Spectral Sequences From A Linear-Algebra Perspective

Senior Thesis

Raymond Chan
University of California, Davis

Supervisor: Erik Carlsson

Introduction / Motivation

Undeniably, the sphere S^n is one of the simplest topological space that one can imagine in their mind (most likely without visualizing them). It is described with one quadratic equation in $\mathbb{R}^{n+1}$. However, we still cannot completely understand their relationships. For example, consider the following object.

Definition 0.1 Given $i, n \geq 0$, we define the **higher homotopy group** $\pi_i(S^n) = [S^i, S^n]$ to be the group of homotopy classes of maps $S^i \to S^n$. We want to keep the introduction simple, so we will not define the group operation here, see [1, Chapter 9] for the detail.

The groups $\pi_i(S^n)$ for $i < n$ are all trivial. However, no known rules exist to compute, in general, $\pi_i(S^n)$ for $i \geq n$ both large. The group $\pi_3(S^2)$ contains the Hopf fibration (see Section 1.6) and hence is nontrivial. In fact, $\pi_3(S^2) = \mathbb{Z}$.

Computations of these (and many other things, of course) eventually leads to the invention of new techniques to try to tackle them. One of them, and unarguably the most well-known one, is the notion of a spectral sequence, and most notably, the Adams spectral sequence. The amount of information a spectral sequence holds is huge. For example, every spectral sequence has three indices to keep tracks with, and the differentials keep changing direction as we "move along" the sequence. As with ordinary sequence (of numbers), a spectral sequence sometimes can only approximate the thing we are trying to compute, instead of actually computing it (and this is what the notion of convergence entails). Just like ordinary sequences, a spectral sequence doesn't need to have each term written explicitly. There is something analogous to the notion of a recurrence relation called the exact couple and is used very often to bypass telling you an "explicit formula" for each term. All in all, a spectral sequence, in most contexts, is often difficult to visualize.

The Leray spectral sequence is a very general, and hence abstract, class of spectral sequences. However, as shown in [2], there is a very particular context in which we can actually visualize it using linear algebra: the Leray spectral sequence associated to a simplicial map. Also, as long as we uphold some finiteness condition (more precisely, every simplex complex is finite), this Leray spectral sequence will be finite, that is, it stabilizes after finitely many terms. In other words, you don't just get an approximation; you get the actual thing you want to compute.

In this paper, we will study this spectral sequence using basic linear algebra. After stating all the preliminary content and the Leray spectral sequence, three elementary examples will be provided to visualize this sequence in practice: the two double covers of S^1, as well as the projection $S^1 \times S^1 \to S^1$. We choose these particular examples simply because they have some of the simplest triangulations (although the projection is a little bit more involved). Many diagrams will also be provided throughout to facilitate the visualization. We will end this paper with some discussion about the Hopf fibration. More precisely, we will give a minimal triangulation of the Hopf fibration and briefly discuss the Leray spectral sequence associated to it.

Contents

1

Spectral Sequence for Simplicial Maps

In this section, we will provide a linear-algebra perspective on this subject by studying one particular spectral sequence in the context of simplicial complexes. Section 1.1 serves as a foundation of this section by providing all the definitions we will need about simplicial homology. Section 1.2 and 1.3 will be devoted to the general theory of spectral sequence and defining the Leray spectral sequence. Section 1.4 and 1.5 will be devoted to computation of the Leray spectral sequences using concrete elementary examples so that they can be done essentially by hand. Section 1.6 consists of a discussion about the Hopf fibration and a simplicial approximation. We end it with an short and informal discussion on a comparison with a well-known result after trying to compute the Leray spectral sequence of Hopf fibration.

1.1 Preliminary: Simplicial Homology

This subsection will serve as the backbone of the entire section. We will begin by a review on the definition of homology. Then we will spend some time to develop the basic notions about simplicial complex. Finally, we will define the simplicial homology, which effectively gives the main linkage between the two parts.

Definition 1.1.1 Fix an ambient ring R. By a **chain complex** (of R-modules) $C_\bullet$, we mean a sequence of R-modules

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots$$

such that $d_i \circ d_{i+1} = 0$ (or equivalently, $\operatorname{im}(d_{i+1}) \subseteq \ker(d_i)$). The maps d_i are called **boundary maps**. Note that the definition can be generalized easily to any abelian category.

Definition 1.1.2 Given a chain complex $C_\bullet$, an n-**cycle** is an element in $\ker(d_n)$ and an n-**boundary** is an element in $\operatorname{im}(d_{n+1})$. The n-**th homology** is the R-module

$$H_n(C_\bullet) = \ker(d_n)/\operatorname{im}(d_{n+1})$$

(i.e., cycle mod boundary). The **total homology** is the R-module $H_*(C_\bullet) = \bigoplus_{i\in\mathbb{Z}} H_i(C_\bullet)$.

To simplify notation, it is accustomed to suppress the subscripts on boundary maps, as it often can be read easily from the context.

Next up, we will discuss the notion of a simplicial complex and simplicial maps for our next important building block.

Definition 1.1.3 An **(abstract) simplicial complex** X consists of a finite set $V(X)$ (called the **vertex** set) together with a collection of subsets $S(X)$ of $V(X)$ satisfying the following conditions:

(i) if $v \in V(X)$ then $\{v\} \in S(X)$;
(ii) if $\sigma \in S(X)$ and $\tau \subseteq \sigma$ is nonempty then $\tau \in S(X)$.

Given an integer p, a p-**simplex** is an element $\sigma \in S(X)$ such that $|\sigma| = p + 1$. We also say in this case that the **dimension** of σ is p. A **subcomplex** of X consists of a subset of $V(X)$ and a subset of $S(X)$ such that they form a simplicial complex on their own.

A simplicial complex is a combinatorial way to study a topological space. More precisely, a simplicial complex is what we get by triangulating a space (if this can actually be done). We can make sense of this formally using the geometric realization construction. The intuition behind geometric realization is simple. Given a p-simplex σ, we can think of it as a p-dimensional tetrahedron whose vertices are elements in σ, whose edges are the subsets of σ with size 2, and (in general) whose i-dimensional faces are the subsets of σ with size $i+1$. Then we can just glue all the tetrahedrons together. More formally speaking:

Definition 1.1.4 (i) A set of points $\{a_1, \cdots, a_n\}$ in $\mathbb{R}^N$ is said to be **affinely dependent** if there exist $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, not all zero, such that $\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$ and $\lambda_1 + \cdots + \lambda_n = 0$. Otherwise, these points are **affinely independent**. In particular, any set of linearly independent points is affinely independent.

(ii) Given an integer p, the **standard p-simplex** Δ^p is the convex hull of $p+1$ affinely independent points in $\mathbb{R}^{p+1}$. Up to homeomorphism, one can simply pick the points $\mathbf{e}_1 = (1, 0, \cdots, 0), \cdots, \mathbf{e}_{p+1} = (0, \cdots, 0, 1)$ in $\mathbb{R}^{p+1}$ to span the convex hull. In particular, Δ^2 is a (solid) triangle and Δ^3 is a (solid) tetrahedron.

Definition 1.1.5 (Geometric Realization) Given an abstract simplicial complex X, we let $N \geq |V(X)|$ and choose an embedding of $V(X)$ into $|V(X)|$ affinely independent points in $\mathbb{R}^N$. Identify each p-simplex of X with the standard p-simplex Δ^p (the convex hull of the corresponding $p+1$ points). The **geometric realization** $|X|$ is the topological space given by the union of all the standard simplices as we run through all simplices in $S(X)$.

Definition 1.1.6 A topological space X is **triangulable** if there is a simplicial complex $\widetilde{X}$ such that $|\widetilde{X}| \cong X$. In this case, we call $\widetilde{X}$ a **triangulation** of X.

Remark 1.1.7 If the reader is familiar with CW-complexes, then the geometric realization can also be visualized in the same way, except we are now attaching (higher-dimensional) tetrahedrons instead of balls. Even better, the geometric realization is in fact a CW-complex, because Δ^p is homeomorphic to the p-ball D^p. Note of terminology: some people uses the word *simplicial complex* to mean the geometric realization of an abstract simplicial complex. But for us, a simplicial complex will always mean the abstract one.

After some formal discussion with trianguable spaces, let's return to the basic and define a couple more things that will be used throughout the paper.

Definition 1.1.8 Given a simplicial complex X and an integer n, the **n-skeleton** of X is the subcomplex X^n consisting of all p-simplices with $p \leq n$. Just as a matter of fact: the skeletons form an increasing **filtration**

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$$

that terminates to the right with X.

Definition 1.1.9 Given two simplicial complexes X and Y, a **simplicial map** $f : X \to Y$ is a function $f : V(X) \to V(Y)$ such that for each simplex $\sigma \in S(X)$, we have $f(\sigma) \in S(Y)$. Simplicial complexes and simplicial maps form a category **SimComp**. With some effort, one can check that the geometric realization defines a functor $| - | :$ **SimComp** $\to$ **Top**.

Next up is just a technical remark.

Remark 1.1.10 (How to get a simplicial map, at least in theory) Let X, Y be simplicial complexes and $\phi : |X| \to |Y|$ a continuous function between their geometric realizations.

(i) For any $v \in X$, we define the **star** of v, denoted by $\operatorname{st} v$, to be the union of all simplices containing v as a vertex. (Note: $\operatorname{st} v$ needs not be a subcomplex in general.)

(ii) A simplicial map $f : X \to Y$ is a **simplicial approximation** of ϕ if for every vertex $v \in V(X)$, we have

$$\phi(\operatorname{st} v) \subseteq \operatorname{st} f(v)$$

(Note: We identify, with an abuse of notation, $\operatorname{st} v$ and $\operatorname{st} f(v)$ with their corresponding subspace in $|X|$ and $|Y|$, respectively.) In this case, we also have $\phi \simeq |f|$, where $\simeq$ indicates homotopy.

(iii) (**Simplicial Approximation Theorem**) For n sufficiently large, there is a simplicial map $f : \mathrm{Sd}^n(X) \to Y$ approximating ϕ, where Sd is the (**berycentric**) **subdivision** functor **SimComp** $\to$ **SimComp** that sends a simplicial complex to a "more-refined version" of itself. More precisely, we can define it as follows. Let **FinPos** be the category of finite posets and monotone functions. Let U : **SimComp** $\to$ **FinPos** be the forgetful functor $X \mapsto S(X)$. Given a (finite) poset $(P, \leq)$, a **flag** of P is a chain of elements $p_1 \leq p_2 \leq \cdots \leq p_m$. Then we can assoiate $(P, \leq)$ with the simplicial complex whose vertex set is P and simplices are the flags in P. This assignment is functorial, so we have a functor $\mathrm{F}\ell$: **FinPos** $\to$ **SimComp**. Then define $\mathrm{Sd} = \mathrm{F}\ell \circ U$. Bypassing all the categorical terms, $\mathrm{Sd}(X)$ is the simplicial complex whose vertices are exactly the simplicies of X and whose simplices (of $\mathrm{Sd}(X)$) are chains of inclusions of simplices of X. We shall also point out (according to [4]) that the canonical map

$$\begin{array}{cccc} g: & V(X) & \to & V(\mathrm{Sd}(X)) \\ & v & \mapsto & \{v\} \end{array}$$

induces a homeomorphism $|g| : |X| \cong |\mathrm{Sd}(X)|$ (not too obviously).

Every simplicial complex can be associated naturally with a chain complex. This ties up a connection between the two seemingly-unrelated concepts that we just introduced.

Definition 1.1.11 Let X be a simplicial complex.

(i) Fix a total order on the vertex set $V(X)$. For a p-simplex σ, the **orientation** of σ is the ordering on the elements of σ given by the total order on $V(X)$.

