The Development of a Physical Theory of Braids

An Extension of the Ropelength Model to Braids

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Abstract

Physical knot theory is an area of study in the field of knot theory which seeks to create physical frameworks with which to study knots. One of the most well known models is the *ropelength model* for knots and links. The ropelength model of a knot or link seeks to model presentations of knots or links that are made of an ideally flexible, thickened rope and to define a tight presentation of a knot or link. It was established by J.W. Alexander in 1923 that every knot or link can be represented as the closure of a braid [1]. With this, we are inspired to extend the ropelength model to braids. In this paper we define a ropelength model for braids and prove the existence of a ropelength minimizing braid presentation within each braid type.

1 Acknowledgements

First and foremost, I would like to thank my Dad and Mom for their constant support of my mathematical endeavors and encouraging me to continue learning mathematics. In addition, I would also like to thank my friends for pretending to be interested when I would talk about braiding rope–I only hope I didn't bore them too much.

I also want to thank Cameron Bjorklund for his time helping me. He gave absolutely great insight and opinions on the subject matter of this thesis. I would also like to thank Cameron for his proof in Section 6 that the sequence R_n converges. This proof has of course been integral to our definition of ropelength for braids taken as elements of the braid group.

Most importantly, I would like to give a huge THANK YOU to Professor Abigail Thompson, without whom this project would not exist. Nearly two years ago professor Thompson had asked me if I wanted to work on a Summer REU defining the roundness of a braids. This proved to be quite a challenging task and required us to develop a physical model of braids to work in, ultimately leading to this project. I greatly appreciate the amount of time, energy and patience Professor Thompson has put into this project by helping me in these past two years. Professor Thompson has been a continuing inspiration to me and has facilitated much growth in me as a budding mathematician. So it is with much gratitude that I repeat myself: Thank you Professor Thompson.

2 Introduction

"Can you tie a knot on a foot long rope that is one inch thick?" This has been a long standing question in knot theory. The question was answered negatively in [4] by employing the *ropelength model of a knot*. [3, 6, 7] developed the ropelength model for knots and links prior to this as a way to mathematically model knots that are made out of a thickened, ideally flexible rope and measure how tightly tied a given knot or link in this model can be tied.

One of the first observations to make about knots and links that are made out of a real rope is that there is a limit to how tightly one can tie the knot or link. This is, of course, due to the thickness of the rope. It is only natural that the ropelength model for knots and links also models this sort of limiting behavior. Before this limiting behavior can be rigorously defined, however, one must first give a reasonable account of the thickness of a given knot or link. Intuitively, a person would buy a length of rope that has a given thickness, then tightly tie the knot out of this length of rope. This is a very natural and valid approach to solving this problem. However from a mathematical perspective, it makes sense to first construct a specific presentation of a knot or link out of a curve (or disjoint union of curves) of zero thickness, then place a normal tube(s) around the presentation whose core is the original curve (or disjoint union of curves). Given this approach, we have to be careful to not make our normal tube too large, else it will intersect its own interior. The interiors of the normal tube should not self-intersect because real world rope does not do this. We note, however, that self-intersection of the normal tube is allowed if it is only along the boundary of the normal tube i.e the outermost tube, since this behavior occurs in knots tied out of physical rope.

[6] gave a useful account of this maximal thickness for a given knot (or link) presentation by employing the *global radius of curvature*. The global radius of curvature for a presentation of a knot or link is a functional that provides a nice theoretical framework to work with the thickness of curves. The global radius of curvature takes advantage of the fact that for three distinct, non-collinear points $x, y, z \in \mathbb{R}^3$, there is a unique circle that passes through the three points–we call the radius of this circle $r(x, y, z)$. The global radius of curvature of a knot or link is defined locally by first fixing a point on x the knot (or link) presentation and taking the infimum over all y and z (with $y \neq z$) on the knot presentation of $r(x, y, z)$. Then the global radius of curvature is defined as the infimum of all x over the local radius of curvature.

Both [6] and [3] showed that the global radius of curvature corresponds to the maximal radius of thickness for an embedded open normal tube around the given knot or link presentation. Using this characterization of thickness is advantageous because it allows for control over the thickness in terms of local curvature and distance between strands of the knot *at the same time*. In addition, this functional is upper semicontinuous, which allows us to have better control over convergence properties which we will address momentarily.

Now with a working account of the maximal thickness of an embedded normal tube, [6, 3] define a notion of how "tight" a knot (or link) presentation is through the *ropelength*, which is defined as follows: Let $L \subset \mathbb{R}^3$ be a parameterized presentation of a knot or link. We define the *ropelength* of L to be

$$Rl(L) := \ell(L)/\Delta[L],$$

where $\ell(K)$ is the total length of L and $\Delta[L]$ is the maximimal thickness of L. The smaller the ropelength, the tighter the presentation is tied. This makes sense with our intuitive notion of tightness since the ropelength of L would be smallest when the total length of rope used to make L is smallest and the thickness of L is largest. This would mean that L is made out of the smallest amount of rope and that there is very little space between the strands once they are thickened.

Using this definition of tightness, [3, 6] then frame the question of existence of ideal knot presentations as "For any knot (or link) type, does there exist a presentation L^* such that $RL(L^*)$ is minimal?" This question itself can actually be rephrased as follows: "For each knot (or link) type, does there exist a presentation $L^* \in \mathcal{L}$ such that

$$Rl(L^*) = \inf_{L \in \mathcal{L}} Rl(L),$$

where $\mathcal{L}$ is the isotopy class of a given knot (or link) type?"

In [7], this is proved for $C^{1,1}$ (once differentiable with Lipschitz derivative) presentations of knots. In [3], a more general result is proved for $C^{1,1}$ presentations of links and several classes of such ropelength minimizing presentations is constructed. This also shows that such presentations are not unique in general. For the purposes of this paper, we are interested in how these existence results were achieved. Since the process was essentially the same in both papers, the next paragraph will solely refer to the process for proving the existence of ropelength minimizers in [3] since it allowed for the existence of ropelength minimizing link types as well.

