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Solutions in the Large for the Nonlinear Hyperbolic Conservation Laws of Gas Dynamics*

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The constraints under which a gas at a certain state will evolve can be given by three partial differential equations which express the conservation of momentum, mass, and energy. In these equations, a particular gas is defined by specifying the constitutive relation $e = e(v, S)$, where e = specific internal energy, v = specific volume, and S = specific entropy. The energy function $e = -\ln v + (S/R)$ describes a polytropic gas for the exponent $\gamma = 1$, and for this choice of $e(V, S)$, global weak solutions for bounded measurable data having finite total variation were given by Nishida in [10]. Here the following general existence theorem is obtained: let $e_\varepsilon(v, S)$ be any smooth one parameter family of energy functions such that at $\varepsilon = 0$ the energy is given by $e_0(v, S) = -\ln v + (S/R)$. It is proven that there exists a constant C independent of ε, such that, if $\varepsilon \cdot$ (total variation of the initial data) $< C$, then there exists a global weak solution to the equations. Since any energy function can be connected to $e_0(V, S)$ by a smooth parameterization, our results give an existence theorem for all the conservation laws of gas dynamics. As a corollary we obtain an existence theorem of Liu, *Indiana Univ. Math. J.* **26**, No. 1 (1977) for polytropic gases. The main point in this argument is that the nonlinear functional used to make the Glimm Scheme converge, depends only on properties of the equations at $\varepsilon = 0$. For general $n \times n$ systems of conservation laws, this technique provides an alternate proof for the interaction estimates in Glimm's 1965 paper. The new result here is that certain interaction differences are bounded by ε as well as by the approaching waves.

Contents.

INTRODUCTION

The existence of a solution to the initial value problem for a one-dimensional system of nonlinear hyperbolic conservation laws was solved by

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Glimm [2] for small variational data. In the following paper, the "Glimm Scheme" is used to obtain existence in the case of large variational data for a class of nonisentropic gas equations. The main idea is to study large variational solutions to ideal gas equations which are near a class of soluble equations. The soluble equations can be viewed as the nonisentropic equations for polytropic gases having $\gamma = 1$.[1] This generalizes the theorems of Nishida and Smoller [7] and Liu [5], who considered this problem for polytropic gases (a family of ideal gases parameterized by $1 < \gamma \leqslant 5/3$). New estimates, required for the proof of existence, are obtained for general $n \times n$ systems.

We first consider the initial value problem for a system of nonlinear hyperbolic partial differential equations in conservation form

$$\begin{aligned} &u_t + f(u)_x = 0, \qquad u = (u_1, \dots, u_n) = u(x, t),\\ &u(x, 0) = u_0(x), \qquad f(u) = (f_1(u), \dots, f_n(u)). \end{aligned} \tag{1}$$

We assume general conditions on f that guarantee unique solutions to Riemann problems (problems in which the initial data $u_0(x)$ are constant to the left and right of $x = 0$); i.e., we assume that the system is strictly hyperbolic, and that the ith characteristic family is either genuinely nonlinear or linearly degenerate; cf. [4]. The system is strictly hyperbolic if the eigenvalues of df, the first Frechet derivative of f, are real and distinct. Denoting the eigenvalues of df as $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and the respective right eigenvectors of df as R_i, we say that the ith characteristic field is genuinely nonlinear [resp. linearly degenerate] if λ_i increases [resp. is constant] along the integral curves of the eigenvectors R_i. It is commonly known that discontinuities form in the solutions of (1) even in the presence of smooth initial data. For this reason we look for weak solutions in the sense of the theory of distributions; i.e., solutions which satisfy

$$\iint_{\substack{-\infty < x < \infty\\ t \geqslant 0}} u\phi_t + f(u)\,\phi_x\,dx\,dt + \int_{-\infty}^{\infty} u(x, 0)\,\phi(x, 0)\,dx = 0$$

for every smooth function $\phi(x, t)$ with compact support. Further "entropy conditions" are required of the solutions in order to select physically relevant weak solutions; cf. [4].

We let $f_\epsilon(u) = f(u, \varepsilon)$, $0 \leqslant \varepsilon \leqslant 1$, be any smooth one parameter family of functions, such that $f_\epsilon(u)$ satisfies the above conditions at each fixed ε. By "smooth" we mean that $f(u, \varepsilon)$ has sufficiently many derivatives with respect

[1] Here "ideal gas" and "polytropic gas" are used in the sense of Courant and Friedrichs [1].

to u, ε. Actually, four derivatives will suffice. We write the parameterization of initial value problems

$$\begin{aligned} u_t + f_\epsilon(u)_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \qquad 0 \leqslant \varepsilon \leqslant 1. \tag{2}$$

In this paper, we first consider the general problem (2) in compact regions U of u-space where Riemann problems $\langle u_L, u_R \rangle$ have unique solutions at every ε in $[0, 1]$. We define the signed strengths of i-waves in a Riemann problem solution, and then consider the difference between the wave strengths in the Riemann problem $\langle u_L, u_R \rangle$, and the strengths in the Riemann problems $\langle u_L, u_M \rangle$, $\langle u_M, u_R \rangle$, where u_L, u_M, u_R are states in U. Writing

$$\begin{aligned} \langle u_L, u_R \rangle &= (c_1, ..., c_n) = c, \\ \langle u_L, u_M \rangle &= (a_1, ..., a_n) = a, \\ \langle u_M, u_R \rangle &= (b_1, ..., b_n) = b, \end{aligned}$$

where the right-hand side denotes the signed strengths of the solution waves, we show that the following interaction estimates hold:

$$\begin{aligned} |c_i(\varepsilon) - c_i(0)| &\leqslant G\varepsilon D, \\ |c_i - a_i - b_i| &\leqslant GD. \end{aligned}$$

Here c is viewed as a function of a, b and ε, D is the sum of the products of the approaching waves among a and b as defined by Glimm [2], and G is a constant depending only on the region U.

In Section 4 we consider the nonisentropic gas dynamic equations (Lagrangian form)

$$\begin{aligned} u_t + p_x &= 0, \qquad \text{where} \quad E = e(v, S) + \tfrac{1}{2}u^2, \\ v_t - u_x &= 0, \qquad p = p(v, S) = -e_v(v, S) = -e_v, \\ E_t + (up)_x &= 0, \end{aligned} \tag{3}$$

and

e = specific internal energy,	E = total energy,
v = specific volume,	S = specific entropy,
p = pressure,	u = velocity.

These equations represent the conservation of momentum, mass and energy, respectively, for one-dimensional gas flow. To guarantee that the system is strictly hyperbolic and either genuinely nonlinear or linear degenerate in each characteristic field, we assume the usual (physically justifiable) conditions on e, S:

$$\begin{aligned} -e_v &= p > 0, & -e_{vvv} &= p_{vv} > 0, \\ -e_{vv} &= p_v < 0, & S &= S(e, v). \end{aligned} \tag{*}$$

We consider smooth parameterizations $e_\epsilon(v, S)$ of energy functions which satisfy (*) at each ε, and which, at $\varepsilon = 0$, reduce to the energy function for the ideal gas given by

$$e_0(v, S) = -a^2 \ln(v) + \frac{a^2 S}{R} + C,$$

where a and C are arbitrary constants, and R is the specific gas constant. By smooth, we mean that $e(v, S, \varepsilon)$ has five derivatives with respect to v, S and ε, so that the nonlinear function in (3) has four derivatives. These systems can be studied because, near $\varepsilon = 0$, the shock–rarefaction curves have a nice structure when viewed in the transformed coordinates

$$(r, s, S) = \Psi(u, p, S) = \left(u + a \ln\left(\frac{1}{p}\right), u - a \ln\left(\frac{1}{p}\right), S\right).$$

This nice structure coupled with the estimates of the preceding sections, implies that the Glimm Scheme converges and allows us to prove the following theorem:

THEOREM 4.1. *Let E be any compact set in rsS-space, and let $N > 1$ be any positive constant. Then there exists a $C > 0$, where $C = C(E, N)$ such that, for every initial data $w_0(x) \subset E$ with total variation $\{w_0(x)\} = V \leqslant N$, if $\varepsilon \cdot V < C$, then there exists a global weak solution to* (3).

The energy function $e_0(v, S)$ can be viewed as the limiting state of a polytropic gas as $\varepsilon \to 0$, where the energy function for a polytropic gas is given by (cf. [1])

$$e = \left\{ v \exp\left(-\frac{S - S_0}{R}\right)\right\}^{-\epsilon} + \text{Constant}.$$

Thus, as a corollary of Theorem 4.1, we obtain an existence theorem of Liu [5], for polytropic gases. We note also that the method of proof here also handles smooth parameterizations of general 2×2 systems [3] which reduce, at $\varepsilon = 0$, to the system studied by Nishida in [6].

1. Preliminaries

We consider the shock–rarefaction and contact discontinuity curves for system (2) in regions of $u\varepsilon$-space where their existence is guaranteed. Note first that if $w = \Psi(u)$ is a regular smooth 1–1 onto transformation of R^n to itself, then shock–rarefaction and contact curves are defined, and Riemann problems can be solved in $Y(U)$ if and only if the corresponding things are true in U. For this reason we can, without loss of generality, discuss the shock–rarefaction and contact curves in regions of w-space where their existence is guaranteed. In such regions we adopt the following notation:

$$w = H_i(t_i, w_L, \varepsilon),$$

where

(i) if the ith characteristic family is genuinely nonlinear, t_i regularly parameterizes the i-shock–rarefaction curve starting at w_L for the system at ε, with $t_i \leqslant 0$ along the shock, $t_i \geqslant 0$ along the rarefaction curve;

(ii) if the ith characteristic family is linearly degenerate, t_i regularly parameterizes the contact curve starting at w_L for the system at ε. For gas dynamics, t_2 can be taken to be the change in entropy S.