(ii) Given an integer p, define $C_p(X)$ to be the $\mathbb{C}$-vector space with basis $\{\sigma \in S(X) : |\sigma| = p+1\}$, the set of all p-simplices. If $\sigma = \{\sigma_0 < \cdots < \sigma_p\}$ is a p-simplex, then we write $[\sigma_0, \cdots, \sigma_p]$ (in this particular order) for the basis element of $C_p(X)$ corresponding to σ. Define a $\mathbb{C}$-linear map $d_p : C_p(X) \to C_{p-1}(X)$ on basis by

$$d_p([\sigma_0, \cdots, \sigma_p]) = \sum_{i=0}^{p} (-1)^i [\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p]$$

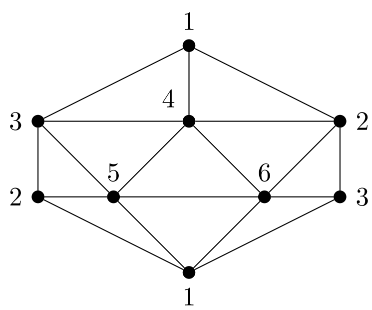
(and extend by $\mathbb{C}$-linearity), where $\widehat{\ }$ indicates omission.

(iii) If $\gamma \in S_{p+1}$ is a permutation of the set $\{0, \cdots, p\}$, then we define

$$[\sigma_{\gamma(0)}, \cdots, \sigma_{\gamma(p)}] = \mathrm{sgn}(\gamma)[\sigma_0, \cdots, \sigma_p],$$

where $\mathrm{sgn}(\gamma)$ (sometimes also written as $(-1)^\gamma$) is the sign of the permutation γ.

Example 1.1.12 The sign rule (iii) above allows bigger flexibility when writing down a basis element in $C_p(X)$. It can be useful in some situation, like identifying cycles within a simplicial complex, which can be used to identify the kernel of the boundary map. For example, in the triangulation



of $\mathbb{R}P^2$ (in which all the triangles are 2-simplices), we can see that the edges $[1,3]$, $[3,2]$, $[2,5]$, $[5,4]$, $[4,1]$ form a loop, so $d([1,3]+[3,2]+[2,5]+[5,4]+[4,1]) = 0$. The sign rule above allows us to convert $[1,3]+[3,2]+[2,5]+[5,4]+[4,1]$ to an actual element $[1,3]-[2,3]+[2,5]-[4,5]-[1,4]$ in C_1 with the cost of minus signs. (For those who wonder: This triangulation is obtained from the fact that $\mathbb{R}P^2$ is the same thing as the disk D^2 but identifying antipodal points on the boundary circle.)

Remark 1.1.13 When the vertices of a simplicial complex is labelled as $1, 2, 3, \cdots$, the orientation of a simplex is, by convention, given by the usual ordering on $\mathbb{N}$. We will label the vertices of a simplicial complex with $\mathbb{N}$ as much as possible, but in some occasion a different labelling makes thing clearer, as we shall see later in this section.

Lemma 1.1.14 With the notation in Definition 1.1.11, $C_\bullet(X)$ defines a chain complex. In particular, the homology of $C_\bullet(X)$ is called the **simplicial homology** of X (or $|X|$). We simply write $H_n(X)$ for the n-th simplicial homology of $C_\bullet(X)$ and write $H_*(X)$ for the total simplicial homology.

Proof Let σ be a p-simplex. Then

$$\begin{aligned}
&d_{p-1}d_p([\sigma_0, \cdots, \sigma_p]) \\
&= d_{p-1}\left(\sum_{i=0}^{p}(-1)^i[\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p]\right) \\
&= \sum_{i=0}^{p}(-1)^i d_{p-1}([\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p]) \\
&= \sum_{i=0}^{p}(-1)^i \left(\sum_{j<i}(-1)^j[\sigma_0, \cdots, \widehat{\sigma_j}, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p] + \sum_{j>i}(-1)^{j-1}[\sigma_0, \cdots, \widehat{\sigma_i}, \cdots, \widehat{\sigma_j}, \cdots, \sigma_p]\right) \\
&= \sum_{i=1}^{p}\sum_{j<i}(-1)^{i+j}[\sigma_0, \cdots, \widehat{\sigma_j}, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p] + \sum_{j=1}^{p}\sum_{j<i}(-1)^{i+j-1}[\sigma_0, \cdots, \widehat{\sigma_j}, \cdots, \widehat{\sigma_i}, \cdots, \sigma_p] \\
&= 0,
\end{aligned}$$

where, in the second-last line, we interchange the indices i and j in the second sum so that it is apparent on why the two sums cancel out. Therefore $d^2 = 0$, as desired. □

Remark 1.1.15 The simplicial homology of a simplicial complex X agrees with the singular homology of $|X|$ (though showing this requires quite a bit of effort). Thus, if Y is another triangulations of $|X|$, then it turns out that $H_n(X) = H_n(Y)$. In particular, the simplicial homology of a trianguable space is well-defined (independent of the choice of triangulation).

1.2 What Is a Spectral Sequence?

In this subsection, we will define, following [5], spectral sequences in the category of finite-dimensional vector spaces. We shall simply note here that the definition can be generalized to any abelian category for the broadest generality.

Definition 1.2.1 Fix an ambient field k. A $\mathbb{Z}$**-graded** (resp., $(\mathbb{Z}, \mathbb{Z})$**-bigraded**) k-vector space E^* (resp., $E^{*,*}$) is a vector space of the form

$$\bigoplus_{p\in\mathbb{Z}} E^p \quad \left(\text{resp., } \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}\right)$$

where each summand is a k-vector space.

All graded (resp., bigraded) objects that we will considered are $\mathbb{Z}$-graded (resp., $(\mathbb{Z}, \mathbb{Z})$-bigraded), so we will drop the prefix from now on. An $\mathbb{N}$-graded (resp., $(\mathbb{N}, \mathbb{N})$-bigraded) vector space is automatically $\mathbb{Z}$-graded (resp., $(\mathbb{Z}, \mathbb{Z})$-bigraded) by inserting the zero vector space in the remaining slots.

Definition 1.2.2 A **spectral sequence** of k-vector spaces of **homological type** consists of

(i) a collection of bigraded vector spaces $E = \{E_r^{*,*}\}_r$ (so yes, a spectral sequence is technically **trigraded**);

(ii) for each p, q, r, there is a (linear) map

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p-r,q+r-1}$$

called a **differential** with **bidegree** $(-r, r-1)$,

such that $d_r \circ d_r = 0$ whenever the composition makes sense (to simplify notation, we have suppressed all the superscripts). We also require

$$E_{r+1}^{p,q} \cong \ker(d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}) / \operatorname{im}(d_r : E_r^{p+r,q-r+1} \to E_r^{p,q}).$$

The index r of $\{E_r^{*,*}\}_r$ can begin at any integer that one finds convenient, but commonly it starts at either 0, 1, or 2. For any fixed r, the bigraded space $E_r^{*,*}$ is called the E_r**-page** of the spectral sequence.

Remark 1.2.3 (i) Note that the E_r-page together with the differentials d_r determine the E_{r+1}-page (up to isomorphism) but not the differentials d_{r+1}.

(ii) It is often useful to think of a spectral sequence as a (infinite) book. The r-th page of the book is just the E_r-page of the spectral sequence, where each $E_r^{p,q}$ constitutes a word in that page. Flipping to the $(r+1)$-th page is equivalent to taking homology at every word on the r-th page.

A spectral sequence of **cohomological type** is define analogously, except now the differentials have bidegree $(r, 1-r)$.

To help us better understand spectral sequence (of homological type), it is often useful to draw it on a grid. We picture $E_r^{p,q}$ sitting in position (p, q) on the E_r-page. Conventionally, (p, q) is meant to understand as a coordinate, in contrast to a matrix entry. For example, the E_0-page looks like

$$\begin{tikzcd}
E_0^{-2,4} \arrow[d, "d"] & E_0^{-1,4} \arrow[d, "d"] & E_0^{0,4} \arrow[d, "d"] & E_0^{1,4} \arrow[d, "d"] & E_0^{2,4} \arrow[d, "d"] \\
E_0^{-2,3} \arrow[d, "d"] & E_0^{-1,3} \arrow[d, "d"] & E_0^{0,3} \arrow[d, "d"] & E_0^{1,3} \arrow[d, "d"] & E_0^{2,3} \arrow[d, "d"] \\
E_0^{-2,2} \arrow[d, "d"] & E_0^{-1,2} \arrow[d, "d"] & E_0^{0,2} \arrow[d, "d"] & E_0^{1,2} \arrow[d, "d"] & E_0^{2,2} \arrow[d, "d"] \\
E_0^{-2,1} \arrow[d, "d"] & E_0^{-1,1} \arrow[d, "d"] & E_0^{0,1} \arrow[d, "d"] & E_0^{1,1} \arrow[d, "d"] & E_0^{2,1} \arrow[d, "d"] \\
E_0^{-2,0} \arrow[d, "d"] & E_0^{-1,0} \arrow[d, "d"] & E_0^{0,0} \arrow[d, "d"] & E_0^{1,0} \arrow[d, "d"] & E_0^{2,0} \arrow[d, "d"] \\
E_0^{-2,-1} & E_0^{-1,-1} & E_0^{0,-1} & E_0^{1,-1} & E_0^{2,-1}
\end{tikzcd}.$$

The E_1-page looks like

$$\begin{tikzcd}
E_0^{-2,4} & E_0^{-1,4} \arrow[l, "d"'] & E_0^{0,4} \arrow[l, "d"'] & E_0^{1,4} \arrow[l, "d"'] & E_0^{2,4} \arrow[l, "d"'] \\
E_0^{-2,3} & E_0^{-1,3} \arrow[l, "d"'] & E_0^{0,3} \arrow[l, "d"'] & E_0^{1,3} \arrow[l, "d"'] & E_0^{2,3} \arrow[l, "d"'] \\
E_0^{-2,2} & E_0^{-1,2} \arrow[l, "d"'] & E_0^{0,2} \arrow[l, "d"'] & E_0^{1,2} \arrow[l, "d"'] & E_0^{2,2} \arrow[l, "d"'] \\
E_0^{-2,1} & E_0^{-1,1} \arrow[l, "d"'] & E_0^{0,1} \arrow[l, "d"'] & E_0^{1,1} \arrow[l, "d"'] & E_0^{2,1} \arrow[l, "d"'] \\
E_0^{-2,0} & E_0^{-1,0} \arrow[l, "d"'] & E_0^{0,0} \arrow[l, "d"'] & E_0^{1,0} \arrow[l, "d"'] & E_0^{2,0} \arrow[l, "d"'] \\
E_0^{-2,-1} & E_0^{-1,-1} \arrow[l, "d"'] & E_0^{0,-1} \arrow[l, "d"'] & E_0^{1,-1} \arrow[l, "d"'] & E_0^{2,-1} \arrow[l, "d"']
\end{tikzcd}.$$

The E_2-page looks like

$$\begin{tikzcd}
E_0^{-2,4} & E_0^{-1,4} & E_0^{0,4} & E_0^{1,4} & E_0^{2,4} \\
E_0^{-2,3} & E_0^{-1,3} & E_0^{0,3} \arrow[ull, "d"] & E_0^{1,3} \arrow[ull, "d"] & E_0^{2,3} \arrow[ull, "d"] \\
E_0^{-2,2} & E_0^{-1,2} & E_0^{0,2} \arrow[ull, "d"] & E_0^{1,2} \arrow[ull, "d"] & E_0^{2,2} \arrow[ull, "d"] \\
E_0^{-2,1} & E_0^{-1,1} & E_0^{0,1} \arrow[ull, "d"] & E_0^{1,1} \arrow[ull, "d"] & E_0^{2,1} \arrow[ull, "d"] \\
E_0^{-2,0} & E_0^{-1,0} & E_0^{0,0} \arrow[ull, "d"] & E_0^{1,0} \arrow[ull, "d"] & E_0^{2,0} \arrow[ull, "d"] \\
E_0^{-2,-1} & E_0^{-1,-1} & E_0^{0,-1} \arrow[ull, "d"] & E_0^{1,-1} \arrow[ull, "d"] & E_0^{2,-1} \arrow[ull, "d"]
\end{tikzcd}.$$

(Of course, these pages extend to infinity in all directions.)