In [3], Cantarella et al. first showed that the thickness functional (obtained by employing the global radius of curvature) was upper semicontinuous. Then it was shown that for a given presentation of a link L with thickness $\tau > 0$, L is $C^{1,1}$ with Lipschitz constant $1/2\tau$. After which, there was a lemma proven which stated that if one takes a sequence of links L_i with thickness $\tau > 0$ which converge to a limiting link L (in the C^0 norm), then $L_i \to L$ in the C^1 norm and L is isotopic to (all but finitely many) of the L_i. It should also be noted that by application of the upper-semicontinuity of thickness, L also has thickness of at least τ. Then using these results, the existence result was proven as follows: Consider the compact space of all $C^{1,1}$ curves with length uniformly bounded by 1. Then, one can consider a sequence of L_i which maximizes thickness over the isotopy class $\mathcal{L}$. Using the uniform boundedness of the lengths and the fact that each L_i is uniformly Lipschitz, one can extract a uniformly convergent subsequence $L_{i_k} \to L_*$ by employing the Arzelà-Ascoli Theorem. By the previously stated lemma L_* is isotopic to (all but finitely many) of the L_{i_k} and must also have thickness equal to the supremal thickness (since thickness is upper semicontinuous). Then L_* is, of course, a ropelength minimizer.

In this paper, we define a similar ropelength model for braids. We will be using the same notion of thickness for braids as was used in [3, 6, 7]. In addition, we will be using the same notion of *ropelength* for braid presentations as was used for links (i.e total length used in all strands divided by maximum thickness). The reason ropelength corresponds to tightness for braids would be the same as why it corresponds for links. We will also be considering a thickness maximizing sequence of braid presentations (for a given braid type) and employing the Arzelà-Ascoli theorem to extract a convergent subsequence, which will show the existence of ropelength minimzing braid presentations. There is however, a slight difference in our method for proving the existence of ropelength minimizing braid presentations and what was used in [3, 6, 7]. The difference arises from a certain subtlety that comes from dealing with braid presentations. This subtlety arises when we have to work with the endpoints of a braid presentation. When working with the typical definition of a braid, two braid presentations can have different endpoints but still be equivalent. This leads to a somewhat problematic viewpoint when we are treating our braid presentations as parameterized curves sitting in $\mathbb{R}^3$. This is because equivalence classes of braid presentations with arbitrary endpoints makes for an infinite maximal thickness since we can just keep considering braids whose endpoints are arbitrarily far apart; a problem that is not experienced by nontrivial presentations of knots or links. At first, one would think the solution to this endpoint problem would be to merely fix endpoints so that they are all evenly spaced in a line and a certain vertical distance apart. However, this is not a very natural definition to consider, because when we make braids out of physical rope and pull them as tight as we can, we have that the endpoints of the braid do not necessarily form a straight line as we can see pictured:

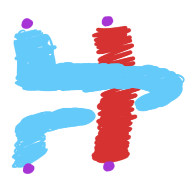

Figure 1: We can see that the perpendicularity of the endpoints and the "flat" behavior of the endpoints being stuck in a row is obviously not ideal.

As such, we are not capturing the maximally tightened behavior of the braid. We believe that the fix for this issue is to do the following: choose the endpoints in any suitable way, maximize thickness over all such presentations of the braid with the same endpoints, then compose this maximized presentation with itself. After which, tighten this composed braid maximally and consider the minimal ropelength of this new braid divided by 2. Next, compose this maximally tightened composed braid with itself to obtain a new braid and consider this new ropelength divded by 3. Repeat this process indefinitely. We will then define the minimal ropelength of an element of the braid group to be the sequence of minimal ropelengths of the composed braids, divided by the number of compositions. The geometric intuition behind this decision is that when we compose the braid (that has fixed endpoints) with itself after an arbitrary number of times, the actual ideal presentation of the braid–as we would see from making it out of rope–is somewhere within this maximally tightened composition; then all we need to do is cut out this ideal presentation. This process of arbitrary composition would, intuitively, allow us to "forget the endpoints" that we have chosen.

With this strategy in mind, we spend sections 3 and 4 of this paper laying out the necessary definitions and results for a ropelength model of braid presentations with fixed endpoints. In the fifth section we lay out our more general process for finding ropelength minimizers in greater detail.

3 Preliminary Definitions

Definition 3.1 (Braid Frame)**.** Let C_2 be a collection of n equally spaced points on the x-axis in $\mathbb{R}^3$. Now let $C_1 = C_2 + \{(0, 0, b)\}$ be the Minkowski sum and $b \in \mathbb{R}$ such that $b > 0$. We call the collection

$$F = \{C_1, C_2\}$$

a *braid frame*.

We give an example of a braid frame.

Example 1. Let $C_1 = \{(1/3, 0, 0), (2/3, 0, 0), (1, 0, 0)\}$ and take $C_2 = \{(1/3, 0, 1), (2/3, 0, 1), (1, 0, 1)\}$. Then we let $F = \{C_1, C_2\}$ and we can picture F as follows:

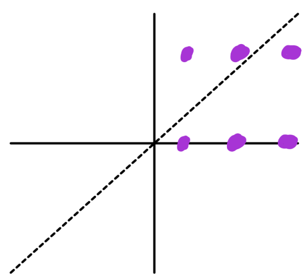

Figure 2: The bottom row of purple points depict the set C_1 and the top row of purple points depict the set C_2. Together, they make the frame $F = \{C_1, C_2\}$.

Definition 3.2 (braid strand, endpoints, and braid presentation)**.** Let $F = \{C_1, C_2\}$ be a braid frame and let $\gamma : [0, 1] \to \mathbb{R}^3$ be a smooth (C^k) curve that is (weakly) monotonic in z such that $\gamma(a) \in C_1$ and $\gamma(b) \in C_2$ where γ is perpendicular to the line containing $\gamma(a)$ and $\gamma(b)$. We call γ a *braid strand* and $\gamma(a)$ and $\gamma(b)$ the endpoints of γ.

If for a given frame F, we have a collection, S, of n-braid strands $\gamma_i : [0, 1] \to \mathbb{R}^3$ ($i \in \{1, ..., n\}$) such that $\gamma_i(t) \neq \gamma_j(s)$ for all $t \in [0, 1]$ and $s \in [0, 1]$ and $i \neq j$, then we call the structure $B = (F, S)$ a *framed presentation of an n-braid* or a *framed braid presentation*.

Example 2. It would be rather cumbersome to give explicit parameterizations of a disjoint union of smooth strands which form a framed braid presentation as we have defined, so we present this example using a picture:

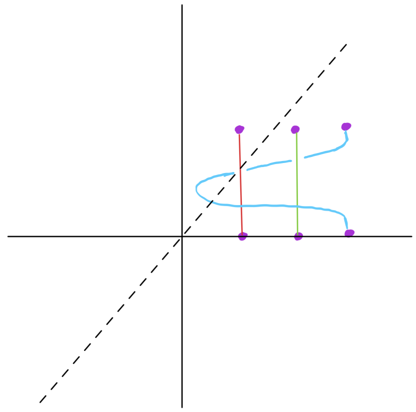

Figure 3: With the red, green, and blue curves, we note that each of them is drawn to be perpendicular to the endpoints. Taken along with F from the previous example we have a framed braid presentation.