For $n \times n$ systems let t_i denote parameterization with respect to arclength. For the gas dynamic equations, t_i will denote a parameterization with respect to a smooth function of ε as well as arclength. We let $t = (t_1, ..., t_n)$, and define

$$w = H(t, w_L, \varepsilon) = H_n(t_n, H_{n-1}(t_{n-1}, ..., H_1(t_1, w_L, \varepsilon), ..., \varepsilon), \varepsilon),$$

$$w = G(a, b, w_L, \varepsilon) = H(b, H(a, w_L \varepsilon), \varepsilon),$$

$$\text{where} \quad a = (a_1, ..., a_n), \quad b = (b_1, ..., b_n).$$

Note that if $w_R = H(t, w_L, \varepsilon)$, then the intermediate states in the definition of H solve the Riemann problem $\langle w_L, w_R \rangle$ at fixed ε. Thus we define t_i to be the *signed strength* and $|t_i|$ to be the *strength* of the i-wave in $\langle w_l, w_R \rangle$ for the system at ε. Moreover, letting $w_M = H(a, w_L, \varepsilon)$, we see that $w_R = G(a, b, w_L, \varepsilon)$ implies that the intermediate states in the definition of G solve the consecutive Riemann problems $\langle w_L, w_M \rangle$, $\langle w_M, w_R \rangle$.

We study the interaction problems for system (2) in compact sets U of w-space in which H, G satisfy the following conditions for ε in $[0, \varepsilon_1]$, some $\varepsilon_1 > 0$. These conditions guarantee the existence of solutions to Riemann problems in U_1 for ε in $[0, \varepsilon_1]$, and also ensure that H, G are well defined on "nicely shaped" domains in the variables t, a, b, w_L, ε. For convenience, we always assume that U_1 is a compact convex open set, so that it makes sense to talk of the derivatives of functions with domain $\mathrm{Cl}(U_1)$ and so that locally

Lipschitz continuous functions on $\mathrm{Cl}(U_1)$ have uniform Lipschitz bounds. For $n \times n$ systems (2), Lemma 1.2 states that such a U_1 exists in a neighborhood of every $u \in R^n$, where $\varepsilon_1 = 1$. For systems (3), we shall show in the next section that every compact set U_1 in $(r, s, S) = w$-space satisfies these conditions for some $\varepsilon_1 > 0$.

Conditions (H). There exists a compact convex open set $U_2 \supset \mathrm{Cl}(U_1)$, and positive constants $\tau < \tau_2$ such that, for ε in $[0, \varepsilon_1]$, the following hold:

(i) H is defined for every (t, w_L, ε) in

$$\Gamma_2 = \{t \in R^n: |t_i| \leqslant \tau_2\} \times \mathrm{Cl}\, U_2 \times [0, \varepsilon_1].$$

(ii) $U_2 \subset \mathrm{Range}_{|t_i| \leqslant \tau_2}$ (H) for each fixed w_L in $\mathrm{Cl}\, U_2$, ε in $[0, \varepsilon_1]$.
$U_1 \subset \mathrm{Range}_{|t_i| \leqslant \tau}$ (H) for each fixed w_L in $\mathrm{Cl}\, U_1$, ε in $[0, \varepsilon_1]$.

(iii) $G(\Gamma) \subset U_2$, where

$$\Gamma = \{a \in R^n: |a_i| \leqslant \tau\} \times \{b \in R^n: |b_i| \leqslant \tau\} \times \mathrm{Cl}\, U_1 \times [0, \varepsilon_1].$$

(iv) H, G are C^2 functions of their arguments, and second derivatives of H, G are locally Lipschitz in Γ_2, Γ, respectively.

(v) H is 1–1 and $|\partial H/\partial t| \neq 0$ in Γ_2 for each fixed w_L, ε.

Note that Condition (ii) implies that Riemann problems $\langle w_L, w\rangle$ are solvable when w_L and w are in U_2, and $\varepsilon \leqslant \varepsilon_1$. Condition (v) then implies that we can solve for t in terms of (w, w_L, ε) in the relation $w = H(t, w_L, \varepsilon)$ when w and w_L are in U_2, and $\varepsilon \leqslant \varepsilon_1$. Denoting this function as $t = B'(w, w_L, \varepsilon)$, we can write

$$t = B'(G(a, b, w_L, \varepsilon), w_L, \varepsilon) = B(a, b, w_L, \varepsilon)$$
$$= (B_1(a, b, w_L, \varepsilon), \ldots, B_n(a, b, w_L, \varepsilon)),$$

where

$$\langle u_L, u_M\rangle = a = (a_1, \ldots, a_n),$$
$$\langle u_M, u_R\rangle = b = (b_1, \ldots, b_n),$$
$$\langle u_L, u_R\rangle = t \equiv c = (c_1, \ldots, c_n).$$

For notational convenience, we let c denote the range variable of the function B, which is defined on Γ. At fixed a, b and w_L, we write

$$c(\varepsilon) = B(a, b, w_L, \varepsilon),$$

where $c(\varepsilon)$ is a function of a, b, w_L. Conditions (iv) and (v) imply the following lemma:

LEMMA 1.1. *Let* U_1 *and* ε_1 *satisfy the Conditions* (H). *Then the function*

$c = B(a, b, w_L, \varepsilon)$ is defined, is C^2, and second derivatives of B are locally Lipschitz in Γ.

Proof of Lemma 1.1. By the chain rule, the composition of C^2 functions with locally Lipschitz second derivatives, is also a C^2 function with locally Lipschitz second derivatives. Thus we show that B' satisfies Lemma 1.1. Consider the equation $P(w, t, w_L, \varepsilon) = w - H(t, w_L, \varepsilon) = 0$ defined on $U_2 \times \Gamma_2$. P is C^2 with locally Lipschitz second derivatives in $U_2 \times \Gamma_2$. Moreover,

$$\left| \frac{\partial P}{\partial t} \right| = \left| \frac{\partial H}{\partial t} \right| \neq 0$$

by condition (v), and thus the implicit function theorem implies that $t = B'(w, w_L, \varepsilon)$ is defined locally and has the same smoothness as P. Since P is also 1–1 at each w, w_L, ε in $U_2 \times U_2 \times [0, \varepsilon_1]$ by (iv), we have that B' is globally defined, is C^2, and has second derivatives which are locally Lipschitz in $U_2 \times U_2 \times [0, \varepsilon_1]$. But $B(a, b, w_L, \varepsilon) = B'(G(a, b, w_L, \varepsilon), w_L, \varepsilon)$, and hence B satisfies the conditions of Lemma 1.1. Q.E.D.

We say that two waves a_i, b_j in a, b approach (cf. Glimm [2]) if $i > j$, or if $i = j$ and not both a_i and b_j are positive (i.e., not both a_i, b_j measure the strengths of rarefaction waves). We define $D = D(a, b)$ to be the sum of the products of the strengths of the approaching waves in a, b, and write this

$$D = \sum_{APP} |a_i|\,|b_j|. \tag{1.0}$$

For $n \times n$ systems of Eqs. (2), we are interested in estimating the difference between the strengths of the outgoing and incoming waves in the interaction function B; hence we study the functions

$$\begin{aligned} e_i = c_i - a_i - b_i &= B_i(a, b, w_L, \varepsilon) - a_i - b_i \\ &= F_i(a, b, w_L, \varepsilon), \end{aligned} \tag{1.01}$$

where again the domain of F_i is Γ. It is easy to check (cf. Glimm [2]) that $e_i = 0$ when $D = 0$ (since the outgoing waves are then the same as the incoming waves in the corresponding interaction). Moreover

$$c_i(\varepsilon) - c_i(0) = e_i(\varepsilon) - e_i(0) = B_i(a, b, w_L, \varepsilon) - B_i(a, b, w_L, 0)$$

is zero when $D = 0$ or when $\varepsilon = 0$. We shall show that this, together with the smoothness properties of B given in Lemma 1.1, imply that

$$c_i = O(a, b, w_L, \varepsilon) \cdot D,$$

where O is uniformly bounded, and locally Lipschitz in ε uniformly in the remaining variables. Since the domain of O is the compact set Γ, this will imply that on Γ,

$$O(a, b, w_L, \varepsilon) - O(a, b, w_L, 0) = O(1)\varepsilon$$

and hence the following estimates hold in Γ for some constant $G > 1$:

$$\begin{aligned} |c_i(\varepsilon) - c_i(0)| &\leqslant G\varepsilon D, \\ |c_i - a_i - b_i| &\leqslant GD. \end{aligned} \tag{1.1}$$

One difficulty is the D changes form depending on whether a_i and b_i are positive an negative. Thus we break the domain Γ up into regions where D has a fixed form; i.e., regions where $\text{sign}(a_i)$ and $\text{sign}(b_i)$ are constant. Thus we let

$$s = (s_1, \ldots, s_{2n}) \qquad \text{where} \quad s_i = + \text{ or } -$$

and define

$$\Gamma_s = \{(a, b, w_L, \varepsilon) \in \Gamma : \text{sign}(a_i) = s_i, \text{sign}(b_i) = s_{i+n}\}.$$

(For convenience, we allow $x = 0$ to satisfy $\text{sign}(x) = s_i$ for every s_i.) Then on Γ_s,

$$\begin{aligned} D_s = \delta_1 |a_1 b_1| + |a_2 b_1| + \delta_2 |a_2 b_2| + |a_3 b_1| + |a_3 b_2| \\ + \delta_3 |a_3 b_3| + \cdots + \delta_n |a_n b_n| \end{aligned}$$

with $\delta_s = (\delta_1, \ldots, \delta_n)$ some sequence of 0's and 1's. We show that $c_i = O(a, b, w_L, \varepsilon) D_s$ on each Γ_s, where O is bounded and locally Lipschitz in (w_L, ε) uniformly in a, b. This will imply that the estimates (1.1) hold in each Γ_s and hence on all of Γ. The proof of the following theorem is the primary goal of the next section:

THEOREM 1.1. *Let $w \in R^n$ with the setting of system* (2). *If U_1 and ε_1 satisfy Conditions* (H), *then the interaction estimates* (1.1) *hold for every w_L, w_M, w_R in U_1, and ε in $[0, \varepsilon_1]$.*

The following lemma, whose proof is left to Apendix I, implies that for $n \times n$ systems (2), interaction estimates (1.1) holds for every ε, in some neighborhood of every $u \in R^n$.