Here is one more terminology to describe a spectral sequence.

Definition 1.2.4 A spectral sequence $\{E_r^{*,*}\}_r$ is **first-quadrant** if in each page, $E_r^{p,q} = 0$ for all $p, q < 0$.

We now state the notion of convergence of a spectral sequence. The one in the following can be generalized to any abelian category. However, this turns out to be way too broad for our purpose. We will then provide an alternative way of convergence that's well-fit to our situation.

Definition 1.2.5 Let $E = \{E_r^{*,*}\}_r$ be a first-quadrant spectral sequence. Then by first-quadrantness, for all $p, q \in \mathbb{Z}$ there is an $r_0 = r_0(p,q)$ sufficiently large such that for any $r \geq r_0$, the $E_r^{p,q}$-spot of the spectral sequence looks like

$$\begin{tikzcd}
0 & & \\
& E_r^{p,q} \arrow[ul, "d"'] & \\
& & 0 \arrow[ul, "d"']
\end{tikzcd}.$$

(The assumption on first-quadrantness can be reasonably weaken to allow finitely many nonzero entries in other quadrants, but many spectral sequences in practice are predominantly first-quadrant.) In this case, we have $E_r^{p,q} = E_{r+1}^{p,q} = \cdots$ (because the kernel of the upper map is everything, while the image of the lower map is 0). Let $E_\infty^{p,q}$ be this common value. We say that the spectral sequence E **converges** to a *graded* vector space H^* if there is a decreasing filtration $F^\bullet$ on H^* such that

$$E_\infty^{p,q} \cong F^p H^{p+q}/F^{p+1}H^{p+q}$$

or an increasing filtration $F^\bullet$ such that

$$E_\infty^{p,q} \cong F^p H^{p+q}/F^{p-1}H^{p+q}.$$

(Note: the increasing-ness and decreasing-ness of a filtration should only be seen as a technicality of the definition, in the sense that the filtration often has finite length ad hence the two notions are equivalent.)

Similar to ordinary sequence of numbers, as $N \to \infty$, the E_N-page approximates H^*. But what if the sequence actually lands onto the limit?

Definition 1.2.6 A spectral sequence $\{E_r^{*,*}\}_r$ **collapses** at the E_N-page if $d_r = 0$ for all $r \geq N$. This will always be the case for us, due to the fact that we will only deal with spectral sequences with finitely many nonzero entries.

Remark 1.2.7 (Convergence of Collapsed Spectral Sequences) Suppose a (first-quadrant or not) spectral sequence $E = \{E_r^{*,*}\}_r$ collapses on the E_N-page. Then every spot of the spectral sequences stabilizes (i.e., $E_N^{p,q} = E_\infty^{p,q}$ for all p, q). From here, it is not hard to deduce that E converges in the sense of Definition 1.2.5. Namely, you take the whole E_N-page as the space H^*. (Note: Every bigraded vector space can be seen as a graded vector space by, for example, $\bigoplus_{p,q\in\mathbb{Z}} E^{p,q} = \bigoplus_{q\in\mathbb{Z}} E^{*,q}$, where for any fixed q, $E^{*,q} = \bigoplus_{p\in\mathbb{Z}} E^{p,q}$.) We will leave it to the interested reader to convince themselves that a suitable filtration $F^\bullet$ of H^* can then be constructed canonically to do the job.

1.3 The Leray Spectral Sequence

Historically, the Leray spectral sequence is one of the earliest spectral sequences ever invented, and it specializes to some other well-known spectral sequences like the *Serre spectral sequence* of a Serre fibration. we now explain an explicit form of the Leray spectral sequence for simplicial maps, as explained in [2]. The description of each E_r-term is given in Theorem 1.3.7. We give a convergence result in Theorem 1.3.9. Proofs of both theorems will not be reproduced in this paper, and the relevant reference is given instead. To see how this Leray spectral sequence is specialized from the general version using Leray cosheaf, see [2, Section 3].

Definition 1.3.1 Let X and Y be simplicial complexes and $f : X \to Y$ a simplicial map. Write Y^n for the n-skeleton of Y and $X_n = f^{-1}(Y^n)$. Given integers p, q, we define $X_{p,q}$ to be the set of all $(p + q)$-simplices in X_p. Let $C_{p,q}$ be the subspace of $C_{p+q}(X)$ spanned by $X_{p,q}$.

The boundary map restricts to various maps between the $C_{p,q}$'s. We will give a proof to the following one so that the reader can see what is happening. The same logic applies to give the next, more general, corollary.

Proposition 1.3.2 The boundary map $d : C_{p+q}(X) \to C_{p+q-1}(X)$ restricts to a map $d : C_{p,q} \to C_{p,q-1}$.

Proof Let $\sigma \in X_{p,q}$, which is a $(p + q)$-simplex in X_p. Let σ' be a simplex obtained from σ by deleting an element. It is thus a $(p + q - 1)$-simplex. Although the dimension of $f(\sigma')$ might be lower than the dimension of $f(\sigma)$, it is still within the same *skeleton* of $f(\sigma)$. Therefore, the restricted map d maps every basis element of $C_{p,q}$ to a linear combination of basis elements of $C_{p,q-1}$, as desired. □

Corollary 1.3.3 For a fixed integer r, the boundary map $d : C_{p+q}(X) \to C_{p+q-1}(X)$ restricts to a map $d : C_{p-r,q+r} \to C_{p-r,q+r-1}$. Therefore, the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p-r,q+r+1} \arrow[r, "d"] \arrow[d, hook] & C_{p-r,q+r} \arrow[r, "d"] \arrow[d, hook] & C_{p-r,q+r-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & C_{p,q+1} \arrow[r, "d"] \arrow[d, hook] & C_{p,q} \arrow[r, "d"] \arrow[d, hook] & C_{p,q-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & C_{p+r,q-r+1} \arrow[r, "d"] \arrow[d, hook] & C_{p+r,q-r} \arrow[r, "d"] \arrow[d, hook] & C_{p+r,q-r-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & C_{p+q+1}(X) \arrow[r, "d"] & C_{p+q}(X) \arrow[r, "d"] & C_{p+q-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

commute.

To define the Leray spectral sequence, we will need to define a few more terms.

Definition 1.3.4 Given integers p, q, r, define $B_r^{p,q}$ to be the subspace of $C_{p,q}$ given by

$$B_r^{p,q} = C_{p,q} \cap d(C_{p+r,q-r+1})$$

and define $Z_r^{p,q}$ to be the subspace of $C_{p,q}$ given by

$$Z_r^{p,q} = C_{p,q} \cap d^{-1}(C_{p-r,q+r-1}),$$

where both d's are taken as the d's appearing in the lowest row in the commutative diagram in Corollary 1.3.3 (i.e., the d that appears in the chain complex $C_\bullet(X)$).

Warning 1.3.5 We want to clear a potentially major mistake the reader might have at this point. Namely, the map d in the definition of $B_r^{p,q}$ can be taken as the d appearing on the third row in the commutative diagram in Corollary 1.3.3. There is no harm to do so. However, the map d in the definition of $Z_r^{p,q}$ must be taken as the chain complex one. There is a problem if one take it as the one on the first row in the commutative diagram in Corollary 1.3.3. Namely, the d's on the first row are obtained via *restrictions* of the d's on the lowest row, so when taking the preimage of $C_{p-r,q+r-1}$, we don't necessarily land perfectly inside $C_{p-r,q+r}$. In fact, (iii) in the following proposition will not be true if we take the wrong d.

Proposition 1.3.6 Given integers p, q, r, we have

(i) $Z_{r-1}^{p-1,q+1} \subseteq Z_r^{p,q}$;
(ii) $B_{r-1}^{p,q} \subseteq B_r^{p,q}$;
(iii) $B_r^{p,q} \subseteq Z_r^{p,q}$.

Proof

(i) Let $z \in Z_{r-1}^{p-1,q+1} = C_{p-1,q+1} \cap d^{-1}(C_{(p-1)-(r-1),(q+1)+(r-1)-1})$. Since $C_{p-1,q+1} \subseteq C_{p,q}$ (every $(p+q)$-simplex in X_{p-1} is automatically in X_p by construction), we have $z \in Z_r^{p,q} = C_{p,q} \cap d^{-1}(C_{p-r,q+r-1})$.

(ii) Let $b \in B_{r-1}^{p,q} = C_{p,q} \cap d(C_{p+(r-1),q-(r-1)+1})$. Then $b \in C_{p,q}$. Find a preimage of b in $C_{p+r-1,q-r+2}$ and write it as a linear combination of basis elements. Each such basis element is, by definition, a $(p+q+1)$-simplex in

$$X_{p+r-1} = f^{-1}(Y^{p+r-1}) \subseteq f^{-1}(Y^{p+r}) = X_{p+r}.$$

Therefore, $b \in d(C_{p+r,q-r+1})$ and hence $b \in B_r^{p,q}$.

(iii) Let $b \in B_r^{p,q} = C_{p,q} \cap d(C_{p+r,q-r+1})$. Then $b \in C_{p,q}$. Find a preimage of b in $C_{p+r,q-r+1}$ and call it a. Then

$$d(b) = d^2(a) = 0 \in C_{p-r,q+r-1}.$$

Therefore, $b \in d^{-1}(C_{p-r,q+r-1})$ and hence $b \in Z_r^{p,q}$. □

We now have everything we need to present the formula for the Leray spectral sequences. In contrast to most spectral sequences in the literature (which only specify their E_2-page), this formula specifies all pages of the Leray spctral sequence.

Theorem 1.3.7 (The Leray Spectral Sequence) ([2, Section 2.2]) Given a simplicial map $f : X \to Y$, there exists a spectral sequence $\{E_r^{*,*}\}_{r\geq 0}$ of homological type with E_r-page

$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p-1,q+1} + B_{r-1}^{p,q}}$$

(where $+$ indicates subspace sum, i.e., it gives the smallest subspace containing both summand). The differentials $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ are given by "restrictions" of the boundary maps of the chain complex $C_\bullet(X)$ (i.e., sending every coset representative via the boundary map d on $C_\bullet(X)$ to another coset representative). We called this sequence the **Leray spectral sequence**.

Most spectral sequences in the literature also come with a convergence statement. Our Leray spectral sequence also comes with a onvergence statement. In our case, the Leray spectral sequence will converge to the total homology of the domain X of the simplicial map. Before we state this formally, here is one more small definition about simplicial complex.

Definition 1.3.8 Given a simplicial complex X, the **dimension** of X, written as $\dim X$, is the largest integer p such that X has at least one p-simplex but has no n-simplices for any $n > p$.

Theorem 1.3.9 ([2, Theorem 2.7]) Let $f : X \to Y$ be a simplicial map and $m = \dim Y$. Let $\{E_r^{*,*}\}_r$ be the Leray spectral sequence associated to f. Then there is a canonical isomorphism of vector spaces

$$H_n(X) \cong \bigoplus_{p=0}^{k} E_{m+1}^{p,n-p}.$$

(Pictorially, we are summing along the antidiagonal $y = -x$.) Hence, the Leray spectral sequence converges to $H_*(X) = \bigoplus_{n\in\mathbb{Z}} H_n(X)$.

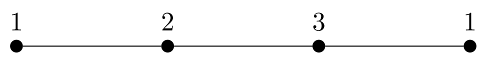
1.4 Two Double Covers of S^1

We will demonstrate the Leray sepctral sequence with two locally-identical maps. Let S^1 be the unit circle on the complex plane. By a **double cover** of S^1, we mean a two-to-one (continuous, of course) mapping onto S^1. Let $f : S^1 \to S^1$ be the double cover given by wrapping around S^1 twice, or equivalently, "doubling the travelling speed." (That is, $z \mapsto z^2$.) Let $g : S^1 \sqcup S^1 \to S^1$ be the double cover that acts as the identity map on each copy of S^1. Note that while the two maps have very distinct global structure, their local structures are identical. More precisely, given a point $z \in S^1$ (in the codomain), the pullback of any small open arc containing z (i.e., a neighborhood of z) along f and g are homeomorphically identical as a disjoint union of two open arcs.