We note that we can have two different framed braid presentations which contain all of the same crossing information such that one presentation can be continuously deformed into the other within the ambient space of $\mathbb{R}^3$. Since the two framed presentations carry all of the same "relevant" information, we define an equivalence on braid presentations.

Definition 3.3 (equivalence of framed braid presentations)**.** Let F be a braid frame and let $B_0 = (F, S_0)$ and $B_1 = (F, S_1)$ be braid presentations. We say B_0 is equivalent to B_1, denoted $B_0 \simeq B_1$, if there exist n ambient isotopies

$$H^i : [0,1] \times [0,1] \to \mathbb{R}^3$$

relative to the respective endpoints of the i-th strands where, after a suitable change of coordinates for each $\alpha_i \in S_0$, there is a $\beta_i \in S_1$ such that $H^i(s,0) = \alpha_i(s)$, $H^i(s,1) = \beta_i(s)$ and the collection $S_t = \{H^i(s,t) : s \in [0,1]\}_{i=1}^n$ forms a framed braid presentation $B_t = (F, S_t)$ for all $t \in [0,1]$.

Example 3. We convey this example of equivalence pictorially:

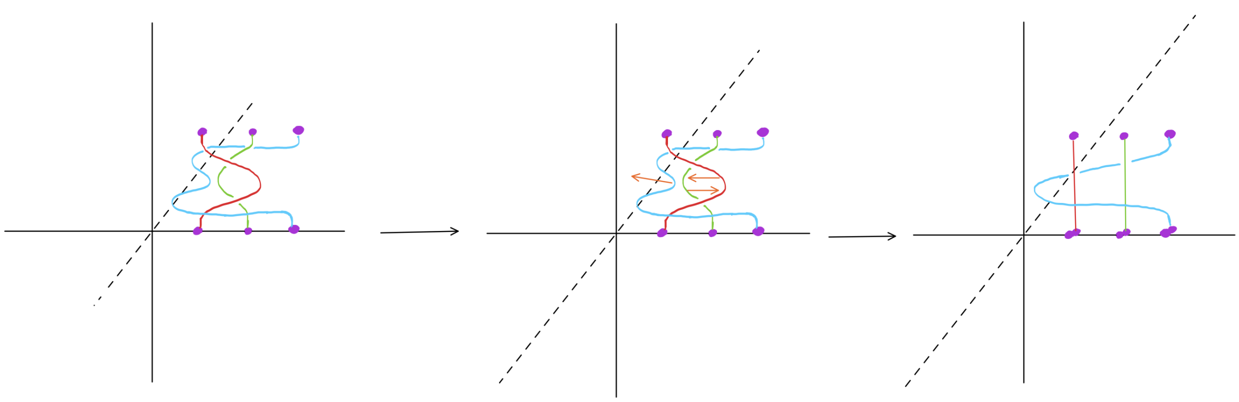

Figure 4: Starting with the framed braid presentation on the far left, we can obtain the framed braid presentation on the far right as depicted by the middle picture. In the middle picture we are continuously pushing the blue strand outward while also continuously pushing the red strand and green strand until they are straight.

Definition 3.4. Let $F = \{C_2, C_1 = C_2 + \{(0,0,b)\}\}$ (for some $b > 0$) be a braid frame. We label the points of C_1 and C_2 from left-to-right by $1, ..., n$. Let $B_1 = (F, S_1)$ and $B_2 = (F, S_2)$ be two braid presentations. We define the *composition of a braid presentation* of B_1 and B_2, denoted $B_1 * B_2$, in the following manner:

Let $\alpha_i : [0,1] \to \mathbb{R}^3$ be the i-th strand of B_1. By the i-th strand of B_1 we mean that $\alpha_i(0)$ is the i-th points of C_1. We note that $\alpha_i(1)$ is the j-th point of C_2. Let $\beta_j : [0,1] \to \mathbb{R}^3$ be the j-th strand of B_2. We reparametrize α_i and β_j so that they are defined on $[0, 1/2]$ and $[1/2, 1]$ respectively (while keeping both strands smooth (C^k)) and we rename the reparametrizations as α_i and β_j. In addition, we translate α_i along the z-axis by $(0,0,b)$ (and again name the resulting curve α_i). Now we construct the curve $\gamma_i : [0,1] \to \mathbb{R}^3$ defined by

$$\begin{cases} \alpha_i(t), t \in [0, 1/2] \\ \beta_j(t), t \in [1/2, 1] \end{cases}$$

(where γ_i is smooth (C^k)). We then consider the frame

$$F^* = \{C_2, C_2 + \{(0,0,2b)\}\} \text{ and the collection } S^* = \{\gamma_i\}_{i=1}^n.$$

We then define $B_1 * B_2$ as

$$B_1 * B_2 := (F^*, S^*).$$

***Example* 4.** We convey this example of braid composition pictorially:

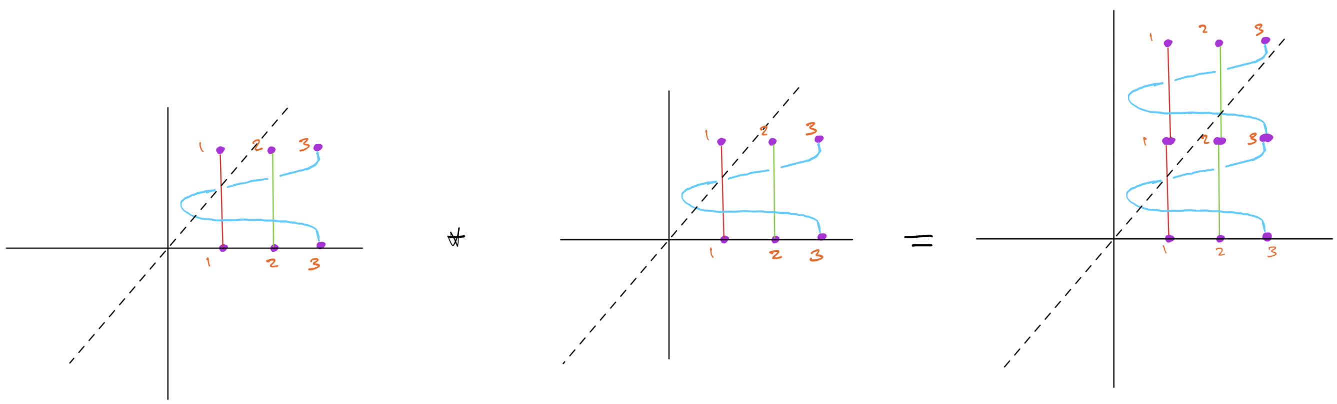

Figure 5: Here we have composed our framed braid presentation (from the previous examples) with itself. Note that where each strand on the top presentation ends is where the bottom presentation begins. This continuation of strands from the first presentation to the second must be carried out in a smooth manner. Note that the resulting composed braid has a different frame than its two component braid presentations. Lastly, we note that in this example since we are composing two braid presentations within the same isotopy class, the order of the composition does not matter. However, in general the composition operation is not commutative.