LEMMA 1.2. *For every $u \in R^n$, there exists a neighborhood U_1 of u such that Conditions* (H) *hold with $\varepsilon_1 = 1$ and $w = u$.*

Theorem 1.1 and Lemma 1.2 together imply the following theorem, which yields among other things, the main interaction estimates of Glimm [2].

THEOREM 1.2. *For every $u \in R^n$, there is a neighborhood U_1 of u such that the interaction estimates* (1.1) *hold for every u_L, u_m, u_R in U_1, and every ε in* $[0, 1]$.

2. INTERACTION ESTIMATES

Let $c = f(a, b, z)$ be defined on the set

$$W = \{a \in R^n : a_i \geqslant 0\} \times \{b \in R^n : b_i \geqslant 0\} \times Z \qquad \text{with } c \in R^1,$$

where $Z \subset R^l$ is compact, convex, and is the closure of its interior. Assume that $f \in C^2$ in $\text{Int}(W)$ with derivatives continuous up to the boundary, and that second derivatives of f are locally Lipschitz in W. Further assume that $c = 0$ when $D = 0$, where

$$D = \delta_1 a_1 b_1 + a_2 b_1 + \delta_2 a_2 b_2 + a_3 b_1 + a_3 b_2 + \delta_3 a_3 b_3 + \cdots + \delta_n a_n b_n, \quad (2.1)$$

where $\delta = (\delta_1, ..., \delta_n)$ is a fixed sequence of 0's and 1's. The main result of this section is the following theorem:

THEOREM 2.1. *If $c = f(a, b, z)$, where f satisfies the conditions above on the domain W, then*

$$c = O(a, b, z) \cdot D,$$

where O is locally bounded, and locally Lipschitz in z, uniformly in a, b.

We prove this theorem with the aid of the following lemmas. We write

$$D = (\delta_1 a_1 + a_2 + \cdots + a_n)\, b_1 + (\delta_2 a_2 + a_3 + \cdots + a_n)\, b_2 + \cdots + \delta_n a_n b_n.$$

Letting the coefficients of b_i be denoted by d_i, we have

$$\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} \delta_1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & \delta_2 & 1 & 1 & \cdots & 1 \\ & & & \vdots & & \\ 0 & 0 & \cdot & \cdot & \cdots & \delta_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix},$$

which we abbreviate as $d = Ma$, and thus $D = \sum_{i=1}^{n} d_i b_i$. This is the setting for the following lemma.

LEMMA 2.1. *There exists nonsingular transformations A, B such that $a' = Aa$, $b' = Bb$ and $D = \sum_{i=1}^{m} a_i' b_i'$ for some $m \leqslant n$, where A, B are matrices all of whose entries are either 0's or 1's. Moreover, the image of a* *(for $a_i \geqslant 0$) under A is the region $a'_1 \geqslant a'_2 \geqslant \cdots a'_m \geqslant 0$ in the first m variables of a', and for $k > m$, $a'_k \geqslant 0$ or $0 \leqslant a'_k \leqslant a'_j$ for some $1 \leqslant j \leqslant m$ determines the range of A in the remaining variables. The image of b (for $b_i \geqslant 0$) under B is the region $b'_i \geqslant 0$ for $1 \leqslant i \leqslant m$, and for $k > m$, $0 \leqslant b'_k \leqslant b'_j$, for some $1 \leqslant j \leqslant m$, determines the range of B in the remaining variables.*

Proof. We define A, B by defining each row of A, B. Thus, let M_k, A_k, and B_k denote the vectors which are respectively the kth rows of M, A, and B. We would like to define $A_k = M_k$, $B_k = e_k =$ the vector with 1 in the kth spot, zeros everywhere else, so that $a'_i = d_i$, $b'_i = b_i$ with $\sum_{k=1}^{n} d_i b_i = D$; but whenever $\delta_i = 0$, $\delta_{i+1} = 1$ we see that $d_i = d_{i+1}$, and hence A would not be nonsingular. However, in this case we can write $d_i b_i + d_{i+1} b_{i+1} = d_i(b_i + b_{i+1})$. To exploit this idea, we partition the sequence $\{\delta_1,..., \delta_n\}$ into consecutive subsequences of a nice form. Without changing the order of the δ_j's, let

$$\{\delta_1,..., \delta_n\} = \underbrace{\{\delta_{m_1},..., \delta_{n_1}\}}_{S_1}, \underbrace{\{\delta_{m_2},..., \delta_{n_2}\}}_{S_2},..., \underbrace{\{\delta_{m_p},..., \delta_{n_p}\}}_{S_p},$$

where, by choosing $S_k = \varnothing$ as needed, we make the following conditions hold on the parity of the indices of $\{S_k\}$:

For i even:

$$S_i = \{0, 0,..., 0, 1\} \text{ or } \varnothing \text{ if } i \neq p.$$
$$\text{If } \delta_n = 0, \text{ make } p \text{ even with } S_p = \{0, 0,..., 0\}.$$

For i odd:

$$S_i = \{1, 1,..., 1\} \text{ or } \varnothing.$$

It is clear that any sequence of 0's and 1's can be so partitioned. We say loosely that $j \in S_k$ if $m_k \leqslant j \leqslant n_k$ for $S_k \neq \varnothing$. With this set up, we define A_k and B_k as follows:

(i) If $k = n_j$ or $k = n_j - 1$ for j even with $S_j \neq \varnothing$, define

$$A_{n_j-1} = M_{n_j-1}, \qquad A_{n_j} = e_{m_j},$$
$$B_{n_j-1} = e_{n_j-1} + e_{n_j}, \qquad B_{n_j} = e_{n_j}.$$

(ii) Otherwise

$$A_k = M_k,$$
$$B_k = e_k.$$

We show that A is nonsingular.

First, every A_k for $k \in S_j$ is either M_k or e_{m_j}. Thus all entries in A_k for $i < m_j$ are zero, and not all entries from m_j to n_j are zero. But a quick check shows that every A_l such that $l \notin S_j$ has either ones in the $m_j - 1$ through n_j entries, or else zeros in the m_j through n_j entries (depending on whether A_l is a row which is respectively above or below A_k). Thus linear combinations of $\{A_l\}$ have either a constant value in the $m_j - 1$ through n_j entries, or else all zeros in the m_j through n_j entries. Thus $A_k \notin \text{Span}\{A_l : l \notin S_j\}$. Therefore A is nonsingular if $\{A_{m_j},..., A_{n_j}\}$ are linearly independent for every j such that $S_j \neq \varnothing$. But this is clearly true for j odd, since then the vectors $A_{m_j},..., A_{n_j}$ are upper triangular with nonzero entries along the diagonal of A. For j even, $A_{m_j},..., A_{n_j}\}$ are linearly independent. Therefore A is a nonsingular matrix all of whose entries are 0's and 1's.

We show that B is nonsingular.

$$B_k = e_k \text{ if } k \neq n_j - 1 \text{ or } n_j \qquad \text{for some } j \text{ even,}$$

while

$$B_{n_j} = e_{n_j}, B_{n_{j-1}} = e_{n_{j+1}} + e_{n_j} \qquad \text{otherwise.}$$

It is thus clear that $\{B_k\}$ are linearly independent, and hence B is a nonsingular matrix of 0's and 1's.

Claim.

$$D = \sum_{\substack{i \text{ such that} \\ a_i' = d_i}} a_i' \, b_i'.$$

We have that $D = \sum_{i=1}^n d_i b_i$. Moreover, for $i \neq n_j$ or $n_j - 1$ for j even, we have

$$a_i' = d_i,$$

$$b_i' = b_i.$$

But in the other case, $d_{n_j - 1} = d_{n_j}$, which implies that

$$d_{n_j-1} b_{n_j-1} + d_{n_j} b_{n_j} = d_{n_{j-1}}(b_{n_j-1} + b_{n_j}) = a'_{n_j-1} b'_{n_j-1},$$

where

$$a'_{n_j-1} = d_{n_j-1}.$$

Hence

$$D = \sum_{i=1}^{n} d_i b_i = \sum_{\substack{i \neq n_j, n_j - 1 \\ \text{for } j \text{ even}}} d_i b_i + \sum_{\substack{i \text{ even} \\ S_j \neq \varnothing}} d_{n_j - 1}(b_{n_j - 1} + b_{n_j})$$

$$= \sum_{\substack{i \neq n_j, n_j - 1 \\ \text{for } j \text{ even}}} a'_i b'_i + \sum_{\substack{j \text{ even} \\ A_j \neq \varnothing}} a'_{n_j - 1} b'_{n_j - 1}$$

$$= \sum_{\substack{i \text{ such that} \\ a'_i = d'_i}} a'_i b'_i,$$

which proves the claim. Note also that if $a'_i \neq d_i$, then $i = n_j$ for some j even, and $a'_{n_j} = a_{m_j}$. Here the value of a_{m_j} is independent of a'_k for $k > m_j$, thus the values of a'_{n_j} are simply constrained to be less than $a'_{m_j - 1}$ in the primed variables. Also in the case, $b'_{n_j} = b_{n_j}$ with $b'_{n_j - 1} = b_{n_j + 1} + b_{n_j}$ and the value of b_{n_j} is independent of any other b'_k. Hence b'_{n_j} is simply constrained to be less than $b'_{n_j - 1}$ in the primed variables.