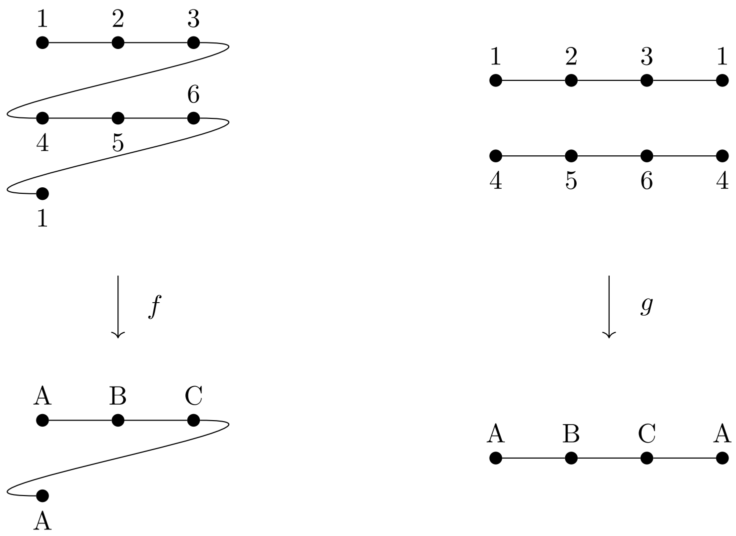
Remark 1.4.1 Such a map, where the pullback of a small neighborhood is a disjoint union of open subsets, each homeomorphic to this small neighborhood, is called a **covering map**. A covering map is automatically a **fibration**, i.e., satisfying the **homotopy lifting property**. See [1, Chapter 3, 7] and [5, Section 4.3] for a more detail treatment.

In this subsection, we will apply the Leray spectral sequence to the two maps above and see how the computation differ.

The circle S^1 has a very simple triangulation:



so that $V(S^1) = \{1, 2, 3\}$ and $S(S^1) = \{1, 2, 3, 12, 23, 13\}$. (To simplify notation, we will omit the unnecessary brackets and commas. For example, by 12, we mean the simplex $\{1, 2\}$. The same notation will be used to indicate a basis element in the chain complex. This would causes no confusion since the number of vertices is small enough.) But since we are working with two-to-one maps, the triangulation of the domains (of f and g) should have twice as many vertices as the codomain. Thus, we arrive at the following simplicial approximation of f and g.



That is, f maps 1 and 4 to A, 2 and 5 to B, and so on.

Now we first compute the Leray spectral sequence for f. Let $X = \mathrm{dom}(f) = S^1$ and $Y = \mathrm{codom}(f) = S^1$. Then

$$Y^0 = \{A, B, C\} \quad \text{and} \quad Y^1 = \{A, B, C, AB, BC, AC\} = Y^2 = Y^3 = \cdots$$

so that

$$X_0 = f^{-1}(Y^0) = \{1, 2, 3, 4, 5, 6\}, \quad X_1 = f^{-1}(Y^1) = \{1, 2, 3, 4, 5, 6, 12, 23, 34, 45, 56, 16\} = X_2 = X_3 = \cdots .$$

Some of the useful nonempty $X_{p,q}$ are listed below:

$$X_{0,0} = \{\text{0-simplices in } X_0\} = \{1, 2, 3, 4, 5, 6\},$$

$$X_{1,-1} = \{\text{0-simplices in } X_1\} = \{1, 2, 3, 4, 5, 6\},$$

$$X_{1,0} = \{\text{1-simplices in } X_1\} = \{12, 23, 34, 45, 56, 16\}.$$

Now we compute the spectral sequence. Note that $E_r^{p,q}$ is, by definition, a quotient of some submodules of $C_{p,q}$ (Theorem 1.3.7), so $E_r^{p,q}$ is automatically 0 whenever $X_{p,q} = \emptyset$. This happens, of course, when $p < 0$ or the simplicial complex X has no $(p+q)$-simplices, so this outright eliminates many unnecessary computations. We thus left with, for each fix $p \geq 0$, $E_r^{p,-p}$ and $E_r^{p,-p+1}$ the only possible nonzero terms in the spectral sequence. We now begin the computation for the E_0-page:

$$E_0^{0,0} = \frac{Z_0^{0,0}}{Z_{-1}^{-1,1} + B_{-1}^{0,0}} = \frac{C_{0,0} \cap d^{-1}(C_{0,-1})}{C_{-1,1} \cap d^{-1}(C_{0,-1}) + C_{0,0} \cap d(C_{-1,2})} = \frac{C_{0,0}}{0+0} = C_{0,0},$$

$$E_0^{0,1} = 0 \quad (\text{since } X_{0,1} = \emptyset)$$

$$E_0^{1,-1} = \frac{Z_0^{1,-1}}{Z_{-1}^{0,0} + B_{-1}^{1,-1}} = \frac{C_{1,-1} \cap d^{-1}(C_{1,-2})}{C_{0,0} \cap d^{-1}(C_{1,-2}) + C_{1,-1} \cap d(C_{0,1})} = \frac{C_{1,-1}}{C_{0,0} + 0} = 0,$$

$$E_0^{1,0} = \frac{Z_0^{1,0}}{Z_{-1}^{0,1} + B_{-1}^{1,0}} = \frac{C_{1,0} \cap d^{-1}(C_{1,-1})}{C_{0,1} \cap d^{-1}(C_{1,-1}) + C_{1,0} \cap d(C_{0,2})} = \frac{C_{1,0}}{0+0} = C_{1,0},$$

and in general, for any $p \geq 1$, we have $C_{p,-p} = C_{p-1,-(p-1)}$, and so

$$\begin{aligned} E_0^{p,-p} &= \frac{Z_0^{p,-p}}{Z_{-1}^{p-1,-p+1} + B_{-1}^{p,-p}} \\ &= \frac{C_{p,-p} \cap d^{-1}(C_{p,-p-1})}{C_{p-1,-p+1} \cap d^{-1}(C_{p,-p-1}) + C_{p,-p} \cap d(C_{p-1,-p+2})} \\ &= \frac{C_{p,-p}}{C_{p-1,-p+1} + \text{some unimportant junk}} \\ &= 0. \end{aligned}$$

(By unimportant junk, we mean they are useless for the computation, as the first summand already contains the entire vector space in the numerator.) Similarly,

$$\begin{aligned} E_0^{p,-p+1} &= \frac{Z_0^{p,-p+1}}{Z_{-1}^{p-1,-p+2} + B_{-1}^{p,-p+1}} \\ &= \frac{C_{p,-p+1} \cap d^{-1}(C_{p,-p})}{C_{p-1,-p+2} \cap d^{-1}(C_{p,-p}) + C_{p,-p+1} \cap d(C_{p-1,-p+3})} \\ &= \frac{C_{p,-p+1}}{C_{p-1,-p+2} + \text{some unimportant junk}} \\ &= 0. \end{aligned}$$

Hence, the E_0-page looks like

$$\begin{tikzcd} 0 \arrow[d, "d"] & 0 \arrow[d, "d"] & 0 \arrow[d, "d"] \\ C_{0,0} \arrow[d, "d"] & C_{1,0} \arrow[d, "d"] & 0 \arrow[d, "d"] \\ 0 & 0 & 0 \end{tikzcd} .$$

Instead of using the formula to compute the E_1-page, we will do this by definition (to better demonstrate how a spectral sequence works). By Definition 1.2.2, the terms in the E_1-page is given by taking homology of the E_0-page, so the E_1-page looks like

$$\begin{tikzcd} 0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\ C_{0,0} & C_{1,0} \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\ 0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"'] \end{tikzcd} .$$

Finally, we compute the E_2-page:

$$E_2^{1,0} = \ker(d : C_{1,0} \to C_{0,0}) = \operatorname{span}\{12 + 23 + 34 + 45 + 56 - 16\},$$

which has dimension 1, and

$$E_2^{0,0} = C_{0,0} / \operatorname{im}(d : C_{1,0} \to C_{0,0}),$$

which has dimension $6 - (6 - 1) = 1$ by the rank-nullity theorem. Our spectral sequence stabilizes here (i.e., $E_2^{0,0} = E_\infty^{0,0}$ and $E_2^{1,0} = E_\infty^{1,0}$), as every spot on the E_2-page and forward will look like the diagram in Definition 1.2.5. By Theorem 1.3.9, we have

$$\dim H_0(S^1) = \dim H_0(X) = \dim E_2^{0,0} = 1,$$
$$\dim H_1(S^1) = \dim H_1(X) = \dim E_2^{1,0} = 1,$$
$$\dim H_n(S^1) = \dim H_n(X) = 0 \quad (n > 1).$$

It is not hard to compute directly (using Definition 1.1.11) that the circle S^1 has 0th and 1st homology $\mathbb{C}$ (and the trivial vector space for any higher homology). This verifies our calculation.

Next up, we consider the map $g : S^1 \sqcup S^1 \to S^1$. Some of the calculation will be identical to the one above, but for ease of readability, we will include them here as well. Let $X = \mathrm{dom}(g)$ and $Y = \mathrm{codom}(g)$. Then

$$X_0 = f^{-1}(Y^0) = \{1,2,3,4,5,6\}, \quad X_1 = f^{-1}(Y^1) = \{1,2,3,4,5,6,12,23,31,45,56,46\} = X_2 = X_3 = \cdots.$$

Some of the useful nonempty $X_{p,q}$ are listed below:

$$X_{0,0} = \{0\text{-simplices in } X_0\} = \{1,2,3,4,5,6\},$$

$$X_{1,-1} = \{0\text{-simplices in } X_1\} = \{1,2,3,4,5,6\},$$

$$X_{1,0} = \{1\text{-simplices in } X_1\} = \{12,23,13,45,56,46\}.$$

Now we compute the spectral sequence. The E_0-page once again has three possibly nonzero terms:

$$E_0^{0,0} = \frac{Z_0^{0,0}}{Z_{-1}^{-1,1} + B_{-1}^{0,0}} = \frac{C_{0,0} \cap d^{-1}(C_{0,-1})}{C_{-1,1} \cap d^{-1}(C_{0,-1}) + C_{0,0} \cap d(C_{-1,2})} = \frac{C_{0,0}}{0+0} = C_{0,0},$$

$$E_0^{0,1} = 0 \quad (\text{since } X_{0,1} = \emptyset)$$

$$E_0^{1,-1} = \frac{Z_0^{1,-1}}{Z_{-1}^{0,0} + B_{-1}^{1,-1}} = \frac{C_{1,-1} \cap d^{-1}(C_{1,-2})}{C_{0,0} \cap d^{-1}(C_{1,-2}) + C_{1,-1} \cap d(C_{0,1})} = \frac{C_{1,-1}}{C_{0,0}+0} = 0,$$

$$E_0^{1,0} = \frac{Z_0^{1,0}}{Z_{-1}^{0,1} + B_{-1}^{1,0}} = \frac{C_{1,0} \cap d^{-1}(C_{1,-1})}{C_{0,1} \cap d^{-1}(C_{1,-1}) + C_{1,0} \cap d(C_{0,2})} = \frac{C_{1,0}}{0+0} = C_{1,0},$$

and in general, for any $p \geq 1$, we have $C_{p,-p} = C_{p-1,-(p-1)}$, and so

$$\begin{aligned} E_0^{p,-p} &= \frac{Z_0^{p,-p}}{Z_{-1}^{p-1,-p+1} + B_{-1}^{p,-p}} \\ &= \frac{C_{p,-p} \cap d^{-1}(C_{p,-p-1})}{C_{p-1,-p+1} \cap d^{-1}(C_{p,-p-1}) + C_{p,-p} \cap d(C_{p-1,-p+2})} \\ &= \frac{C_{p,-p}}{C_{p-1,-p+1} + \text{some unimportant junk}} \\ &= 0, \end{aligned}$$

and similarly,

$$\begin{aligned} E_0^{p,-p+1} &= \frac{Z_0^{p,-p+1}}{Z_{-1}^{p-1,-p+2} + B_{-1}^{p,-p+1}} \\ &= \frac{C_{p,-p+1} \cap d^{-1}(C_{p,-p})}{C_{p-1,-p+2} \cap d^{-1}(C_{p,-p}) + C_{p,-p+1} \cap d(C_{p-1,-p+3})} \\ &= \frac{C_{p,-p+1}}{C_{p-1,-p+2} + \text{some unimportant junk}} \\ &= 0. \end{aligned}$$