We can easily observe that since all our γ_i in the above definition were smooth, and the strands α_i and β_i were monotonic, had endpoints in the frame, and met the frame perpendicularly, that each γ_i is a braid strand of $B_1 * B_2$.

Now we turn our attention towards developing a ropelength model for our framed braid presentations. In order to define a maximum thickness for a given framed presentation, we require the following geometric notion also used in [6]. For any three non-collinear points $x, y, z \in \mathbb{R}^3$, there is a unique circle sitting in $\mathbb{R}^3$ containing x, y, and z. We denote the radius of this circle by $r(x,y,z)$. As in [6], we can actually continuously extend $r(x,y,z)$ if it is defined on a smooth (C^k) curve. We recount this construction here: Let $\gamma : [a,b] \to \mathbb{R}^3$ be a simple, smooth (C^k) such that $x = \gamma(t), y = \gamma(s)$, and $z = \gamma(u)$ for some $t, s, u \in [a,b]$. Then define

$$r(x,y,y) := \lim_{u \to s} r(x,y,z).$$

We can define other such extensions of r (such as $r(x, x, y)$ or $r(x, z, z)$) similarly. As a note, we can also consider cases such as

$$r(x, x, x) := \lim_{s,u \to t} r(x, y, z),$$

and we note that $r(x, x, x)$ is actually the radius of curvature for the curve. Now we can give our definition of maximal thickness for a framed braid presentation.

Definition 3.5. Let $B = (F, S)$ be a framed braid presentation. We define the *maximum thickness for a framed braid presentation*, $\Delta[B]$, in terms of the *local thickness* $\Delta_x(B)$ where x is a point on any strand, γ_i and

$$\Delta_x(B) := \inf_{y,z \in S} r(x, y, z) \text{ and } \Delta[B] := \inf_{x \in S} \Delta_x(B).$$

As a sanity check, we would like to show that this definition corresponds to a maximum thickness for a given framed braid presentation. Using [3, 6], we know that for any C^1 presentation of a link L, the maximal thickness defined from the global radius of curvature is equal to the normal injectivity radius (i.e the largest radius of an injective tube composed of open disks normal to the curve centered on the curve). Let $B = (F, S)$ be a framed braid presentation, where $\Delta[B] = \tau > 0$. We smoothly close the braid into a presentation of a link, L, in such a manner where $\Delta[L] = \tau$, and immediately apply the result from [3] to know that the the functional, Δ, corresponds to the thickness of the link. Since $\Delta[B] = \tau$ was not changed by closing up our braid, we conclude that the global radius of curvature defined over a framed braid presentation corresponds to the maximal thickness of the presentation.

Now that we have a notion of maximum thickness for a given framed presentation of a braid, we would like to numerically quantify just how tightly wound a given framed presentation is. For this, we turn to [3, 6, 7]. These papers describe the ropelength model for a given presentation of a knot (or link). The ropelength of a presentation of a knot or link is the total length of the strands divided by the maximimum thickness of the presentation. We now define the corresponding version of ropelength for a framed braid presentation as follows:

Definition 3.6. Let F be a braid frame with $B = (F, S)$ a framed braid presentation. In addition, let $\Delta[B]$ be the maximum thickness of B, and let $\ell(B)$ be the sum of the lengths of all strands of B. The *ropelength of* B is defined as

$$Rl(B) := \ell(B)/\Delta[B].$$

We now direct our attention to the following situation: in the real world, when we buy rope of a given thickness and length, it is obvious that for a given braid we can make a tightest version of that braid out of rope (given that we have enough rope). In order to have a somewhat physically accurate model for braids, we would like our model to allow us to have presentations of braids that can be maximally tightened for every framed braid type. So we seek to answer the following question: "For every (framed) braid type, does there exist a tightest (framed) braid presentation?" Of course, using our new definitions we can rephrase the question as follows: "For every (framed) isotopy class, $\mathcal{B}_F$, does there exist a (framed) braid presentation $B^* \in \mathcal{B}$ such that

$$\inf_{B \in \mathcal{B}} Rl(B) = Rl(B^*)?\text{”}$$

If such a B^* exists, we call it the *ropelength minimizer* of $\mathcal{B}_F$. We note that we can alternatively denote the ropelength minimizer for a given isotopy class for a framed presentation as $Rl(\mathcal{B}_F)$

In order to prove that there exist ropelength minimizers for framed presentations, we are going to have to consider sequences of framed braid presentations. We give a definition of a sequence of framed braid presentations and what it means for a sequence of braid presentations to converge.

Definition 3.7. Let F be a frame and for each $i \in \{1, ..., n\}$ let $\{\gamma_m^i\}_{i=1}^n$ be a sequence of curves such that the disjoint union $S_n = \{\gamma_m^i\}_{i=1}^n$ forms a framed braid presentation

$$B_m = (F, S_m)$$

for each $m \in \mathbb{N}$. We call B_m a *sequence of framed braid presentations*. If each γ_m^i converges (uniformly) to a curve γ_*^i, then we we say B_m converges (uniformly) to $B = (F, S_*)$ where $S_* = \{\gamma_*^i\}_{i=1}^n$ and we express this as $B_m \to B$.

It is important to note that if we have a convergent sequence of framed braid presentations $B_m \to B$, the limiting framed collection of curves B need not necessarily be a framed braid presentation. This is because we can have a sequence in which the braid strands are never touching but converge to two braid strands which do intersect, thereby making the collection to which it converges not a framed braid presentation. However we actually have great control over the convergence if we consider sequences of framed braids, B_m that have thickness $\Delta[B_m] \geq \tau > 0$ for all m. Then the collection of framed curves to which it converges must be a braid presentation. We prove this fact later in section 4.

4 Basic Results

Our goal is to show that for every framed isotopy class of a braid, there is a ropelength minimizing framed presentation. In order to show this, we need to take some detours.