Now, without loss of generality, reorder $a'_1, \ldots, a'_n$ and $b'_1, \ldots, b'_n$ so that the a'_i which are equal to d_i are listed first, and in order of increasing i. Then A, B remain nonsingular, and D is then given by

$$D = \sum_{i=1}^{m} a'_i b'_i \qquad \text{some } m \leqslant n.$$

Moreover, letting j and k index the original sequences, and letting i index the reordered sequences, if $i < m$ we have

$$a'_i = \delta_j a_j + a_{j+1} + ,\ldots, + a_n,$$

$$a'_{i+1} = \delta_k a_k + a_{k+1} + ,\ldots, a_n,$$

for some $k > j$, where, if $k = j + 1$, not both $\delta_j = 0$ and $\delta_k = 1$. Thus $a'_i \geqslant a'_{i+1}$. Hence the image of $\{a \text{ in } R^n\colon a_i \geqslant 0\}$ under A is given by $a'_1 \geqslant a'_2 \geqslant ,\ldots, \geqslant a'_m$ in the first m variables of a'. With the same reordering given for the indices of the coordinates of a' above, if $i \leqslant m$, either $b'_i = b'_j$ or $b_i = b_k + b_{k+1}$, where such a j is not equal to such a k or $k + 1$ for $i \leqslant m$. Thus any positive values for $b'_1, \ldots, b'_m$ can be obtained in the image of $\{b \in R^n\colon b_i \geqslant 0\}$ under B. The note after the claim above now completes the proof of Lemma 2.1.

LEMMA 2.2. *Let $f(x, y, z)$ be a real valued C^2 function with locally Lipschitz second derivatives defined on $R^{n^+} \times R^{m^+} \times Z$, where $R^{n^+} = \{x \in R^n\colon x_i \geqslant 0\}$, $R^{m^+} = \{y \in R^m\colon y_i \geqslant 0\}$ and Z is a convex set in R^l which is the closure of its interior. Then we can write*

$$f(x, y, z) = f(0, y, z) + O_1(x, y, z)\,\|x\|$$

and

$$O_1(x, y, z) = O_1(x, 0, z) + O_2(x, y, z) \| y \|,$$

where $O_2(x, y, z)$ is locally bounded and locally Lipschitz in z, uniformly in x and y.

Proof. First, let $g(t)$ be a real valued C^N function of the real variable t for $t \geqslant 0$. Then by Taylor's theorem,

$$g(t) = g(0) + g'(0)t + ,..., + g^{(N-1)}(0)t + R_N,$$

where

$$R_N = \frac{1}{N!} \int_0^t (t-s)^{N-1} f^{(N-2)}(s)\, ds.$$

Letting $s = ut$, $ds = t\, du$, we can write

$$R_N = \left\{ \frac{1}{N!} \int_0^1 (1-u)^{N-1} f^{(N)}(ut)\, du \right\} t^N,$$

and so for $N = 1$, we have

$$g(t) = g(0) + \left\{ \int_0^1 g'(ut)\, du \right\} t. \tag{T$_1$}$$

Now let $\bar{x}$ be a unit vector in the domain of the variable x. For fixed $\bar{x}$, define

$$F(t, y, z) = f(t\bar{x}, y, z).$$

Then by (T_1), we can write

$$F(t, y, z) = f(0, y, z) + O_1(t, y, z) \cdot t,$$

where

$$O_1(t, y, z) = \int_0^1 F_{(1)}(ut, y, z)\, du,$$

where the subscript "(1)" denotes the partial derivative with respect to the first slot variable. Thus, for $x = t\bar{x}$ we have

$$f(x, y, z) = f(0, y, z) + O_1(x, y, z) \| x \|,$$

where

$$O_1(x, y, z) = \int_0^1 \frac{\partial}{\partial(ut)} f(ut\bar{x}, y, z)\, du$$

or

$$O_1(x, y, z) = \int_0^1 \nabla_{(1)} f(ux, y, z) \cdot \bar{x}\, du, \tag{T$_2$}$$

where $\nabla_{(1)} f$ denotes the gradient of f with respect to the first slot variable. Since $O_1(x, y, z)$ is uniformly bounded by $2 \|\nabla_{(1)} f(0, y, z)\|$ near $x = 0$, we can differentiate with respect to y through the integral sign in (T_1). Fixing $\bar{x}$ again, choose a fixed unit vector $\bar{y}$ in the domain of the variable y, and define

$$O_1(x, s, z) = O_1(x, s\bar{y}, z).$$

Using Taylor's theorem again, we obtain

$$O_1(x, s, z) = O_1(x, 0, z) + O_2(x, s, z) \cdot s,$$

where, for $y = s\bar{y}$, we have

$$O_2(x, y, z) = \int_0^1 \nabla_{(2)} O_1(x, vy, z) \cdot \bar{y}\, dv,$$

where $\nabla_{(2)} O_1$ denotes the gradient of O_1 with respect to the second slot variable. Substituting (T_2) into the last equation yields

$$O_2(x, y, z) = \int_0^1 \int_0^1 \nabla_{(2)} \{\nabla_{(1)} f(ux, vy, z) \cdot \bar{x}\} \cdot \bar{y}\, du\, dv$$

or

$$O_2(x, y, z) = \int_0^1 \int_0^1 \bar{x} \cdot \left[\frac{\partial^2 f}{\partial(ux_i)\, \partial(vy_j)} (ux, vy, z)\right] \cdot \bar{y}^t\, du\, dv, \tag{T$_3$}$$

where the expression in brackets denotes the $n \times m$ matrix with (i, j)th entry given. Thus, $O_2(x, y, z)$ is uniformly bounded by $2 \|(\partial^2 f/\partial x_i \partial y_j)(0, 0, z)\|$ near $x = 0$, $y = 0$. That $O_2(x, y, z)$ is locally Lipschitz in z, uniformly in x and y, follows immediately from the hypothesis that

$$\left[\frac{\partial^2 f}{\partial x_i\, \partial y_j} (x, y, z)\right],$$

a second derivative of f, is locally Lipschitz; i.e., on compact sets,

$$\begin{aligned}|O_2(x, y, z_2) - O_2(x, y, z_1)| &= \left| \int_0^1 \int_0^1 \bar{x} \cdot \left\{ \left[\frac{\partial^2 f}{\partial(ux_i)\,\partial(vy_j)} (ux, vy, z_2) \right] \right.\right. \\ &\quad \left.\left. - \left[\frac{\partial^2 f}{\partial(ux_i)\,\partial(vy_j)} (ux, vy, z_1) \right] \right\} \cdot \bar{y}^t \, du\, dv \right| \\ &\leqslant \int_0^1 \int_0^1 \left\| \left[\frac{\partial^2 f}{\partial(ux_i)\,\partial(vy_j)} (ux, vy, z_2) \right] \right. \\ &\quad \left. - \left[\frac{\partial^2 f}{\partial(ux_i)\,\partial(vy_j)} (ux, vy, z_1) \right] \right\| du\, dv \\ &\leqslant K \|z_2 - z_1\|\end{aligned}$$

since

$$\left[\frac{\partial^2 f}{\partial x_i\, \partial y_j} (x, y, z) \right]$$

is uniformly Lipschitz continuous on compact sets. This completes the proof of Lemma 2.2.

Note that the proof of Lemma 2.2 also goes through if f satisfies the conditions of this lemma on the closure of a domain of the form $\mathscr{A} \times \mathscr{B} \times Z$, where $\mathscr{A}$ [resp. $\mathscr{B}$] is a convex subset of R^{n^+} [resp. R^{m^+}], which is the closure of its interior and contains $x = 0$ [resp. $y = 0$].

LEMMA 2.3. *Let $c = f(a, b, \alpha, \beta, \gamma, z)$ be a real valued function which is C^2 and has locally Lipschitz second derivatives on a domain Y of the following form*:

$$\begin{aligned}&a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0, && 1 \leqslant i \leqslant n,\\ &b_i \geqslant 0, && 1 \leqslant i \leqslant n,\\ &0 \leqslant \alpha_i \leqslant a_i \text{ or } \alpha_i \equiv 0, && 1 \leqslant i \leqslant n,\\ &0 \leqslant \beta_i \leqslant b_i \text{ or } \beta_i \equiv 0, && 1 \leqslant i \leqslant n,\\ &\gamma_i \geqslant a_{k(i)} \text{ or } \gamma_i \leqslant \gamma_j \text{ for } \gamma_j \geqslant a_{k(j)} \text{ or } \gamma_i \equiv 0, && 1 \leqslant i \leqslant m,\\ &z \in Z \textit{ a convex open set in } R^l. &&\end{aligned}$$

Then if $c = 0$ when the inner product $a \cdot b = 0$ in Y, then

$$c = O(a, b, \alpha, \beta, \gamma, z) a \cdot b$$

where in Y, $O(a, b, \gamma, z)$ is locally bounded and locally Lipschitz in z, *uniformly in the remaining variables. (The main point here is that Y is convex, and further, if $(a, b, \alpha, \beta, \gamma, z)$ is in Y, then the point obtained by letting $b_i = \beta_i = 0$ or $a = \alpha = 0$ without changing the other entries, is also in Y.)*