Hence, the E_0-page looks like

$$\begin{tikzcd}
0 \arrow[d, "d"] & 0 \arrow[d, "d"] & 0 \arrow[d, "d"] \\
C_{0,0} \arrow[d, "d"] & C_{1,0} \arrow[d, "d"] & 0 \arrow[d, "d"] \\
0 & 0 & 0
\end{tikzcd}.$$

By definition, the terms in the E_1-page is given by taking homology of the E_0-page, so the E_1-page looks like

$$\begin{tikzcd}
0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\
C_{0,0} & C_{1,0} \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\
0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"']
\end{tikzcd}.$$

And here is where things get a bit different. We compute the E_2-page:

$$E_2^{1,0} = \ker(d : C_{1,0} \to C_{0,0}) = \operatorname{span}\{12 + 23 - 13, 45 + 56 - 46\},$$

which has dimension 2, and

$$E_2^{0,0} = C_{0,0}/\operatorname{im}(d : C_{1,0} \to C_{0,0}),$$

which has dimension $6 - (6 - 2) = 2$ by the rank-nullity theorem. Our spectral sequence stabilizes here. By Theorem 1.3.9, we have

$$\begin{aligned}
\dim(H_0(S^1 \sqcup S^1)) &= \dim(H_0(X)) = \dim(E_2^{0,0}) = 2, \\
\dim(H_1(S^1 \sqcup S^1)) &= \dim(H_1(X)) = \dim(E_2^{1,0}) = 2, \\
\dim(H_n(S^1 \sqcup S^1)) &= \dim(H_n(X)) = 0 \quad (n > 1).
\end{aligned}$$

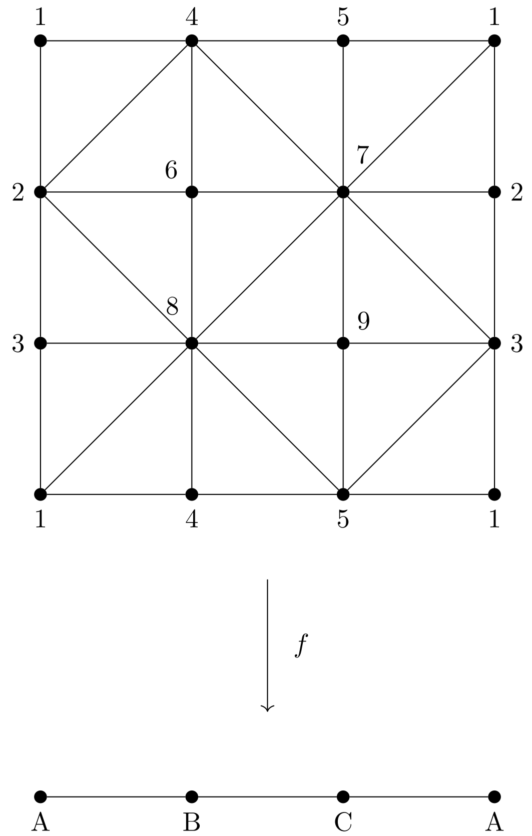
It is a general fact that if X and Y are spaces (or simplicial complexes), then

$$H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(Y)$$

This verifies our calculation.

1.5 A More Complicated Example: $S^1 \times S^1 \to S^1$

In this subsection, we will demonstrate the Leray spectral sequence with a more complicated example. The one we pick here is the projection map $f : S^1 \times S^1 \to S^1$ from the torus to the circle. We begin by providing a simplicial approximation of f.



As usual, let $X = \mathrm{dom}(f)$ and $Y = \mathrm{codom}(f)$. Then

$$Y^0 = \{A, B, C\}, \quad Y^1 = \{A, B, C, AB, BC, AC\}$$

and so

$$X_0 = f^{-1}(Y^0) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 23, 13, 46, 68, 48, 57, 79, 59\},$$

$$X_1 = f^{-1}(Y^1) = \left\{ \begin{array}{c} 1, 2, 3, 4, 5, 6, 7, 8, 9 \\ 12, 13, 14, 15, 17, 18, 23, 24, 26, 27, 28, 35, 37, 38, 39, 45, 46, 47, 48, 57, 58, 59, 67, 68, 78, 79, 89, \\ 124, 246, 268, 238, 138, 148, 457, 467, 678, 789, 589, 458, 157, 127, 239, 379, 359, 135 \end{array} \right\}.$$

Some of the nonempty $X_{p,q}$ are listed below:

$$X_{0,0} = \{0\text{-simplices in } X_0\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$X_{0,1} = \{1\text{-simplices in } X_0\} = \{12, 23, 13, 46, 68, 48, 57, 79, 59\},$$

$$X_{1,-1} = \{0\text{-simplices in } X_1\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$X_{1,0} = \{1\text{-simplices in } X_1\} = \left\{ \begin{array}{c} 12, 13, 14, 15, 17, 18, 23, 24, 26, 27, 28, 35, 37, 38, 39, \\ 45, 46, 47, 48, 57, 58, 59, 67, 68, 78, 79, 89 \end{array} \right\},$$

$$X_{1,1} = \{2\text{-simplices in } X_1\} = \left\{ \begin{array}{c} 124, 246, 268, 238, 138, 148, \\ 457, 467, 678, 789, 589, 458, \\ 157, 127, 237, 379, 359, 135 \end{array} \right\},$$

$$X_{2,0} = \{2\text{-simplices in } X_2\} = \left\{ \begin{array}{c} 124, 246, 268, 238, 138, 148, \\ 457, 467, 678, 789, 589, 458, \\ 157, 127, 237, 379, 359, 135 \end{array} \right\}.$$

Let's compute the E_0-page:

$$E_0^{0,0} = \frac{Z_0^{0,0}}{Z_{-1}^{-1,1} + B_{-1}^{0,0}} = \frac{C_{0,0} \cap d^{-1}(C_{0,-1})}{C_{-1,1} \cap d^{-1}(C_{0,-1}) + C_{0,0} \cap d(C_{-1,2})} = \frac{C_{0,0}}{0 + 0} = C_{0,0},$$

$$E_0^{0,1} = \frac{Z_0^{0,1}}{Z_{-1}^{-1,2} + B_{-1}^{0,1}} = \frac{C_{0,1} \cap d^{-1}(C_{0,0})}{C_{-1,2} \cap d^{-1}(C_{0,0}) + C_{0,1} \cap d(C_{-1,3})} = \frac{C_{0,1}}{0+0} = C_{0,1},$$

$$E_0^{1,-1} = \frac{Z_0^{1,-1}}{Z_{-1}^{0,0} + B_{-1}^{1,-1}} = \frac{C_{1,-1} \cap d^{-1}(C_{1,-2})}{C_{0,0} \cap d^{-1}(C_{1,-2}) + C_{1,-1} \cap d(C_{0,1})} = \frac{C_{1,-1}}{C_{0,0}+0} = 0 \quad (\text{since } X_{1,-1} = X_{0,0}),$$

$$E_0^{1,0} = \frac{Z_0^{1,0}}{Z_{-1}^{0,1} + B_{-1}^{1,0}} = \frac{C_{1,0} \cap d^{-1}(C_{1,-1})}{C_{0,1} \cap d^{-1}(C_{1,-1}) + C_{1,0} \cap d(C_{0,2})} = \frac{C_{1,0}}{C_{0,1}+0} = C_{1,0}/C_{0,1},$$

$$E_0^{1,1} = \frac{Z_0^{1,1}}{Z_{-1}^{0,2} + B_{-1}^{1,1}} = \frac{C_{1,1} \cap d^{-1}(C_{1,0})}{C_{0,2} \cap d^{-1}(C_{1,0}) + C_{1,1} \cap d(C_{0,3})} = \frac{C_{1,1}}{0+0} = C_{1,1},$$

$$E_0^{2,0} = \frac{Z_0^{2,0}}{Z_{-1}^{1,1} + B_{-1}^{2,0}} = \frac{C_{2,0} \cap d^{-1}(C_{2,-1})}{C_{1,1} \cap d^{-1}(C_{2,-1}) + C_{2,0} \cap d(C_{1,2})} = \frac{C_{2,0}}{C_{1,1}+0} = 0 \quad (\text{since } X_{2,0} = X_{1,1}).$$

The reader should convince themselves that the remaining terms on the E_0-page are all 0. Thus, the E_0-page looks like

$$\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow d & \downarrow d & \downarrow d \\
C_{0,1} & C_{1,1} & 0 \\
\downarrow d & \downarrow d & \downarrow d \\
C_{0,0} & C_{1,0}/C_{0,1} & 0 \\
\downarrow d & \downarrow d & \downarrow d \\
0 & 0 & 0
\end{array},$$

where in the middle column, the map $d : C_{1,1} \to C_{1,0}/C_{0,1}$ is the composition

$$C_{1,1} \xrightarrow{d} C_{1,0} \xrightarrow{\text{quotient}} C_{1,0}/C_{0,1}.$$

Now we compute the E_1-page by taking homology. We will do it column by column.

$$E_1^{0,1} = \ker(d : C_{0,1} \to C_{0,0}) = \operatorname{span}\{12 + 23 - 13, 46 + 68 - 48, 57 + 79 - 59\},$$

$$E_1^{0,0} = C_{0,0}/\operatorname{im}(d : C_{0,1} \to C_{0,0}) = C_{0,0}/\operatorname{span}\{1-2, 2-3, 4-6, 6-8, 5-7, 7-9\},$$

$$\begin{aligned}
E_1^{1,1} &= \ker(d : C_{1,1} \to C_{1,0}/C_{0,1}) \\
&= \ker(d : C_{1,1} \to C_{1,0}) + C_{1,1} \cap d^{-1}(C_{0,1}) \\
&= \operatorname{span}\left\{ \begin{array}{c} 124 - 246 - 268 + 238 - 138 + 148 \\ -457 + 467 - 678 + 789 - 589 + 458 \\ +157 - 127 - 237 + 379 - 359 + 135 \end{array} \right\} + \operatorname{span}\left\{ \begin{array}{c} 124 - 246 - 268 + 238 - 138 + 148, \\ 457 - 467 + 678 - 789 + 589 - 458, \\ 157 - 127 - 237 + 379 - 359 + 135 \end{array} \right\} \\
&= \operatorname{span}\left\{ \begin{array}{c} 124 - 246 - 268 + 238 - 138 + 148, \\ 457 - 467 + 678 - 789 + 589 - 458, \\ 157 - 127 - 237 + 379 - 359 + 135 \end{array} \right\},
\end{aligned}$$

and the most complicated term in E_1-page:

$$
\begin{aligned}
E_1^{1,0} &= \frac{C_{1,0}/C_{0,1}}{\operatorname{im}(d : C_{1,1} \to C_{1,0}/C_{0,1})} \\
&= \frac{C_{1,0}/C_{0,1}}{(\operatorname{im}(d : C_{1,1} \to C_{1,0}) + C_{0,1})/C_{0,1}} \\
&= \frac{C_{1,0}}{\operatorname{im}(d : C_{1,1} \to C_{1,0}) + C_{0,1}} \qquad \text{(3rd isomorphism theorem)} \\
&= C_{1,0} \Big/ \operatorname{span} \left\{ \begin{array}{c} 12, 23, 13, 46, 68, 48, 57, 79, 59, \\ 24 - 14 + 12, 46 - 26 + 24, 68 - 28 + 26, \\ 38 - 28 + 23, 38 - 18 + 13, 48 - 18 + 14, \\ 57 - 47 + 45, 67 - 47 + 46, 78 - 68 + 67, \\ 89 - 79 + 78, 89 - 59 + 58, 59 - 48 + 45, \\ 57 - 17 + 15, 27 - 17 + 12, 37 - 27 + 23, \\ 79 - 39 + 37, 59 - 39 + 35, 35 - 15 + 13 \end{array} \right\} \\
&= C_{1,0} \Big/ \operatorname{span} \left\{ \begin{array}{c} 12, 23, 13, 46, 68, 48, 57, 79, 59, \\ 24 - 14, -26 + 24, -28 + 26, 38 - 28, 38 - 18, -18 + 14, \\ -47 + 45, 67 - 47, 78 + 67, 89 + 78, 89 + 58, -48 + 45, \\ -17 + 15, 27 - 17, 37 - 27, -39 + 37, -39 + 35, 35 - 15 \end{array} \right\}.
\end{aligned}
$$