***Lemma** 5.* Let $\mathcal{B}_F$ be a framed isotopy class of a braid. Then

$$\Delta[\mathcal{B}_F] := \sup_{B \in \mathcal{B}_F} \Delta[B]$$

exists and is finite.

Proof. Since our frame is fixed, we see that we cannot make any of the strands thicker than half of the distance between two consecutive top (or bottom) endpoints else we would have that the normal tubes would self intersect along interiors. Because of this, we have that $\Delta[B]$ is bounded for all $B \in \mathcal{B}_F$. By the completeness of the real numbers we conclude $\Delta[\mathcal{B}_F]$ exists and is finite. □

The importance of this result is that we can consider a maximizing sequence $\{B_m\}_{m=1}^{\infty} \subset \mathcal{B}_F$ such that $\Delta[B_m] \to \Delta[\mathcal{B}_F]$.

We note that in [3] it was proven that maximum thickness is upper semicontinuous with respect to the C^0 topology on the space of $C^{0,1}$ curves. Then in [3], it was shown that for a given presentation of a link L with thickness $\tau > 0$, L is $C^{1,1}$ with Lipschitz constant $1/2\tau$. We can obtain a similar result for framed braid presentations as follows:

***Lemma** 6.* If $B = (F, S)$ is a framed braid presentation with thickness $\tau = \Delta[B] > 0$, then each curve $\gamma^i \in S$ is $C^{1,1}$ with Lipschitz constant $1/2\tau$.

Proof. We smoothly close the braid into a presentation of a link, L, in such a manner that $\Delta[L] = \tau$. We can then apply the result from Lemma 4 of [3] to conclude that the presentation L must be $C^{1,1}$ with Lipschitz constant $1/2\tau$. Then we can recover B from L by cutting it at the proper points to conclude that it is also $C^{1,1}$ with Lipschitz constant $1/2\tau$. □

***Lemma** 7.* Let F be a braid frame and let $B_m \to B$ be a uniformly convergent sequence of framed braid presentations such that each $\Delta[B_m] \geq \tau > 0$ for all $m \in \mathbb{N}$. Then B is a framed braid presentation with $\Delta[B] \geq \tau$ and is equivalent to (all but finitely many) B_m.

Proof. We first want to show that B is actually a framed braid presentation. Since $\Delta[B_m] \geq \tau > 0$ for all m, we then have by the semicontinuity of Δ we know

$$\Delta[B] \geq \tau > 0.$$

Then since $\Delta[B] > 0$, we know that none of the strands of B intersect each other. In addition, if we let $\gamma_m^i : [a_i, b_i] \to \mathbb{R}^3$ be the i-th strand of the m-th framed presentation B_m such that $\gamma_m^i \to \gamma^i$. Since each $\gamma_m^i(a_i) = \gamma_1^i(a_i)$ and $\gamma_m^i(b_i) = \gamma_1^i(b_i)$ for all $m \in \mathbb{N}$, we must have that

$$\gamma^i(a_i) = \gamma_1^i(a_i) \text{ and } \gamma_i(b_i) = \gamma_1^i(b_i).$$

So each γ^i respects the frame F. In addition, we have that since each γ_m^i must be weakly monotonic in the z-axis. Hence (wlog) for $t \leq s$, we have

$$\pi_z(\gamma_m^i(t)) \leq \pi_z(\gamma_m^i(s)).$$

Then by limit inequality of real numbers, we have

$$\lim_{m\to\infty} \pi_z(\gamma_m^i(t)) \leq \lim_{m\to\infty} \pi_z(\gamma_m^i(s)),$$

and so we conclude for $s \leq t$ and

$$\pi_z(\gamma^i(t)) \leq \pi_z(\gamma^i(s)).$$

Thus B is a framed collection of strands that do not intersect and are monotonic in the z-axis, hence $B = (F, \{\gamma^i\}_{i=1})$ is a framed braid presentation. We now prove that B is equivalent to all but finitely many B_m. Since $\Delta[B] \geq \tau > 0$, each curve of B must be surrounded by an embedded normal tube of diameter τ. In addition, all but finitely many of the strands of the B_m lie within the respective i-th normal tube since the convergence is assumed to be uniform. By the C^1 convergence of the previous lemma, we also have that the all but finitely many of the strands are tranverse to each normal disk of the normal tube. Each γ_m^i is isotopic to γ^i by straight line homotopy within the normal disk. Hence the result is proven. □

Now we can answer our existence question for ropelength minimizers from framed braid presentations.

Theorem 8. Fix a nonzero length $l \in \mathbb{R}$ and pick a frame F. Consider all framed presentations isotopic to B of total length at most l; call this class $\mathcal{B}_F$. Then there exists a framed ropelength minimizer $B^* \in \mathcal{B}_F$

Proof. By Lemma 5 we know that thickness is bounded and so $\Delta[\mathcal{B}_F]$ is finite. Hence there must exist a sequence $B_m \in \mathcal{B}_F$ that maximizes thickness, i.e $\Delta[B_m] \to \Delta[\mathcal{B}_F]$ as $m \to \infty$. Now take each B_m and smoothly connect arcs of finite length stemming from each point of F onto each B_m such that

1. The thickness of this new object is the same as $\Delta[B_m]$ for all m.

2. The resulting object can be represented as a smooth embedding of a circle.

3. The total lengths of the resulting sequence remains uniformly bounded.

We demonstrate these rules with the following picture:

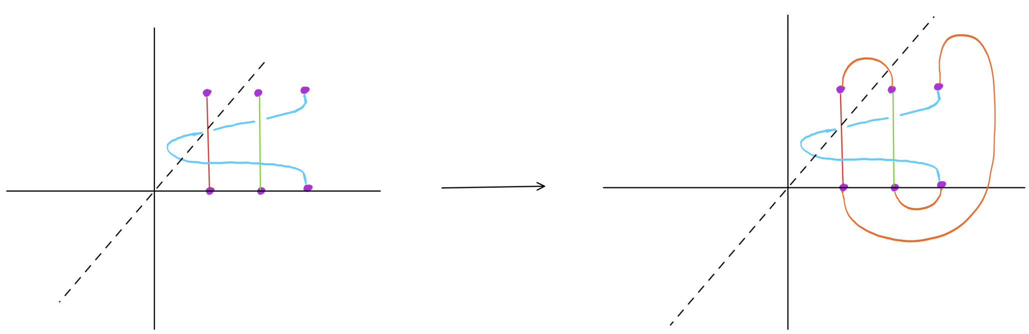

Figure 6: Starting with a framed braid presentation (left), we smoothly connect arcs around the endpoints of each B_m (pictured in orange) so that we obtain a knot presentation, K_m. We connect these arcs in such as way that $\Delta[K_m] = \Delta[B_m]$ and the lengths of K_m are uniformly bounded. We note that it does not matter whether or not the resulting knot is trivial, we just require a sequence of knots.