Proof. We prove this by induction on n. For case $n = 1$ we have that $c = f(a, b, \alpha, \beta, \gamma, z)$ satisfies

$$f(0, b, 0, \beta, \gamma, z) = 0 = f(a, 0, \alpha, 0, \gamma, z)$$

in Y a convex set. Since Lemma 2.2 applies to $f(a, b, \alpha, \beta, \gamma, z)$ with $x = (a, \alpha)$ and $y = (b, \beta)$, we can write

$$f(a, b, \alpha, \beta, \gamma, z) = f(0, b, 0, \beta, \gamma, z) + O_1(a, b, \alpha, \beta, \gamma, z) \|(a, \alpha)\|,$$

where

$$O_1(a, b, \alpha, \beta, \gamma, z) = O_1(a, 0, \alpha, 0, \gamma, z) + O_2(a, b, \alpha, \beta, \gamma, z) \|(b, \beta)\|$$

and where $O_2(a, b, \alpha, \beta, \gamma, z)$ is locally bounded, and locally Lipschitz in z, uniformly in the remaining variables. Since

$$0 = O_1(a, 0, b, 0, \gamma, z) \|(a, \alpha)\|$$

we have that

$$f(a, b, \alpha, \beta, \gamma, z) = O_2(a, b, \alpha, \beta, \gamma, z) \|(a, \alpha)\| \, \|(b, \beta)\|.$$

But since $b \geqslant \beta$ and $a \geqslant \alpha$ in Y, we have

$$\frac{\|(b, \beta)\|}{b} = \sqrt{1 + \left(\frac{\beta}{b}\right)^2}, \qquad \frac{\|(a, \alpha)\|}{a} = \sqrt{1 + \left(\frac{\alpha}{a}\right)^2}$$

where each of these is bounded by $\sqrt{2}$ in Y. Therefore

$$\begin{aligned} f(a, b, \alpha, \beta, \gamma, z) &= \left\{ O_2(a, b, \alpha, \beta, \gamma, z) \sqrt{1 + \left(\frac{\beta}{b}\right)^2} \sqrt{1 + \left(\frac{\alpha}{a}\right)^2} \right\} a \cdot b \\ &= O_2 O_3 a \cdot b = O(a, b, \alpha, \beta, \gamma, z) \, a \cdot b. \end{aligned}$$

Here O_3 is bounded by 2 in Y and does not involves the z variables, and so $O_2 O_3$ as well as O_2 is locally bounded and locally Lipschitz in z, uniformly in the remaining variables. This proves case $n = 1$.

We now prove case $p = n$. Assume Lemma 2.3 is true for $p \leqslant n - 1$, and let

$$\begin{aligned} c &= f(a, b, \alpha, \beta, \gamma, z) \\ &= f(a_1, ..., a_n, b_1, ..., b_n, \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n, \gamma, z) \end{aligned}$$

satisfy the hypotheses of the lemma. Then Lemma 2.2 applies to f with $x=(b_1,\beta_1)$ and $y=(a,\alpha)$, and so we can write

$$f(a,b,\alpha,\beta,\gamma,z)=f(a,0,b_2,\ldots,b_n,\alpha,0,\beta_2,\ldots,\beta_n,\gamma,z) + O_1(a,b,\alpha,\beta,\gamma,z)\,\|(b_1,\beta_1)\|, \tag{A}$$

where

$$O_1(a,b,\alpha,\beta,\gamma,z)=O_1(0,b,0,\beta,\gamma,z)+O_2(a,b,\alpha,\beta,\gamma,z)\,\|(a,\alpha)\| \tag{B}$$

and where O_2 is locally bounded and locally Lipschitz in z, uniformly in the remaining variables. But

$$\begin{aligned} f(a_1,\ldots,a_n,0,b_2,\ldots,b_n,\alpha_1,\ldots,\alpha_n,0,\beta_2,\ldots,\beta_n,\gamma,z) &= g(a_2,\ldots,a_n,b_2,\ldots,b_n,\alpha_2,\ldots,\alpha_n,\beta_2,\ldots,\beta_n,a_1,\alpha_1,\gamma,z) \\ &= g(a',b',\alpha',\beta',\gamma',z), \end{aligned}$$

where

$$\begin{aligned} a' &= (a_2,\ldots,a_n) \text{ is defined on } a_2 \geqslant a_3 \geqslant ,\ldots, \geqslant a_n, \\ b' &= (b_2,\ldots,b_n) \text{ is defined on } b_i \geqslant 0, \\ \alpha' &= (\alpha_2,\ldots,\alpha_n) \text{ is defined on } 0 \leqslant \alpha_i \leqslant a_i \text{ or } \alpha_i \equiv 0, \\ \beta' &= (\beta_2,\ldots,\beta_n) \text{ is defined on } 0 \leqslant \beta_i \leqslant b_i \text{ or } \beta_i \equiv 0, \\ \gamma' &= (a_1,\alpha_1,\gamma) \text{ is defined for } \alpha_1 \leqslant a_1 \text{ for } a_1 \geqslant a_2. \end{aligned}$$

Since $g(a',b',\alpha',\beta',\gamma',z)=0$ when $a'\cdot b'=0$, g satisfies the hypotheses of this lemma for $p=n-1$. So by the induction hypothesis

$$\begin{aligned} g(a',b',\alpha',\beta',\gamma',z) &= O_3'(a',b',\alpha',\beta',\gamma',z)\,a'\cdot b' \\ &= O_3(a,b,\alpha,\beta,\gamma,z)\sum_{i=2}^{n} a_i b_i, \end{aligned} \tag{C}$$

where O_3' and hence O_3 is locally bounded and locally Lipschitz in z, uniformly in the remaining variables on Y. Since $a\cdot b=0$ when $a=0$, we have from (A) that

$$0=O_1(0,b,0,\beta,\gamma,z)\,\|(b_1,\beta_1)\|$$

and so combining (A), (B), and (C) yields

$$\begin{aligned} f(a,b,\alpha,\beta,\gamma,z) &= O_3(a,b,\alpha,\beta,\gamma,z)\sum_{i=2}^{n} a_i b_i \\ &\quad + O_2(a,b,\alpha,\beta,\gamma,z)\,\|(a,\alpha)\|\,\|(b_1,\beta_1)\|. \end{aligned} \tag{D}$$

But in Y, $a_1 \geqslant a_i$ and $a_1 \geqslant \alpha_i$ for $1 \leqslant i \leqslant n$, and so

$$\frac{\|(a, \alpha)\|}{a_1} = \frac{\sqrt{a_1^2 + \cdots + a_n^2 + \alpha_1^2 + \cdots + \alpha_n^2}}{a_1} = \sqrt{\sum_{i=1}^{n} \left(\frac{a_i}{a_1}\right)^2 + \sum_{i=1}^{n} \left(\frac{\alpha_i}{a_1}\right)^2} \leqslant \sqrt{2n}. \tag{E}$$

Moreover, $b_1 \geqslant \beta_1$, and so

$$\frac{\|(b_1, \beta_1)\|}{b_1} = \sqrt{1 + \left(\frac{\beta_1}{b_1}\right)^2} \leqslant \sqrt{2}. \tag{F}$$

Therefore, putting (E) and (F) into (D) yields

$$\begin{aligned} f(a, b, \alpha, \beta, \gamma, z) &= O_3 \sum_{i=2}^{n} a_i b_i + \left\{ O_2 \frac{\|(b_1, \beta_1)\|}{b_1} \frac{\|(a, \alpha)\|}{a_{i1}} \right\} a_1 b_1 \\ &= \left\{ O_3 \sum_{i=2}^{n} \frac{a_i b_i}{a \cdot b} + O_2 \frac{\|(b_1, \beta_1)\| \, \|(a, \alpha)\|}{a_1 b_1} \frac{a_1 b_1}{a \cdot b} \right\} a \cdot b \\ &= O(a, b, \alpha, \beta, \gamma, z)\, a \cdot b. \end{aligned}$$

But O_3 and O_2 are locally bounded and locally Lipschitz in z, uniformly in the remaining variables, while the other terms in $O(a, b, \alpha, \beta, \gamma, z)$ are locally bounded and do not involve the z variables. Thus $O(a, b, \alpha, \beta, \gamma, z)$ is locally bounded and locally Lipschitz in z, uniformly in the remaining variables. This completes the proof of Lemma 2.3.