Note that the six elements in each of the last three row in that giant set above are linearly dependent. (The linear combinations needed to prove this are not hard to obtain by hand; the coefficients are all ± 1.) Thus,

$$
E_1^{1,0} = C_{1,0} \Big/ \operatorname{span} \left\{ \begin{array}{c} 12, 23, 13, 46, 68, 48, 57, 79, 59, \\ 24 - 14, -26 + 24, -28 + 26, 38 - 28, 38 - 18, \\ -47 + 45, 67 - 47, 78 + 67, 89 + 78, 89 + 58, \\ -17 + 15, 27 - 17, 37 - 27, -39 + 37, -39 + 35 \end{array} \right\}.
$$

The dimension of each term is

$$
\begin{aligned}
\dim E_1^{0,1} &= 3, \\
\dim E_1^{0,0} &= 9 - 6 = 3, \\
\dim E_1^{1,1} &= 3, \\
\dim E_1^{1,0} &= 27 - 24 = 3.
\end{aligned}
$$

The E_1-page looks like

$$
\begin{tikzcd}
0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\
E_1^{0,1} & E_1^{1,1} \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\
E_1^{0,0} & E_1^{1,0} \arrow[l, "d"'] & 0 \arrow[l, "d"'] \\
0 & 0 \arrow[l, "d"'] & 0 \arrow[l, "d"']
\end{tikzcd}
.
$$

Now we compute half of the E_2-page. Since all we need for the homology is the dimension of each term in the page, there is no need to compute the entire page, and we can simply invoke the rank-nullity theorem. The $E_2^{1,1}$ term is not hard:

$$
E_2^{1,1} = \ker(d : E_1^{1,1} \to E_1^{0,1}) = \operatorname{span} \left\{ \begin{array}{c} (124 - 246 - 268 + 238 - 138 + 148) \\ -(457 - 467 + 678 - 789 + 589 - 458) \\ +(157 - 127 - 237 + 379 - 359 + 135) \end{array} \right\}.
$$

To compute $E_2^{1,0}$, we shall first fix a convenient basis of $E_1^{1,0}$:

$$E_1^{1,0} = C_{1,0} \Big/ \operatorname{span} \left\{ \begin{array}{c} 12, 23, 13, 46, 68, 48, 57, 79, 59, \\ 24-14, -26+24, -28+26, 38-28, 38-18, \\ -47+45, 67-47, 78+67, 89+78, 89+58, \\ -17+15, 27-17, 37-27, -39+37, -39+35 \end{array} \right\}$$

$$= \operatorname{span}\{14, 47, 17\}.$$

(Here 14 really means the equivalence class of 14, but this technicality plays no role in the computation, so we shall simply ignore that.) Then

$$E_2^{1,0} = \ker(d : E_1^{1,0} \to E_1^{0,0}) = \operatorname{span}\{14 + 47 - 17\}.$$

Hence,

$$\dim E_2^{1,1} = \dim E_2^{1,0} = 1.$$

Using the rank-nullity theorem, we conclude that

$$\dim E_2^{0,1} = \dim E_2^{0,0} = 3 - 2 = 1.$$

By Theorem 1.3.9,

$$\begin{aligned} \dim H_0(S^1 \times S^1) &= 1, \\ \dim H_1(S^1 \times S^1) &= 1 + 1 = 2, \\ \dim H_2(S^1 \times S^1) &= 1. \end{aligned}$$

The homology of the n-torus $(S^1)^{\times n}$ in general is

$$H_r((S^1)^{\times n}) = \mathbb{C}^{\binom{n}{r}}$$

Put $n = 2$, this verifies our lengthy computation.

1.6 The Hopf fibration and Its Minimal Triangulation

The Hopf fibration is a certain interesting surjective map $S^3 \to S^2$ which exhibit some nontrivial behaviors. For example, the fiber of every point of S^2 is just a circle S^1. That is, we have a fiber bundle $S^1 \hookrightarrow S^3 \to S^2$. Thus, S^3 can be think of as a family of circles parametrized by the sphere. However, S^3 and $S^2 \times S^1$ are very distinctive as manifolds. Intuitively, the Hopf fibration has a twists (in 4 dimension) that we can't visualize directly. This twisting is analogous to how a Möbius stripe M is a fiber bundle of S^1 with fiber $I = [0, 1]$ (i.e., $I \hookrightarrow M \to S^1$) but is globally distinctive from $S^1 \times I$, i.e., a cylinder.

This subsection is divided into three parts. In the first part, for completeness reason only, we will define the Hopf fibration. While the map is often written in terms of quaternions, there is a surprisingly simple way to do it using matrix Lie groups. The second part will be about the minimal triangulation of the Hopf fibration given by [3]. We will also explain how to this triangulation allows us to visualize the twist of the Hopf fibration. The last part will be an open-ended discussion on the Leray spectral sequence of the Hopf fibration.

Definition 1.6.1 Define (or recall, if you wish) the matrix Lie groups

$$\mathrm{SU}(2) = \{A \in M_2(\mathbb{C}) : A^* A = I, \det(A) = 1\}$$

(the 2×2 **special unitary** group) where $*$ indicates conjugate transpose, and

$$U(1) = \{\alpha \in \mathbb{C} : \overline{\alpha}\alpha = 1\}$$

(the 1×1 **unitary** group). To make them into Lie groups, note that each of them carries a manifold structure as a subspace of $\mathbb{C}^4$ (with the standard topology).

Lemma 1.6.2 We have $\mathrm{U}(1) \cong S^1$ and $\mathrm{SU}(2) \cong S^3$ as manifolds.

Proof The first one is obvious, so we only prove the second one. Recall the usual Hermitian form on $\mathbb{C}^2$

$$\begin{array}{cccc} \langle -,- \rangle : & \mathbb{C}^2 \times \mathbb{C}^2 & \to & \mathbb{C} \\ & ((x_1,x_2),(y_1,y_2)) & \mapsto & \overline{x_1}y_1 + \overline{x_2}y_2 \end{array}$$

and recall that a matrix A is unitary if and only if its columns are orthonormal with respect to $\langle -,- \rangle$ (or equivalently, A preserves this Hermitian form). Thus, we have

$$\begin{aligned} \mathrm{SU}(2) &= \{A \in M_2(\mathbb{C}) : A^*A = I, \det(A) = 1\} \\ &= \left\{ \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix} \in M_2(\mathbb{C}) : \overline{\alpha}\alpha + \overline{\beta}\beta = 1, \overline{x}x + \overline{y}y = 1, \overline{\alpha}x + \overline{\beta}y = 0, \alpha y - \beta x = 1 \right\} \end{aligned}$$

From the third equation, we have $x = -k\overline{\beta}$ and $y = k\overline{\alpha}$ for some k. Plug this into the last equation, we obtain $k\overline{\alpha}\alpha + k\overline{\beta}\beta = 1$. Thus, $k = 1$ by the first equation. Hence,

$$\begin{aligned} \mathrm{SU}(2) &= \left\{ \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &= \left\{ \begin{bmatrix} a+bi & -c+di \\ c+di & a-bi \end{bmatrix} : a,b,c,d \in \mathbb{R}, a^2+b^2+c^2+d^2 = 1 \right\} \\ &\cong S^3 \end{aligned}$$

as desired. □

Note that SU(2) contains a copy of U(1) via the diagonal matrices

$$\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} : \overline{\alpha}\alpha = 1 \right\}$$

With an abuse of notation, we will write U(1) for this particular subgroup as well. It also happens to be a maximal abelian connected compact subgroup (i.e., **maximal torus**) of SU(2).

Before we can define the Hopf fibration, we will need one more identification. We will not introduce the notion of spin groups and so we will not provide a proof to this. For a proof, see [6].

Fact 1.6.3 $\mathrm{SU}(2)/U(1) \cong S^2$ as manifolds.

Definition 1.6.4 The **Hopf fibration** is the canonical quotient map

$$\eta : S^3 \cong \mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$$

The fiber of each point in $\mathrm{SU}(2)/\mathrm{U}(1)$ is just a coset and hence is homeomorphic to $\mathrm{U}(1) \cong S^1$ as a manifold.

Fact 1.6.5 $\eta : S^3 \to S^2$ is a fiber bundle with fiber S^1, i.e., locally isomorphic to the trivial fiber bundle over small neighborhoods.

Some reader might like a more explicit version of the Hopf fibration in terms of coordinates, so here is one.

Definition 1.6.6 The Hopf fibration is the map

$$\begin{array}{cccc} \eta : & S^3 & \to & S^2 \\ & (a,b,c,d) & \mapsto & (2ab+2cd, 2ad-2bc, a^2-b^2+c^2-d^2) \end{array}$$

However, with this definition, it is not immediate that this map is surjective, let alone a fiber bundle with fiber S^1.

Now we give a triangulation of this map due to [3]. We will elaborate the triangulation in detail and provide the entire list of simplices. Since this triangulation is much larger than what we have seen before, we will use Euler characteristics as a (probabilistic) indicator that we get everything correctly.

Definition 1.6.7 Given a simplicial complex X, let $X(p) = \{\text{all } p\text{-simplices of X}\}$. The number

$$\chi(X) = \sum_{p \geq 0} (-1)^p |X(p)|$$

is called the **Euler characteristics** of X.

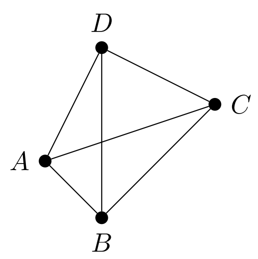
Fact 1.6.8 Let X be a simplicial complex. Then

(i) $\chi(X) = \chi(|X|)$, where $\chi(|X|)$ is the Euler characteristic as a CW-complex;

(ii) $\chi(X) = \sum_{p\geq 0}(-1)^p \dim_{\mathbb{C}} H_p(X)$;

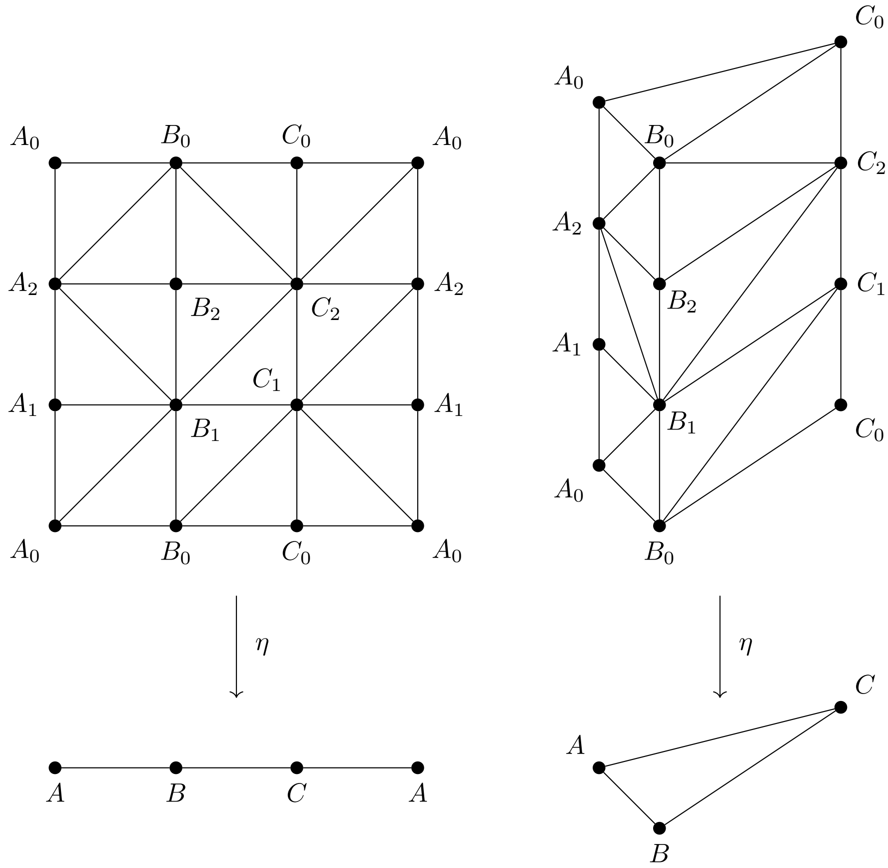
(iii) $\chi(S^1) = 0$, $\chi(S^2) = 2$, and $\chi(S^3) = 0$.