We call the resulting sequence of knot presentations K_m. We note that $\Delta[K_m]$, the maximal thickness of these knot presentations, must approach $\Delta[\mathcal{B}_F]$ as $m \to \infty$. We also note that we can parameterize each K_m be a function $\gamma_m : S^1 \to \mathbb{R}^3$. Now we have a sequence of knots which has uniformly bounded lengths, so we can apply the Arzelà-Ascoli Theorem as stated in [8] to extract a uniformly convergent subsequence $\gamma_{m_k} \to \gamma_*$. Since we are considering a thickness maximizing sequence, after sufficiently large M we must have that there is some $\tau \in \mathbb{R}$ such that $\Delta[\gamma_{m_k}] \geq \tau > 0$ for all $k > M$. We apply Lemma 6 from [3] to assert that $\Delta[\gamma_*] \geq \tau > 0$, so we know that the strands of γ_* are non-intersecting. We point out that by construction, F must have remained constant in each γ_{m_k}. Hence we can easily cut out a framed collection of strands from each γ_{m_k} and γ_* that we call B_{m_k} and B_* respectively. We must have that $B_{m_k} \in \mathcal{B}_F$ for all m_k. We now want to show that B_* is our ropelength minimizer. We note that B_{m_k} converges uniformly to B_* by construction of these subsequences. Since each B_{m_k} is a framed braid presentation equivalent to $\mathcal{B}_F$ and $\Delta[B_{m_k}] \geq \tau > 0$ for some $\tau \in \mathbb{R}$ and sufficiently large k, we can apply Lemma 7 to conclude that B_* must also be a framed braid presentation that is equivalent to all (but finitely many) of the B_{m_k}. Since length is lower semicontinous and thickness is upper semicontinuous, we must have that

$$Rl(B_*) \leq l/\Delta[\mathcal{B}_F],$$

and in particular that

$$\Delta[B_*] \geq \mathcal{B}_F.$$

Since all but finitely many of the B_{m_k} are equivalent to B_*, we conclude that B_* must be in $\mathcal{B}_F$. Then since $\Delta[\mathcal{B}_F]$ is supremal, we must have

$$\Delta[B_*] \leq \Delta[\mathcal{B}_F].$$

So $\Delta[B_*] = \Delta[\mathcal{B}_F]$. Hence we have shown the existence of a framed ropelength minimizer. □

5 A Definition of Ropelength For More General Braids

We would like to define minimal ropelength for any element of the braid group, which would be more closely to the braids we would make out of physical rope. To do this we consider the following construction:

Let $\mathcal{B}_F$ be an isotopy class of framed braids whose presentations are total length at most l. Now using Theorem 8, we know that there exists a framed ropelength minimizing presentation $B_* \in \mathcal{B}_F$, and we consider $Rl(B_*)$, the minimal ropelength of $\mathcal{B}_F$. Using our definition of braid composition in conjunction with our notion of framed equivalence, we can define the equivalence class $\mathcal{B}^n_{F^n} = \mathcal{B}_F * ... * \mathcal{B}_F$ inductively (where F^n is the natural frame after the n-th composition). We consider the sequence

$$R_n = \frac{Rl(\mathcal{B}^n_{F^n})}{n}$$

as $n \to \infty$. We want to show that the sequence R_n converges. First we will show that R_n always contains a convergent subsequence. We note that since the thickness of our presentations are always greater than zero and the total lengths of our presentations are always non-zero, we must have $0 < R_n$ for all n. Next, we note that since $Rl(\mathcal{B}^n_{F^n})$ is infimal, $Rl(\mathcal{B}_F) \leq nRl(\mathcal{B}_F)$, where $nRl(\mathcal{B}_F)$ corresponds to the ropelength of the composition of presentations given by

$$B^n_* \in \mathcal{B}^n_F,$$

where $B_* \in \mathcal{B}_F$ is the ropelength minimizer of $\mathcal{B}_F$. Hence

$$R_n = \frac{Rl(\mathcal{B}^n_{F^n})}{n} \leq Rl(\mathcal{B}_F),$$

for all n. Thus we have

$$0 < R_n \leq Rl(\mathcal{B}_F), \text{ for all } n \in \mathbb{N}.$$

So by the Bolzano-Weierstrass theorem, we know that there must exist some convergent subsequence $R_{n_k} \to R^*$. Now we claim that $\limsup_{n\to\infty} R_n = R^*$. We let $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that

$$R_m < R^* + \frac{\varepsilon}{2}.$$

Now choose $k \in \mathbb{N}$ such that

$$\frac{Rl(\mathcal{B}_F)}{k} < \frac{\varepsilon}{2}.$$

Select $n > km$. Then by the division algorithm we write

$$n = lm + r, \; l \geq k, 0 \leq r \leq m.$$

Now since $Rl(\mathcal{B}_F^n)$ is infimal and we are composing braid presentations, we can obtain the following inequality

$$R_n = R_{lm+r} \leq \frac{R_{lm} + rRl(\mathcal{B}_F)}{lm+r},$$

as the tightest presentation of the lm-times composed braid composed with r copies of the ropelength minimizer for $\mathcal{B}_F$ cannot be tighter than the ropelength minimizer for $\mathcal{B}_{F^n}^n$. Then since $lm + r > lm$ and $r \leq m$, we have

$$R_n \leq \frac{R_{lm} + rRl(\mathcal{B}_F)}{lm+r} \leq R_{lm} + \frac{mRl(\mathcal{B}_F)}{lm}.$$

By our assumptions on l, k, r, and m we obtain

$$R_n \leq R_{lm} + \frac{mRl(\mathcal{B}_F)}{lm} \leq R^* + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = R^* + \varepsilon.$$

Hence if we have a convergent subsequence that converges to R^*, then the $\limsup$ of the sequence must converge to R^* as well. Now assume that the $\liminf$ of the sequence is not equal to the $\limsup$ of the sequence. By this assumption we can certainly extract a subsequence R_{n_j} such that

$$R_{n_j} \to \liminf_{n\to\infty} R_n,$$

but this would mean $\liminf_{n\to\infty} R_n = \limsup_{n\to\infty} R_n$ per the above result. Thus, our sequence converges.