COROLLARY. *Let $c = f(a, b, \alpha, \beta, \gamma, z)$ satisfy the conditions of Lemma 2.3 except that $c \leqslant 0$ when $a \cdot b = 0$. Then for every compact set V in Y, there exists a function $O(a, b, \alpha, \beta, \gamma, z)$ defined on Y, such that $O(a, \beta, \alpha, \beta, \gamma, z)$ is locally bounded, is locally Lipschitz in z uniformly in the remaining variables, and such that in V,*

$$c \leqslant O(a, b, \alpha, \beta, \gamma, z)\, a \cdot b.$$

Proof. We prove this by induction on n. For case $n = 1$, we have that $c = f(a, b, \alpha, \beta, \gamma, z) \leqslant 0$ when $a = \alpha = 0$ or $b = \beta = 0$. Define

$$\begin{aligned} h(a, b, \alpha, \beta, \gamma, z) &= f(a, 0, \alpha, 0, \gamma, z) \\ &\quad + \frac{b}{a + b} \{ f(0, b, 0, \beta, \gamma, z) - f(a, 0, \alpha, 0, \gamma, z) \}. \end{aligned}$$

Since h takes the average of f between $(0, b, 0, \beta, \gamma, z)$ and $(a, 0, \alpha, 0, \gamma, z)$ at fixed (γ, z), and since f is negative when $a=0$ or $b=0$ in Y, we can conclude that

$$h(a, b, \alpha, \beta, \gamma, z) \leqslant 0,$$

and moreover, h agrees with f when $a=\alpha=0$ or $b=\beta=0$. Since

$$\left\{\frac{f(0, b, 0, \beta, \gamma, z) - f(a, 0, \alpha, 0, \gamma, z)}{a+b}\right\}$$

has the smoothness properties of f away from $a=b=0$, and loses at most one derivative at $a=b=0$, it can easily be shown that

$$\frac{b}{a+b}\{f(0, b, 0, \beta, \gamma, z) - f(a, 0, \alpha, 0, \gamma, z)\}$$

and hence h, has the same smoothness as f in Y. Thus,

$$g(a, b, \alpha, \beta, \gamma, z) = f(a, b, \alpha, \beta, \gamma, z) - h(a, b, \alpha, \beta, \gamma, z)$$

dominates f, has the smoothness of f, and vanishes when $a=\alpha=0$ or $b=\beta=0$. Therefore, by Lemma 2.3,

$$g(a, b, \alpha, \beta, \gamma, z) = O(a, b, \alpha, \beta, \gamma, z)\, ab$$

and so

$$f(a, b, \alpha, \beta, \gamma, z) \leqslant O(a, b, \alpha, \beta, \gamma, z)\, ab,$$

where $O(a, b, \alpha, \beta, \gamma, z)$ is locally bounded and locally Lipschitz in z, uniformly in the remaining variables on Y. This completes the proof of the corollary for case $n=1$.

We now prove the case $p=n$. Assume the corollary is true for $p<n$ and let

$$\begin{aligned} c &= f(a, b, \alpha, \beta, \gamma, z) \\ &= f(a_1, ..., a_n, b_1, ..., b_n, \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n, \gamma, z) \end{aligned}$$

satisfy the hypotheses of the corollary, so that, in particular, $c \leqslant 0$ when the inner product $a \cdot b = 0$. Let

$$f_1(a, b, \alpha, \beta, \gamma, z) = f(a_1, ..., a_n, 0, b_2, ..., b_n, \alpha_1, ..., \alpha_n, 0, \beta_2, ..., \beta_n, \gamma, z),$$

$$f_2(a, b, \alpha, \beta, \gamma, z) = f(a_1, ..., a_{n-1}, 0, b_1, ..., b_n, \alpha_1, ..., \alpha_{n-1}, 0, \beta_1, ..., \beta_n, \gamma, z).$$

But $f_1(a, b, \alpha, \beta, \gamma, z)$ [respectively $f_2(a, b, \alpha, \beta, \gamma, z)$] is negative when

$$\sum_{i=2}^{n} a_i b_i = 0 \left[\text{respectively } \sum_{i=1}^{n-1} a_i b_i = 0\right],$$

and so with a renaming of the variables as in the proof of Lemma 2.3, f_i satisfies the conditions of the corollary for $p = n - 1$; i.e.,

$$f_1(a, b, \alpha, \beta, \gamma, z) \leqslant O_1(a, b, \alpha, \beta, \gamma, z) \sum_{i=2}^{n} a_i b_i,$$

$$f_2(a, b, \alpha, \beta, \gamma, z) \leqslant O_2(a, b, \alpha, \beta, \gamma, z) \sum_{i=1}^{n-1} a_i b_i,$$

where in particular, O_i are locally bounded in Y. These local bounds yield a uniform bound $G > 1$ on the compact set V, and so we can conclude that

$$f_i(a, b, \alpha, \beta, \gamma, z) \leqslant G a \cdot b \tag{A}$$

in V. Now consider the function

$$\begin{aligned} h(a, b, \alpha, \beta, \gamma, z) = f_1(a, b, \alpha, \beta, \gamma, z) & \\ + \frac{b_1}{b_1 + a_n} \{f_2(a, b, \alpha, \beta, \gamma, z) - f_1(a, b, \alpha, \beta, \gamma, z)\} & \\ - G a \cdot b. & \end{aligned}$$

As in the case $n = 1$, h has the same smoothness properties as f, and by (A), h is negative in V. Moreover, h agrees with f in Y when $a \cdot b = 0$. That is, assuming $a \cdot b = 0$ in Y, we must have $a_1 b_1 = 0$. If $a_1 = 0$, then since $a_1 \geqslant a_n \geqslant \alpha_n$, $a_n = \alpha_n = 0$, implying that $f(a, b, \alpha, \beta, \gamma, z) = f_2(a, b, \alpha, \beta, \gamma, z) = h(a, b, \alpha, \beta, \gamma, z)$. If $b_1 = 0$, then $f(a, b, \alpha, \beta, \gamma, z) = f_1(a, b, \alpha, \beta, \gamma, z) = h(a, b, \alpha, \beta, \gamma, z)$.) Therefore we can write

$$g(a, b, \alpha, \beta, \gamma, z) = f(a, b, \alpha, \beta, \gamma, z) - h(a, b, \alpha, \beta, \gamma, z),$$

where g vanishes in Y when the inner product $a \cdot b = 0$, and where g has the same smoothness as f. Therefore, by Lemma 2.3,

$$g(a, b, \alpha, \beta, \gamma, z) = O(a, b, \alpha, \beta, \gamma, z)\, a \cdot b,$$

where $O(a, b, \alpha, \beta, \gamma, z)$ is defined on Y, is locally bounded, and locally Lipschitz in z uniformly in the remaining variables. Since h is negative in V, we conclude that

$$f(a, b, \alpha, \beta, \gamma, z) \leqslant O(a, b, \alpha, \beta, \gamma, z)\, a \cdot b$$

in V. This completes the proof of case $p=n$, and so completes the proof of the corollary.

Proof of Theorem 2.1. We are given the real valued function $c=f(a,b,z)$ which is C^2 in $\mathrm{Int}(W)$ with derivatives continuous up to boundary, and which has locally Lipschitz second order derivatives in W. Further, $c=0$ when $D=0$, where D is a quadratic term of form given in (2.1). By Lemma 2.1, we can write $a'=Aa$ and $b'=Bb$, where A and B are nonsingular transformations satisfying $D=a'_1b'_1+\cdots+a'_mb'_m$ in W, and such that the image of $\{a\in R^n: a_i\geqslant 0\}$, $\{b\in R^n: b_i\geqslant 0\}$ under A, B in the first m variables of a', b', respectively, is the set given by

$$a'_1\geqslant a'_2\geqslant\cdots\geqslant a'_m\geqslant 0,$$
$$b'_k\geqslant 0,\qquad 1\leqslant k\leqslant m.$$

Moreover, for $k>m$, we can assume without loss of generality that the image is given by

$$0\leqslant a'_k\leqslant a'_j\qquad \text{for some } 1\leqslant j\leqslant m,$$
$$0\leqslant b'_k\leqslant b'_j\qquad \text{for some } 1\leqslant j\leqslant m$$

(this follows since, when $a'_k\geqslant 0$ defines the image of a'_k under A for $k>m$, a case which happens only when $\delta_1=0$ and $\delta_2=1$, the value of a'_k is independent of the other a'_j and yet does not appear in the expression for D. In this case a'_k can be incorporated into the z-variables). Let Σ denote this image set in the variables a' and b'.

Let

$$a''=(a'_1,\ldots,a'_m),\qquad b''=(b'_1,\ldots,b'_m),$$

$$\begin{aligned}
\alpha=(\alpha_1,\ldots,\alpha_m)\qquad &\text{where } \alpha_j=a'_k &&\text{if } a'_k\leqslant a'_j \text{ for } j\leqslant m,\ k>m\\
&\qquad\quad\ =0 &&\text{otherwise},\\
\beta=(\beta_1,\ldots,\beta_m)\qquad &\text{where } \beta_j=b'_k &&\text{if } b'_k\leqslant b'_j \text{ for } j\leqslant m,\ k>m\\
&\qquad\quad\ =0 &&\text{otherwise.}
\end{aligned}\tag{2.2}$$

Let Σ' be the domain of (a'',b'',α,β). Note that $\Sigma'=\Sigma$ in the nonzero variables of (a'',b'',α,β). Let $Y'=\Sigma'\times Z$. Then Y' satisfies the conditions for the domain Y of Lemma 2.3. Thus we have

$$c=f(a,b,z)=f(A^{-1}a',B^{-1}b',z)=g(a'',b'',\alpha,\beta,z),$$

where

$$(a'',b'',\alpha,\beta,z)\in Y'\qquad\text{and}\qquad c=0 \text{ when } a''\cdot b''=0.$$

Thus by Lemma 2.3, $c = O_1(a'', b'', \alpha, \beta, z)\, a'' \cdot b''$, where O_1 is locally bounded and locally Lipschitz in z uniformly in a'', b'', α, β. Thus

$$\begin{aligned} c &= O_1(a'', b'', \alpha, \beta, z)\, a'' \cdot b'' = O_2(a', b', z)\, a'' \cdot b'' = O_2(Aa, Bb, z) \cdot D \\ &= O(a, b, z) \cdot D \qquad \text{for } (a, b, z) \in W. \end{aligned}$$

Since O_2 is obtained from O_1 by deleting the zero variables among α and β, it is clear that O_2 is locally bounded and locally Lipschitz in z uniformly in a', b'. Let $Y = \Sigma \times Z$ be the image of W under the bijective map Φ, where

$$\Phi : W \to Y,$$
$$\Phi(a, b, z) = (Aa, Bb, z).$$

Let U be a compact convex open subset of W. Then since nonsingular linear transformations preserve compactness and convexity, we have $\Phi(U) = V$, a compact convex open set in Y. Thus, if $(a, b, z_i) \in U$ for $i = 1, 2$, and K, M are respectively the uniform bound and uniform Lipschitz bound in z for V, we have

$$|O(a, b, z_i)| = |O(A^{-1}a', B^{-1}b', z_i)| \leqslant M,$$
$$|O(a, b, z_2) - O(a, b, z_1)| = |O_2(a', b', z_2) - O_2(a', b', z_1)| \leqslant K \|z_2 - z_1\|.$$

Thus $O(a, b, z)$ is locally bounded and locally Lipschitz in z uniformly in a, b inside W. This proves Theorem 2.1.