We now begin describing the triangulation in detail. Note that we will not justify why this is indeed a simplicial approximation of the Hopf map (as it is beyond the scope of this paper). The sphere S^2 can be triangulated into a tetrahedron $ABCD$.



(such that all the triangles ABC, ABD, ACD, BCD are 2-simplices but the tetrahedron $ABCD$ itself is not a 3-simplex). We shall occasionally say that ABC is the *southern* part of S^2 while referring the rest as the *northern* part. Then we can triangulate S^3 by first triangulating η^{-1}(southern part), then triangulating η^{-1}(northern part). Then we glue them together "along the boundary". The triangle ABC is nothing but a disk D^2, and it turns out the (restriction of the) Hopf map is a trivial fiber bundle over this disk, so $\eta^{-1}(ABC)$ is the same thing as $D^2 \times S^1$, i.e., a solid torus. Similarly, η^{-1}(northern part) is also a solid torus.

We first describe the triangulation of $\eta^{-1}(ABC)$, which is encoded into the following diagram



In this diagram, we have a solid cylinder on the right (which becomes our solid torus upon identifying the top and the bottom), but we intentionally omit the back side to ensure a readable diagram. Thus, it is what you will see if you stand in front of the solid cylinder. The diagram on the left is what you get by cutting the sidewall of the cylinder along the verticl line $A_0A_1A_2A_0$ and flatten it out, and so the rightmost column (bounded by the C's and the A's) gives the back side of the cylinder. The cylinder has three "levels", which each there are three tetrahedrons (3-simplices). For example, the toppest level has three tetrahedrons $A_0B_0C_0C_2$, $A_0A_2B_0C_0$, and $A_2B_0B_2C_2$. From this diagram, we arrive the following lists of simplices. Here we use the most canonical ordering $A_0 < A_1 < A_2 < B_0 < B_1 < B_2 < C_0 < C_1 < C_2$ on the vertices when writing each simplex, and simplices are listed in the corresponding lexicographical order.

$$\text{0-simplices} : \{A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2\}$$

$$\text{1-simplices} : \left\{ \begin{array}{c} A_0A_1, A_0A_2, A_0B_0, A_0B_1, A_0C_0, A_0C_1, A_0C_2, A_1A_2, A_1B_1, A_1C_1, \\ A_2B_0, A_2B_1, A_2B_2, A_2C_1, A_2C_2, B_0B_1, B_0B_2, B_0C_0, B_0C_1, B_0C_2, \\ B_1B_2, B_1C_1, B_1C_2, B_2C_2, C_0C_1, C_0C_2, C_1C_2 \end{array} \right\}$$

$$\text{2-simplices} : \left\{ \begin{array}{c} A_0A_1B_1, A_0A_1C_1, A_0A_2B_0, A_0A_2C_2, A_0B_0B_1, A_0B_0C_0, \\ A_0B_0C_1, A_0B_0C_2, A_0B_1C_1, A_0C_0C_1, A_0C_0C_2, A_1A_2B_1, \\ A_1A_2C_1, A_1B_1C_1, A_2B_0B_2, A_2B_0C_2, A_2B_1B_2, A_2B_1C_1, \\ A_2B_1C_2, A_2B_2C_2, A_2C_1C_2, B_0B_1C_1, B_0B_2C_2, B_0C_0C_1, \\ B_0C_0C_2, B_1B_2C_2, B_1C_1C_2 \end{array} \right\}$$

$$\text{3-simplices} : \left\{ \begin{array}{c} A_0A_1B_1C_1, A_0A_2B_0C_2, A_0B_0B_1C_1, \\ A_0B_0C_0C_1, A_0B_0C_0C_2, A_1A_2B_1C_1, \\ A_2B_0B_2C_2, A_2B_1B_2C_2, A_2B_1C_1C_2 \end{array} \right\}$$

The Euler characteristic gives $9 - 27 + 27 - 9 = 0$. To see why this is what we are supposed to get, we will use the following fact:

Fact 1.6.9 If $f : X \to Y$ is a homotopy equivalence of topological spaces, then the induced map

$$H_n(f) : H_n(X) \to H_n(Y)$$

on (singular) homology is an isomorphism.

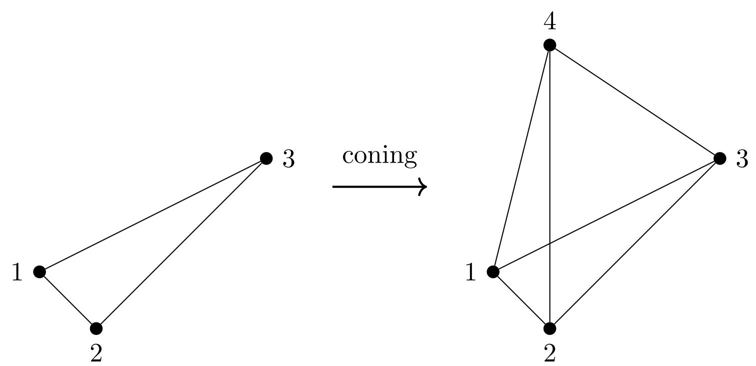
Since a solid torus $D^2 \times S^1$ is homotopy equivalent to S^1 (by just contracting D^2 to its center), it has the same homology as S^1, so by (ii) and (iii) in Fact 1.6.8, we have $\chi(D^2 \times S^1) = \chi(S^1) = 0$, as expected.

Next we triangulate η^{-1}(northern part). The triangulation of this solid torus, however, is not that straightforward compared to the previous one. The main idea here is to "upgrade" a triangulation of a hollow torus (i.e., $S^{\times} S^1$) to a triangulation of a solid torus. This upgrade is done by a process called coning over a vertex. More precisely:

Definition 1.6.10 Given a simplicial complex X and a extra vertex v, we can form a new simplicial complex Y with $V(Y) = V(X) \sqcup \{v\}$ and whose simplicies consist of all subsets of $V(Y)$ of the form $\sigma \in S(X)$, or $\sigma \sqcup \{v\}$ with $\sigma \in S(X) \sqcup \{\emptyset\}$. We call this process **coning** over v.

In other word, coning over a new vertex v gives a larger simplicial complex whose new simplices are given by adding v to each of the simplex in the original simplicial complex. Note that in the definition, we have $\sigma \in S(X) \sqcup \{\emptyset\}$ to simply ensure that $\{v\}$ is also a simplex, in order to satisfy the axioms in Definition 1.1.3. (Thus, the addition of the empty set is a mere technicality that the reader should not be bothered to much with.)

Example 1.6.11 Consider the triangulation of S^1 with simplicies $\{1, 2, 3, 12, 23, 13\}$. Let 4 be our new vertex. Then coning S^1 over 4 gives the simplicial complex $\{1, 2, 3, 12, 23, 13, 4, 14, 24, 34, 124, 234, 134\}$



This gives us a disk D^2 (not a sphere S^2, as the triangle 123 is not a simplex). Intuitively, you can squish the top vertex onto the plane, so it is a disk. More generally, and pictorially, when coning over a vertex, you draw an edge from the new vertex to all other original vertices and "solidifying" some new faces (e.g., 12 gives a new face 124). Also, note that when coning over a new vertex, the dimension jumps up by 1, so you should first, pictorially, embed your complex to a higher dimension.

Remark 1.6.12 (Coning and geometric realization) It is not a surprise that the cone of a simplicial complex is related to the cone of a topological space. We now state this more precisely. Let X be a topological space. The (**unreduced**) **cone** on X is the topological space

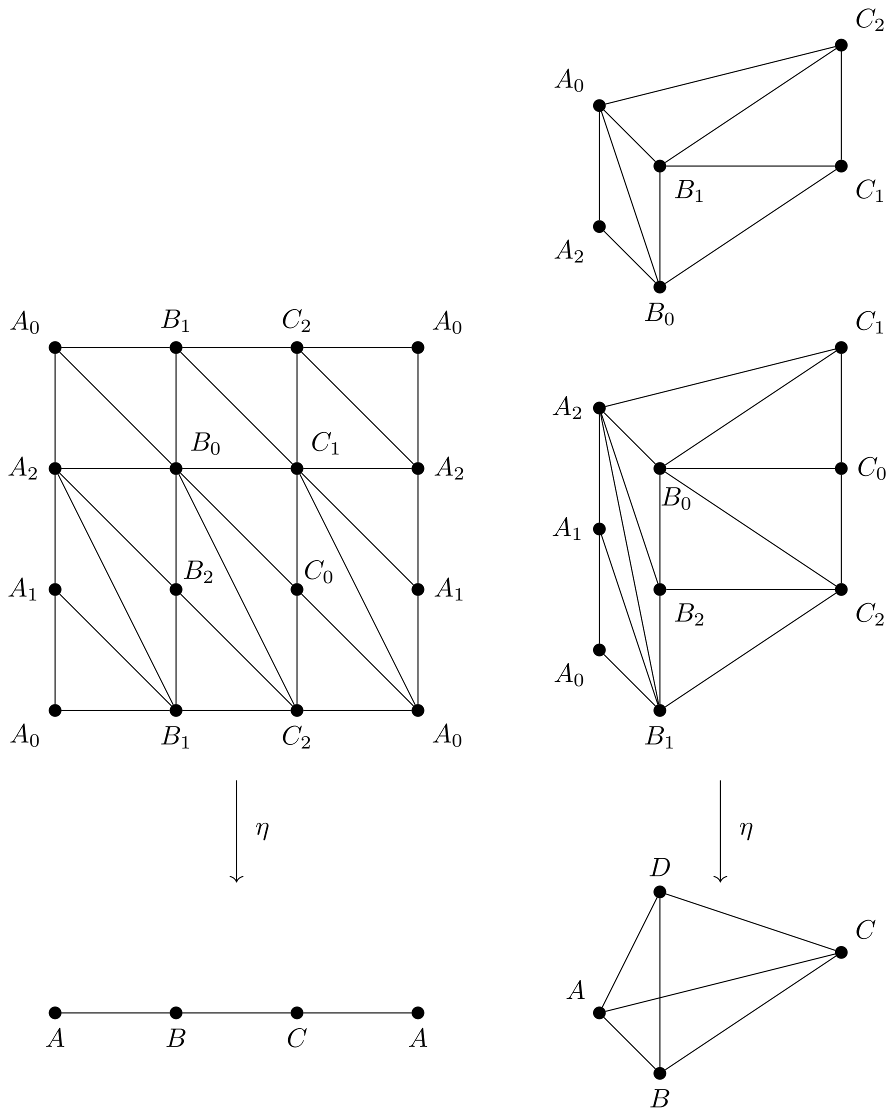
$$CX = X \times [0, 1]/X \times \{1\}.$$

(For those who are interested: The **reduced cone** is only defined when X has a basepoint $*$; it further identifies points on the line $\{*\} \times [0, 1]$ to make the resulting space having a natural basepoint.) Now let Y be a simplicial complex. Suppose you cone Y over a new vertex and write CY for this new simplicial complex. Then we have

$$|CY| \cong C|Y|$$

as topological spaces (or CW-complexes, if one wishes).