As we just saw, for any isotopy class $\mathcal{B}_F$, the sequence $R_n = Rl(\mathcal{B}_F^n)/n$ converges. Since this sequence has a well defined value for all isotopy classes, we define the *minimal ropelength of a word in the braid group* as the limit R_n. We demonstrate the intuition behind this construction in the following picture:

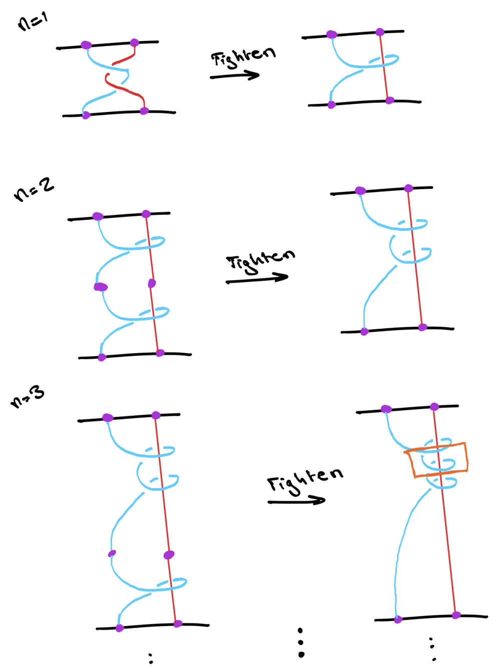

Figure 7: We begin the construction by drawing the $n = 1$ case. On the left, we took any presentation of our braid and applied Theorem 8, resulting in the image on the right (we omitted the thicknesses for a clearer picture). In the $n = 2$ case, we took the previous minimizer and composed it with itself (left). Then we took the ropelength minimizer from the isotopy class of the resulting braid (right). By this construction we can see that the minimal ropelength of the resulting braid must be at least double the minimal ropelength of the previous braid. We continue this process infinitely. Using the same principle, we note that the minimum ropelength of the current step is always at most the sum of the minimum ropelengths of the previous two steps. We included an orange box in the $n = 3$ case which demonstrates the intuition of "cutting the out the ideal braid".

As stated in the introduction, we consider this repeated composition construction in order to "forget the endpoints", thereby allowing us to achieve the tightest braid presentation as it would be made from rope. Although we have not yet managed it, we believe that this construction will not depend on the positioning of the endpoints. In other words, we can pick two presentations that are equivalent (in the traditional sense) and after sufficient compositions and tightenings, we will be able to cut out identical braid presentations from the braid. We also believe that we will be able to find the ideal presentation as the middle part of the third composition of our construction–however this is yet to be shown as well.

We have also yet to find any ropelength minimizers, framed or otherwise. This is because computing such ropelength minimizers is actually rather challenging. We do however note that once we are able to find ropelength minimizing framed braid presentations, we will then easily be able to find ropelength minimizing ideal presentations. We turn our attention in the next section to a discusion on framed ropelength minimizers.

6 Ropelength Minimizers

Now that we have a definition of a minimal ropelength for both framed and non-framed braids, the question becomes "how do we find such ropelength minimizers?" Not surprisingly, this question has proven to be very difficult to answer in even the simplest cases. Hence the question of finding such minimizers exceeds the scope of this paper.

With this in mind, we turn our attention to the braid σ_1^n in the braid group on two strands. Although we have yet to actually find the ropelength minimizer for σ_1^n, by constructing models made of real rope, we arrive at two possible candidates for what the ropelength minimizer would look like. This first possible candidate is where one strand is straight, acting as a core, and the other strand wraps around that strand (staying as tightly wrapped as possible) n-times. From here-on-out we call this candidate the "single twist candidate" since out of the two strands, one is being twisted around the other. The other candidate is what we call the "double twist candidate" and is given by letting both the strands twist around each other equally (in a tightest manner). We give a picture of the single and double twist candidates in the follow figure.

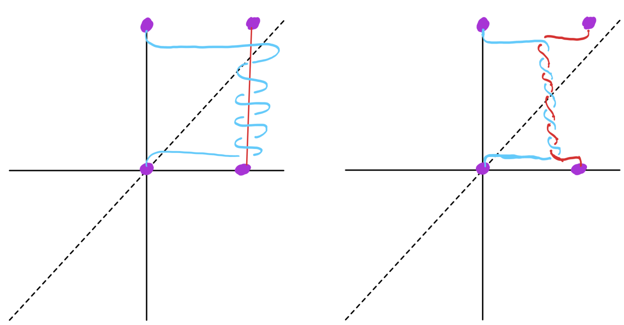

Figure 8: On the left we draw the ideal framed single twist candidate (unthickened so we can better see the behavior of the strands). We note that the red strand should be perfectly straight with the blue strand tightly wrapped around it. On the right, we have drawn the ideal framed double-twist candidate (again, unthickened so we can better see the strands' behavior). We note that both the red strand and the blue strand are moving around each other in a symmetric manner.

Between these two candidates, we conjecture that the single-twist candidate is the ropelength minimizing presentation for σ_1^n. The intuition behind our conjecture is that by allowing one strand to be completely straight, it will use less of the thickened rope than by allowing both strands to move around each other.

Since our model seeks to describe braids made out of real rope, we decided to construct our candidates out of thickened rope and experimentally determine which of the two has a shorter total length. The experiment had the following instructions:

1. Take a piece of rope and measure its length precisely by stretching it and measuring it. Fold the rope in half, stick a pencil in the fold. Hold the ends of the rope tightly. Twist the pencil as many times as you can, counting the total number of half-twists. This, of course, would obtain the double twist candidate
2. Take the rope, fold it over, stick a pencil in the fold. As tightly as you can, keep one strand straight and wrap the other strand around it the same number of half-twists as step 1. Once you have done this, mark the ends of the rope and measure the amount of rope it took to create this presentation.
3. Compare the two total lengths. Which is shorter?

After carrying out this experiment with a length of $29\frac{7}{8}$" long and $\frac{1}{8}$" diameter rope we came to the following data:

Candidate	Diameter	Length After 36 Half-Twists
Single Twist	$1/8''$	$29\frac{7}{8}''$
Double Twist	$1/8''$	$18\frac{1}{4}''$

Meaning that there is a approximately 38.9% decrease in the amount of rope needed to make the single-twist candidate over the double-twist candidate. This particular experiment (although admittedly limited) confirms our intuition that the single-twist candidate is the likely ropelength minimizer for σ_1^n.

7 Where Do We Go From Here?

In this section, we would like to discuss some potential subjects to investigate with our ropelength model. From the previous section, we know that one potential area of interest is to construct ropelength minimizers for non-trivial braids of two or more strands.