COROLLARY 1. *Let $f(a, b, z)$ be C^2 with locally Lipschitz second derivatives in the domain*

$$V = \{a \in R^n : 0 \leqslant a_i \leqslant \tau\} \times \{b \in R^n : 0 \leqslant b_i \leqslant \tau\} \times Z,$$

where τ is a given positive constant, and $Z \subset R^l$ is compact, convex and the closure of its interior. Then if $c = 0$ when $D = 0$ for some D as given in (2.1), *then*

$$c = O(a, b, z) \cdot D,$$

where O is uniformly bounded and uniformly Lipschitz in z on V.

Proof. The proof of Theorem 2.1 carries through in this restricted domain to imply that $c = O(a, b, z) \cdot D$, where O is locally bounded and locally Lipschitz in z uniformly in a, b. Since V is a compact set, these local bounds imply uniform bounds in V.

COROLLARY 2. *Let $c = f(a, b, z)$ be C^2 with locally Lipschitz second* *derivatives in the domain V above. If $c \leqslant 0$ when $D = 0$ for some D as given in* (2.1), *then there exists a positive constant $G \geqslant 1$ such that*

$$c \leqslant GD$$

in V.

Proof. Applying the corollary to Lemma 2.3 in the proof of Theorem 2.1 yields the result that $c \leqslant O(a, b, z) \cdot D$ in V, where $O(a, b, z)$ is locally bounded. This proof carries through under the conditions of Corollary 2; i.e., where we only assume that $c \leqslant 0$ when $a \cdot b = 0$ in V above. The local bounds on $O(a, b, z)$ then imply a uniform bound over the compact set V, which proves that $c \leqslant GD$ in V, for some $G > 1$.

Proof of Theorem 1.1. We have from (1.01)

$$e_i = c_i - a_i - b_i = B_i(a, b, w_L, \varepsilon) - a_i - b_i = F_i(a, b, w_L, \varepsilon).$$

By Lemma 1.1, B_i is C^2 with locally Lipschitz second order derivatives in Γ, and hence so is F_i. We apply Corollary 1 of Theorem 2.1 to each $\Gamma_s \subset \Gamma$. Let

$$\begin{aligned} \tilde{a} &= (|a_1|, |a_2|, ..., |a_n|), \\ \tilde{b} &= (|b_1|, |b_2|, ..., |b_n|), \end{aligned} \tag{2.3}$$

and let

$$V = \{\tilde{a} \in R^n : 0 \leqslant \tilde{a}_i \leqslant \tau\} \times \{\tilde{b} \in R^n : 0 \leqslant \tilde{b}_i \leqslant \tau\} \times \text{Cl}\, U_1 \times [0, \varepsilon_1],$$

and write

$$e_i = F_i(a, b, w_L, \varepsilon) = f_i(\tilde{a}, \tilde{b}, w_L, \varepsilon) \text{ on } \Gamma_s.$$

Since $\text{sign}(a_i)$ and $\text{sign}(b_i)$ are constant on Γ_s, $f_i \in C^2$ with locally Lipschitz second derivatives in V. Hence, Corollary 1 of Theorem 2.1 applies to f_i with $z = (w_L, \varepsilon)$, and so

$$e_i = O(\tilde{a}, \tilde{b}, w_L, \varepsilon)\, D_s,$$

where O is uniformly bounded and uniformly Lipschitz with respect to ε in V. Let G_s be the maximum of these two uniform bounds and let $G = \sup_s\{G_s\}$. Then if $(a, b, w_L, \varepsilon) \in \Gamma$, it is also in some Γ_s, and hence we have

$$|e_i| = |f_i(\tilde{a}, \tilde{b}, w_L, \varepsilon)| \leqslant G_s D_s \leqslant G D_s = GD$$

and also

$$\begin{aligned}|c_i(\varepsilon)-c_i(0)| &= |f_i(\tilde{a},\tilde{b},w_L,\varepsilon)-f_i(\tilde{a},\tilde{b},w_L,0)|\\ &= |O(\tilde{a},\tilde{b},w_L,\varepsilon)-O(\tilde{a},\tilde{b},w_L,0)|\,D_s\\ &\leqslant G_s\varepsilon D_s \leqslant G\varepsilon D_s = G\varepsilon D\end{aligned}$$

since $D=D_s$ if $(a, b, w_L, \varepsilon)\in\Gamma_s$. This proves Theorem 1.1.

In the next section, we need estimates on the difference between the strengths of incoming and outgoing shock and rarefaction waves in the interaction function B. Thus define

$$\begin{aligned}R(a_i) &= 0 \qquad \text{if} \quad a_i\leqslant 0\\ &= a_i \qquad \text{if} \quad a_i\geqslant 0\end{aligned}$$

$$\begin{aligned}\gamma(a_i) &= |a_i| \quad \text{if} \quad a_i\leqslant 0\\ &= 0 \qquad \text{if} \quad a_i\geqslant 0.\end{aligned}$$

We call $R(a_i)$ the strength of the ith rarefaction wave in a, and we call $\gamma(a_i)$ the strength of the ith shock wave in a. The following two theorems are needed to obtain certain interaction estimates needed in the next section.

THEOREM 2.2. *For $c_i=B_i(a,b,w_L,\varepsilon)$ defined and C^2 with locally Lipschitz second derivatives in the compact set Γ, we have*

$$R(c_i(\varepsilon))-R(c_i(0))\leqslant G\varepsilon D,$$
$$\gamma(c_i(\varepsilon))-\gamma(c_i(0))\leqslant G\varepsilon D,$$

where $G\varepsilon D$ is a uniform bound on Γ.

Proof. We do the case for rarefaction waves. By Theorem 1.1 we have

$$|c_i(\varepsilon)-c_i(0)|\leqslant G\varepsilon D$$

on Γ. Thus

$$c_i(\varepsilon)-R(c_i(0))\leqslant c_i(\varepsilon)-c_i(0)\leqslant G\varepsilon D.$$

But if $c_i(\varepsilon)\geqslant 0$, then $c_i(\varepsilon)=R(c_i(\varepsilon))$ and we have

$$R(c_i(\varepsilon))-R(c_i(0))\leqslant G\varepsilon D,$$

while if $c_i(\varepsilon)\leqslant 0$, then $R(c_i(\varepsilon))=0$ and we have

$$R(c_i(\varepsilon))-R(c_i(0))\leqslant 0\leqslant G\varepsilon D.$$

This completes the proof of Theorem 2.2.

THEOREM 2.3. *Consider* $c_i = B_i(a, b, w_L, \varepsilon)$ *defined and* C^2 *with locally Lipschitz second derivatives in the compact set* Γ. *Assume that on each* Γ_s *we have*

$$R(c_i) - R(a_i) - R(b_i) \leqslant 0 \qquad \text{when} \quad D'_s = 0,$$

where D'_s *is a sum of form* (2.1) *among* $a' = (a_{i_1}, ..., a_{i_m})$ *and* $b' = (b_{j_1}, ..., b_{j_m})$, *where* $\{i_k\}$, $\{j_k\}$, m *and* D'_s *depend on* s.

Then for some $G > 1$ *we have*

$$R(c_i) - R(a_i) - R(b_i) \leqslant GD',$$

$$\gamma(c_i) - \gamma(a_i) - \gamma(b_i) \leqslant GD',$$

where $D' = D'_s$ *on* Γ_s.

Proof. We do the case for rarefaction waves. Let

$$c_i - R(a_i) - R(b_i) = B_i(a, b, w_L, \varepsilon) - R(a_i) - R(b_i) = h_i(a, b, w_L, \varepsilon).$$

We apply Corollary 2 of Theorem 2.1 to each $\Gamma_s \subset \Gamma$. For each s, we can write

$$h_i(a, b, w_i, \varepsilon) = f_i(\tilde{a}', \tilde{b}', \gamma, u_L, \varepsilon) = f_i(\tilde{a}', \tilde{b}', z)$$

where $z = (\gamma, u_L, \varepsilon)$, γ is the vector of all components of a, b not appearing in a', b', and $\tilde{a}'_i = |a'_i|$. Now set

$$V_s = \{\tilde{a}' \in R^m : 0 \leqslant \tilde{a}'_i \leqslant \tau\} \times \{\tilde{b}' \in R^m : 0 \leqslant \tilde{b}'_i \leqslant \tau\} \times Z.$$

Since $\text{sign}(a_i)$ and $\text{sign}(b_i)$ are constant on Γ_s, $f_i \in C^2$ with locally Lipschitz second derivatives in V_s. Hence, Corollary 2 of Theorem 2.1 applies to f_i, and so

$$c_i - R(a_i) - R(b_i) \leqslant G_s D'_s,$$

where $G_s D_s$ is a bound which holds uniformly in each V_s, and so $h_i(a, b, w_L, \varepsilon)$ is uniformly bounded by $G_s \cdot D_s$ on each Γ_s. Let $G = \sup_s\{G_s\}$. Then if $(a, b, w_L, \varepsilon) \in \Gamma$, then $(a, b, w_L, \varepsilon) \in \Gamma_s$ for some s, and so we have

$$c_i - R(a_i) - R(b_i) \leqslant G_s D'_s \leqslant GD'. \tag{2.4}$$

Now if $c_i \geqslant 0$, then $R(c_i) = c_i$ and so by (2.4)

$$R(c_i) - R(a_i) - R(b_i) \leqslant GD',$$

while if $c_i \leqslant 0$, then $R(c_i) = 0$ and so

$$R(c_i) - R(a_i) - R(b_i) \leqslant 0 \leqslant GD'.$$

This completes the proof of Theorem 2.3.

