$\begin{aligned}
&\omega_1 = t_1 t_2 \dots t_{\mu_1 - 1}, \\
&\omega_2 = t_{\mu_1} t_{\mu_1+1} \dots t_{\mu_1+\mu_2-2}, \\
&\dots\dots \\
&\omega_r = t_{m-\mu_r+2} t_{m-\mu_r+3} \dots t_m.
\end{aligned}$$

For each permutation which appears in X_μ has at most $n - l(\mu)$ transpositions. By previous fact, we know that if σ_λ appears in X_μ then $n - l(\lambda) \leq n - l(\mu)$, that is $l(\mu) \leq l(\lambda)$. If $l(\mu) < l(\lambda)$, we are done. If $l(\mu) = l(\lambda)$, we denote $\mu = (\mu_1, \dots, \mu_t)$ and $\lambda = (\lambda_1, \dots, \lambda_t)$. If σ_λ appears in the monomial $X_{i_1}^{\mu_1-1} X_{i_2}^{\mu_2-1} \dots X_{i_t}^{\mu_t-1}$. Then the previous fact implies that all $\mu_j - 1$ transpositions from X_{i_j} contribute to one cycle of σ_λ containing i_j. Thus, $\lambda_l \geq \mu_l$ for all $\lambda_l \neq 1$. Then we have $\mu \trianglelefteq \lambda$ in dominance order which implies $\mu \leq \lambda$ in lexicographic order. □

Thus, from Claim 2 and Claim 3, we know that $\{X_\mu\}$ for $\mu \vdash n$ are linearly independent and they will span $Z(\mathbb{C}[S_n])$. We proved the lemma. □

Proof of Theorem 6.1 part (2). One direction is straightforward. If $\lambda_M \neq \lambda_{M'}$ which means they have different content vectors as multisets, they will be inflated from different irreducible representations of S_n. Conversely, if $M \ncong M'$, then there exists an element of $Z(\mathbb{C}[S_n])$ acting on M and M' differently. From previous lemma, ev' is surjective so we have an element $f(X_1, \dots, X_n) \in Z(H(n))$ acting on N and N' by different scalar which means they have different weights as multisets for a weight vector. Thus, $\lambda_M \neq \lambda_{M'}$. □

Now we know that if v is a weight vector of $N = \operatorname{Infl} M$ where M is irreducible S_n representation with weight $(a_1, a_2, \dots, a_n)$, then $\varphi_w v$, $w \in S_n$ will exhaust all possible weight vectors and give us weight $(a_{w^{-1}(1)}, a_{w^{-1}(2)}, \dots, a_{w^{-1}(n)})$ whenever $\varphi_w v \neq 0$; equivalently, whenever operation ⑤ is feasible. $H(n)$-module N will not contain other weights because of Theorem 6.1 part 2. Thus we establish a bijection between $\{$weights of $\operatorname{Infl} V^\lambda\}$ and SYT of shape λ. We denote the set of all weights from inflation modules of all irreducible S_n representations: $\mathrm{IW}(n)$. Thus from previous statement and Proposition 5.4, we conclude

Theorem 6.3. $\mathrm{IW}(n) = \mathrm{VW}(n)$. *Therefore, three properties listed in Theorem 1.34 are indeed the sufficient and necessary condition for a tuple to be the weight of an inflated module.*

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