Although we spent the previous section talking about the challenges of finding ropelength minimizers, there is another interesting question involving ropelength minimizers: "are ropelength minimizing braid presentations unique?" In [3], Cantarella et al. showed that the ropelength minimizing presentations for knots and links are not necessarily unique by constructing two different classes of ropelength minimizing presentations of links. The reason we ask this question is because we are finding ropelength minimizing presentation of braids by first considering convergent subsequences of framed presentations that minimize ropelength, then taking the limit of repeated compositions of these framed minimizers. Since we are only considering convergent subsequences of these framed presentations, there is a chance that we can extract two subsequences that converge to two different ropelength minimizing presentations within the same isotopy class. By answering this question, we can determine how much control we have over the convergence of these framed subsequences. If the answer is that the presentations are not unique, this opens up other similar questions such as "are there any braid types that whose minimizers are unique?" and will inspire some interesting questions.

In addition to finding ropelength minimizing presentations, there are several other avenues that we are also considering as we go forward. One task is to define the "roundness" of a braid. As mentioned in the Acknowledgements section, this project was based on a Summer REU which sought to define how "round" a given presentation of a braid is. The question was based on the observation that when some braids are made out of physical rope, they can lie flat on a table, whereas other braids will appear much more round and cannot be laid flat. An example of this is taking the standard three stranded braid that is used for braiding hair given by the word $(\sigma_2^{-1}\sigma_1)^n$, and the standard four stranded found in challah bread given by $(\sigma_3\sigma_2^2\sigma_1^{-1}\sigma_2^{-2})^n$. Making these braids as tight as possible out of physical rope, one will find that the standard hair braid looks fairly "flat" and the challah braid looks fairly "round." A good starting point for trying to define the roundness of a braid would be to look at the convex hulls of the cross sections of a ropelength minimzing presentation of the braid, measure the eccentricities of these convex hulls, then average using an integral. However, this will not necessarily work in general as one needs to account for how the convex hull can rotate along the cross sections.

Another interesting question to look at is "how does the ropelength of of a braid presentation relate to the ropelength of a knot or link?" This is actually a fairly broad question as it stands and we can choose to interpret it in a few possible ways. One possible interpretation of the question is "Given a presentation of a knot or link, how does the ropelength of the knot or link compare to the ropelength of the corresponding (framed) braid whose closure is the knot or link?" When considering this question, we would probably want to look at the corresponding braids (equivalent through Markov moves) that have a minimal number of strands. Otherwise, there would be an unnecessary length of rope which could lead to poor bounds on the ropelengh of the knot or link. Another interpretation of the question is "Given a ropelength minimizing presentation of a braid, if we close up the braid in a 'natural' way, can we obtain a ropelength minimizing presentation of the corresponding knot or link?" Of course, the first step to answering the question would be to decide what is meant by the "natural way". It should be obvious that any 'natural' way of closing up the braid would require not adding any length of rope to the presentation; so we cannot use the same method that we used in proving Lemma 6. As a result, we believe that the best way to define a natural presentation would be to take the presentation and, in such a way without changing the thickness and length of rope used, curve the braid until its top endpoints meet its bottom endpoints. This account of a 'natural' braid closure is only intuitive, of course, and it has been very challenging to give a rigorous construction of this process. The reason why we are interested in investigating these two questions is because, in general, computing ropelength and minimal ropelength is very hard for a given knot presentation. As such, it is rather helpful to create bounds on the miniminal ropelength. [4, 5] and many other such papers have been written on creating tighter bounds for minimal ropelength of knots and links, and there is a possibility of creating a tighter bound here. This question is interesting to explore because it might allow for the ropelength model of braids to contribute to existing research.

As we saw, there are many interesting questions that one can ask about the ropelength model for braids. However, much like in physical knot theory, we can create other theories that model different physical phenomenon. Besides the ropelength model for knots and links, there are also so-called "knot energy models" The inspiration for knot energies comes from the following situation as described in [9]: Make a knot out of a conductive wire, then run a current through the wire. Due to Coulomb's law, the strands of the knot will repel each other. This repelling force, in turn, creates an ideal presentation of a the knot. Knot energies are models which seek to model these sorts of situations in order to find other ideal conformations of knots and links. As it turns out, inventor Alexander Graham Bell discovered in [2] that in an (analog) telephone transmission circuit, when transmitting cables are twisted around one another the electrical disturbance from their inductive action is reduced. In other words, the transmitted signal is cleaner when the two transmitting cables form a two stranded braid. This braiding action seems to be a natural way to define braid energies, and minimizing these braid energies will give ideal presentations of such braids. Many interesting questions can arise depending on the type of cross-talk interference being chosen such as: are there cross-talk minimizing braid presentations for each isotopy class of braids? And also what is the optimal way to braid n strands so that cross-talk is minimized? Would this be unique? We can also ask: "What is the relationship between the ideal braid-energy model of a given braid and the ideal ropelength model of the braid?" We conjecture that given the nature of the twisted pair, the ideal presentations of both of these models will likely correspond.

As we can see, there are many interesting questions that can be asked when we develop physical theories of braids.

8 Conclusion

In this paper, we defined a ropelength model for braids based on the works of [3, 6, 7]. We adapted this model by first creating a so-called "framed braid presentation" which required a choice of endpoints. After which we developed a ropelength model for these framed presentations by defining the maximimal thickness for a presentation through employment of an upper-semicontinuous functional created from the global radius of curvature. Then we showed that there exist ropelength minimizing braid presentations on a given frame. However, since frames are problematic for an accurate physical model, we defined the actual ropelength minimizing braid presentation by using a sequence that composes the braid with itself an arbitrary number of times and tightening the resulting braid after each composition. This was done so we can "forget the endpoints" and "cut the ideal presentation" somewhere along the middle of this maximally tightened braid that has been composed with itself some arbitrary number of times. We acknowledge that we have yet to define either of these notions of "forgetting the endpoints" and "cutting out the ideal braid", however the notions make intuitive sense. We showed that this construction is well defined for all braid types, however we also yet to prove that this presentation will be the same regardless of choice of frame. We also talked about the challenges of finding the ropelength minimizers for even the simplest of braid types, i.e those of the form σ_1^n for arbitrary $n \in \mathbb{N}$. We noted that in order to find the desired form of ropelength minimizing presentations, it is likely sufficient to study the framed ropelength minimizers first. We also stated that we believe the framed ropelength minimizer of σ_1^n is likely to be given by a core of rope with the second rope wrapping around it n times, and we gave an experimental and intuitive justification of why we believe this to be true. Lastly, we mentioned potential areas of development and possible uses for this model as well as the development of other, similar physical models for braids.

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