First, we begin by giving a triangulation of a torus $S^1 \times S^1$. This triangulation is slightly different than the one given in Section 1.5. The vertices will be labelled in such a way that the twisting of the Hopf fibration is evident. For example, as we move from the A's column to the B's column, we (sort of) translate the labelling to the right, roll them down by 1 (similar to a conveyor belt, if that makes sense), and change the letter from A to B, so that $A_0A_2A_1$ becomes $B_1B_0B_2$. Thus, when we eventually glue this to the previous solid torus along the torus boundary, we have a twist!



A few things to note in this diagram:

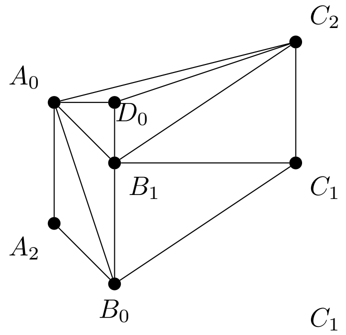
(i) The reason for splitting the (hollow) torus on the right into two pieces will be clear very soon.

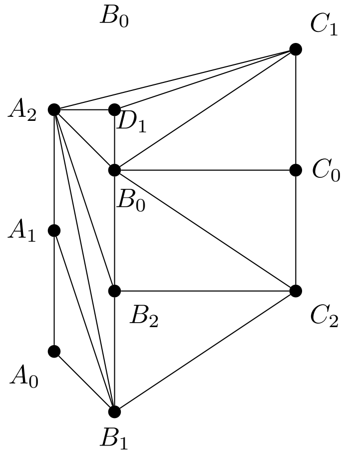
(ii) The vertex D does not have any fiber just yet. Once we finish our upgrade, it will have fiber S^1.

(iii) Only the triangles on the side walls are counted as 2-simplices and there are no 3-simplices. (In particular, the triangles $A_0B_1C_2$ and $A_2B_0C_1$ are not 2-simplices yet. They are both just 3 pieces of 1-simplices.)

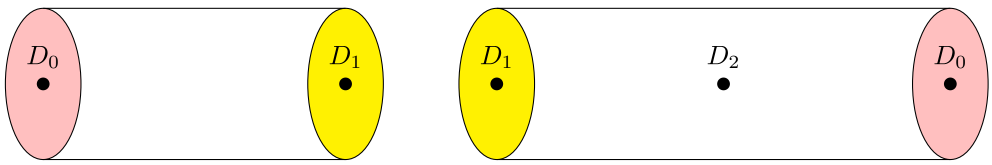
(iv) At the bottom-right, every triangle except ABC is part of the complex. Remember, this is the northern part of our sphere S^2.

Now we upgrade this to a triangulation of a solid torus. We begin by "filling" the two hollow triangles mentioned in (iii) above (i.e., capping the cylinders). This is easily done by coning them with, respectively, new internal vertices D_0 and D_1:





To finish off the upgrade, we cone the upper (enclosed) cylinder over D_1 and cone the lower (enclosed) cylinder over another new vertex D_2 sit inside the hollow bit. Thus pictorially, we have something that looks like



(where disks with the same color are glued together to get the torus). Now note that the fiber of vertex D is indeed $D_0D_1D_2D_0$, i.e., a circle. We then arrived at the following list of simplices.

$$\text{0-simplices} : \{A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2, D_0, D_1, D_2\}$$

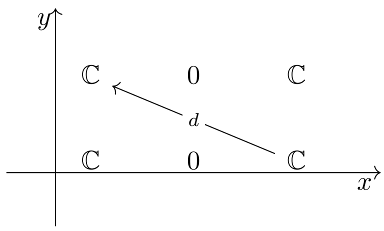
$$\text{1-simplices} : \left\{ \begin{array}{c} A_0A_1, A_2A_2, A_0B_0, A_0B_1, A_0C_0, A_0C_1, A_0C_2, A_0D_0, A_0D_1, A_0D_2, \\ A_1A_2, A_1B_1, A_1C_1, A_1D_2, A_2B_0, A_2B_1, A_2B_2, A_2C_1, A_2C_2, A_2D_1, A_2D_2, \\ B_0B_1, B_0B_2, B_0C_0, B_0C_1, B_0D_1, B_0D_2, B_1B_2, B_1C_1, B_1C_2, B_1D_0, B_1D_1, B_1D_2, \\ B_2C_2, B_2D_2, C_0C_1, C_0C_2, C_0D_2, C_1C_2, C_1D_1, C_1D_2, C_2D_0, C_2D_1, C_2D_2, D_0D_1, D_0D_2, D_1D_2 \end{array} \right\}$$

$$\text{2-simplices} : \left\{ \begin{array}{c} A_0A_1B_1, A_0A_1C_1, A_0A_1D_2, A_0A_2B_0, A_0A_2C_2, A_0A_2D_1, \\ A_0B_0B_1, A_0B_0D_1, A_0B_1D_1, A_0C_0C_1, A_0C_0C_2, A_0C_0D_2, \\ A_0C_1D_2, A_0C_2D_0, A_0C_2D_1, A_0C_2D_2, A_0D_0D_1, A_0D_0D_2, \\ A_1A_2B_1, A_1A_2C_1, A_1A_2D_2, A_1B_1D_2, A_1C_1D_2, A_2B_0B_2, \\ A_2B_0D_1, A_2B_0D_2, A_2B_1B_2, A_2B_1D_2, A_2B_2D_2, A_2C_1C_2, \\ A_2C_1D_1, A_2C_1D_2, A_2C_2D_1, A_2D_1D_2, B_0B_1C_1, B_0B_1D_1, \\ B_0B_2C_2, B_0B_2D_2, B_0C_0C_1, B_0C_0C_2, B_0C_0D_2, B_0C_1D_1, \\ B_0C_1D_2, B_0D_1D_2, B_1B_2C_2, B_1B_2D_2, B_1C_1C_2, B_1C_1D_1, \\ B_1C_2D_0, B_1C_2D_2, B_1D_0D_2, B_2C_2D_2, C_0C_1D_2, C_0C_2D_2, \\ C_1C_2D_1, C_1D_1D_2, C_2D_0D_1, C_2D_0D_2 \end{array} \right\}$$

$$\text{3-simplices} : \left\{ \begin{matrix} A_0A_1B_1D_2, A_0A_1C_1D_2, A_0A_2B_0D_1, A_0A_2C_2D_1, \\ A_0B_0B_1D_1, A_0B_1D_0D_1, A_0C_0C_1D_2, A_0C_0C_2D_2, \\ A_0C_2D_0D_1, A_1A_2B_1D_2, A_1A_2C_1D_2, A_2B_0B_2D_2, \\ A_2B_0D_1D_2, A_2B_1B_2D_2, A_2C_1C_2D_1, A_2C_1D_1D_2, \\ B_0B_1C_1D_1, B_0B_2C_2D_2, B_0C_0C_1D_2, B_0C_0C_2D_2, \\ B_1B_2C_2D_2, B_1C_1C_2D_1, B_1C_2D_0D_1, \end{matrix} \right\}$$

Computing the Euler characteristic gives $12-47+58-23=0$. Then the p-simplices of the triangulation of S^3 is simply given by the union of the p-simplices of both the southern and the northern part.

We will end this subsection (and this paper) with a unsolved problem. What would happen when we compute the spectral sequence using the Hopf fibration? This is actually a very well-known spectral sequence. The E_2-page should look like



.

and 0 everywhere else. We want to compute this particular nontrivial arrow $d : E_2^{2,0} \to E_2^{0,1}$ in terms of a (1×1) matrix. Why is it nontrivial? It is a well-known fact that

$$H_n(S^3) = \begin{cases} \mathbb{C} & \text{if } n = 0, 3 \\ 0 & \text{otherwise} \end{cases}.$$

Thus the two $\mathbb{C}$'s in the diagram connected by d must be killed off after computing the homology, as the sequence collapses at E_3-page (yes, 1 page earlier than what Theorem 1.3.9 says). Hence, we must have

$$\ker(d) = \operatorname{coker}(d) = 0,$$

that is, d is an isomorphism. Any nonzero element in a 1-dimensional vector space is a basis, so to describe d, it suffices to just find one such element and apply d to that element. For this, let's return to the formula of $E_r^{p,q}$ in Theorem 1.3.7:

$$\begin{aligned} E_r^{p,q} &= \frac{Z_r^{p,q}}{Z_{r-1}^{p-1,q+1} + B_{r-1}^{p,q}} \\ &= \frac{C_{p,q} \cap d^{-1}(C_{p-r,q+r-1})}{C_{p-1,q+1} \cap d^{-1}(C_{p-r,q+r-1}) + C_{p,q} \cap d(C_{p+r-1,q-r+2})} \end{aligned}$$

so that we have

$$E_2^{2,0} = \frac{C_{2,0} \cap d^{-1}(C_{0,1})}{C_{1,1} \cap d^{-1}(C_{0,1}) + C_{2,0} \cap d(C_{3,0})},$$

$$E_2^{0,1} = \frac{C_{0,1} \cap d^{-1}(C_{-2,2})}{C_{-1,2} \cap d^{-1}(C_{-2,2}) + C_{0,1} \cap d(C_{1,1})}.$$

The hurdle here is to find something nonzero in $E_2^{2,0}$.

To see where the issue is, we let Y to be the simplicial complex $ABCD$ of S^2 above and $X = \eta^{-1}(Y)$ the simplicial complex of S^3. Note that

$$C_{0,1} = \operatorname{span}_{\mathbb{C}}\{\text{1-simplices in } \eta^{-1}(Y^0)\} = \operatorname{span}_{\mathbb{C}}\{A_0A_1, A_1A_2, A_0A_2, \cdots, D_0D_1, D_1D_2, D_0D_2\}$$

(i.e., the span of all the vertical edges). Since $Y^2 = Y$, the space $C_{2,0}$ is just the span of all 2-simplices in X. Thus, $C_{2,0} \cap d^{-1}(C_{0,1})$ consists of linear combinations of 2-simplices such that, after applying the boundary map d, only vertical edges remains. For example,

$$A_0D_0D_1 - A_0A_2D_1 + A_2D_1D_2 + A_1A_2D_2 + A_0A_1D_2 - A_0D_0D_2$$

is such an element, as applying d gives

$$D_0D_1 - A_0A_2 + D_1D_2 + A_1A_2 + A_0A_1 - D_0D_2.$$

However, any such element is automatically in

$$C_{1,1} = \operatorname{span}_{\mathbb{C}}\{\text{2-simplices in } \eta^{-1}(Y^1)\} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{array}{c} \text{2-simplices of } X \text{ whose vertex letters} \\ \text{repeat twice (after dropping all subscripts)} \end{array} \right\},$$

so this element gets killed by the quotient in $E_2^{2,0}$. (In other words, we have $C_{2,0} \cap d^{-1}(C_{0,1}) \subseteq C_{1,1} \cap d^{-1}(C_{0,1})$.) Assuming the calculation above is correct, the only way for this contradiction to occur is that the E_2-page of the Leray spectral sequence does not looks like as stated above. The author was not able to provide a detail calculation for each page due to time constraint. Such a calculation, due to the huge number of simplices, is best done using computer software like Maple instead of by hand.

References

[1] J.P. May, *A Concise Course in Algebraic Topology*, http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

[2] A. Patel and D. Sauriol, *Leray Spectral Sequence for Simplicial Maps* (arXiv:1912.08288).

[3] K. V. Madahar and K. S. Sarkaria, *A Minimal Triangulation of the Hopf Map and its Applications*, Geometriae Dedicata **105** (2000), p. 105–114.

[4] nLab contributors, *Subdivision*.

[5] J. McCleary, *A User's Guide to Spectral Sequences*, Cambridge University Press, 2001.

[6] M. Albanese, *Homogeneous space and quotient space for spin groups*, https://math.stackexchange.com/q/2873974 (version: 2018-08-06).