3. Gas Dynamic Equations

In this section we study the one-dimensional Lagrangian equations of motion (3) for gas dynamics. This system of equations is determined by specifying the constitutive relation $e = e(v, S)$. We are primarily concerned with energy functions for ideal gases, and for this reason Sections 2–4 of Courant and Friedricks [1] are summarized in the following paragraph.

An ideal gas is one that satisfies the equation of state

$$pv = RT, \tag{3.1}$$

where $p = -e_v$ = pressure, v = specific volume, T = temperature and R = specific gas constant. Moreover, the internal energy gained by the gas during a change of state is equal to the heat contributed to the gas plus the work done on the gas by compressive action of the pressure forces. This fact is expressed by the fundamental relation

$$de = T\,ds - p\,dv. \tag{3.2}$$

For any given ideal gas, the choice of units for p, v, e, and S together with the molecular weight of the gas determines the constant R. We can take the relation

$$T = \frac{pv}{R} \tag{3.3}$$

as the definition of temperature. It is important to note that, since we can only measure changes in energy and entropy, values for e and S at a particular thermodynamic state of the gas are determined only after we choose arbitrary ground states for zero energy and entropy. With these choices, we obtain from (3.2) and (3.3)

$$de = \frac{pv}{R}\,dS - p\,dv, \tag{3.4}$$

which yields the linear partial differential equation for $e(v, S)$

$$Re_S + ve_v = 0, \tag{3.5}$$

the general solution of which is

$$e = h(v \exp(-S/R)), \tag{3.6}$$

where h is an arbitrary differentiable function. It is also generally true for actual media that

$$\begin{aligned} p(v, S) &= -e_v(v, S) > 0, \\ p_v(v, S) &= -e_{vv}(v, S) < 0, \\ p_{vv}(v, S) &= -e_{vvv}(v, S) > 0. \end{aligned} \tag{3.7}$$

Hence, for ideal gases, (3.7) is expressed by

$$\begin{aligned} h'(x) &< 0, \\ h''(x) &> 0, \\ h'''(x) &< 0. \end{aligned} \tag{3.8}$$

Any gas whose energy function satisfies (3.1)–(3.8) we call an ideal gas. A gas is polytropic if further

$$e = c_v T + C_0 \tag{3.9}$$

for some constant $c_v > 0$. The energy function for a polytropic gas is given by

$$e = \left\{ v \exp\left(-\frac{S - S_0}{R} \right) \right\}^{-\epsilon} + C_0, \tag{3.10}$$

where $h(x) = \exp(-\varepsilon S_0/R)x^{-\epsilon} + C_0$. For any choice of S_0 and C_0 a gas with energy function (3.10) satisfies (3.9) with

$$c_v = \frac{R}{\varepsilon}. \tag{3.11}$$

For a given polytropic gas, the value of S_0 is dependent upon our choice of the ground state for entropy, and the arbitrary constant C_0 in (3.10) is required to adjust for a preassigned choice of ground state for energy.[1]

For any ideal gas, system (3) can be written

$$U_t + f(U)_x = 0, \tag{3.12}$$

[1] The constant C_0 is usually taken to be zero [1], but below we make another choice.

where $U = (u, v, E)^t$ and $f(U) = (p, -u, up)^t$. $(p, -u, up)^t$. $(p, -u, up)^t$ is indeed a function of U since (3.6) and (3.8) imply that $S = S(e, v)$ and hence $p = p(v, S) = p(u, v, E)$. The eigenvalues, eigenvectors and Riemann invariants for df are given in (u, v, S) coordinates below. Note that eigenvalues and Riemann invariants can be computed in any transformed coordinates, but eigenvectors and jump conditions are determined by the variables that yield the conservation form (3.12) of the equations.

$$\begin{aligned}
&\lambda_1 = -\sqrt{e_{vv}}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sqrt{e_{vv}}, \\
&R_1 = \left(1, \frac{1}{\sqrt{e_{vv}}}, u + \frac{e_v}{\sqrt{e_{vv}}}\right), \\
&R_2 = (0, e_{vS}, e_v e_{vS} - e_S e_{vv}), \\
&R_3 = \left(1, -\frac{1}{\sqrt{e_{vv}}}, u - \frac{e_v}{\sqrt{e_{vv}}}\right),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\text{1-Riemann invariants } S,\ u - \int^v \sqrt{-p_v}\, dv, \\
&\text{2-Riemann invariant } u,\ p, \\
&\text{3-Riemann invariants } S,\ u + \int^v \sqrt{-p_v}\, dv.
\end{aligned} \tag{3.13}$$

A quick check shows that the equations are genuinely nonlinear in the 1, 3-characteristic fields, and linearly degenerate in the 2-field.

The jump conditions which determine the shock curves for system (3) must be computed from the conservation form of the equations. Letting $[u] = u - u_L$, etc., these are

$$\begin{aligned}
\sigma[v] &= [p], \\
\sigma[v] &= -[u], \\
\sigma[E] &= [pu].
\end{aligned} \tag{3.14}$$

Eliminating the shock speed σ in the first two equations and requiring the usual entropy conditions (cf. [4]) yields

$$u - u_L = -\sqrt{(p - p_L)(v_L - v)} \tag{3.15}$$

and eliminating u and σ from the third equation yields the Hugoniot relation,

$$0 = e - e_L + \tfrac{1}{2}(p + p_L)(v - v_L). \tag{3.16}$$

Define the transformation

$$r = u + a \ln \frac{1}{p}, \qquad s = u - a \ln \frac{1}{p}, \tag{3.17}$$

for $a > 0$ arbitrary. $(u, p) \to (r, s)$ defines a one to one and onto regular C^∞ transformation from $R \times R^+ \to R \times R$. Since $p_v \neq 0$, the transformation

$$\Psi: (u, p, S) \to (r, s, S) \tag{3.18}$$

determines a one to one and onto regular C^∞ mapping from the domain of the variables (u, v, E) to R^3. Hence we can view the shock-rarefaction and contact discontinuity curves of system (3) for any ideal gas, in rsS-space. Note that since S is a 1- and 3-Riemann invariant, and since u, p are 2-Riemann invariants, 1, 3-rarefaction curves in rsS-space lie at constant S, and the 2-contact discontinuity curves are vertical lines parallel to the S axis. We let

$$w = (r, s, S). \tag{3.20}$$

For arbitrary constant a and C, consider now the energy function

$$\begin{aligned} e_0(v, S) &= -a^2 \ln(v \exp(-S/R)) + C \\ &= -a^2 \ln v + \frac{a^2 S}{R} + C. \end{aligned} \tag{3.21}$$

$e_0(v, S)$ satisfies conditions (3.1)–(3.5) for an ideal gas with $h(x) = -a^2 \ln x + C$. We now study properties of the shock–rarefaction curves in rsS-space for system (3) with energy function $e_0(v, S)$. Since we are soon to consider smooth parameterizations $e_\epsilon(v, s) = e(v, S, \varepsilon)$ which reduce to (3.21) at $\varepsilon = 0$, we call system (3) with $e = e_0(v, S)$ the system at $\varepsilon = 0$. We use these special properties together with the estimates of section 2 to prove Theorem 4.1. A corollary of Theorem 4.1 is a Theorem by Liu [5] on the existence of solutions to Eqs. (3) for polytropic gases.

Note first that $p(v, S) = -e_v(v, S) = a^2/v$ for $e_0(v, S)$. Hence Eqs. (3) decouple, and the first two equations here describe the 2×2 system studied by Nishida in [6]. Note also that r is a 3-Riemann invariant while s is a 1-Riemann invariant for $e_0(v, S)$. Thus 1-rarefaction curves lie parallel to the r-axis and 3-rarefaction curves lie parallel to the s-axis. Moreover, the shock conditions (3.15) and (3.16) yield, respectively,

$$\begin{aligned} u - u_L &= -a \frac{|v - v_L|}{\sqrt{v v_L}}, \\ S - S_L &= R \left\{ \ln \frac{v}{v_L} - \frac{v^2 - v_L^2}{2 v v_L} \right\}. \end{aligned} \tag{3.22}$$

1-shocks are parameterized for $0 < v/v_L \leqslant 1$ while 3-shocks are parameterized for $0 < v_L/v \leqslant 1$. By writing conditions (3.22) in terms of r, s we obtain

1-shock curves:

$$r - r_L = a\frac{\alpha - 1}{\sqrt{\alpha}} + a \ln \alpha,$$

$$s - s_L = a\frac{\alpha - 1}{\sqrt{\alpha}} - a \ln \alpha, \qquad 0 < \frac{v}{v_L} = \alpha \leqslant 1,$$

$$S - S_L = R\left\{\ln \alpha - \frac{\alpha^2 - 1}{2\alpha}\right\};$$

3-shock curves: (3.23)

$$r - r_L = a\frac{\alpha - 1}{\sqrt{\alpha}} - a \ln \alpha,$$

$$s - s_L = a\frac{\alpha - 1}{\sqrt{\alpha}} + a \ln \alpha, \qquad 0 < \frac{v_L}{v} = \alpha < 1,$$

$$S - S_L = -R\left\{\ln \alpha - \frac{\alpha^2 - 1}{2\alpha}\right\}.$$

Thus the shock–rarefaction curves are all the same except that entropy decreases along 3-shocks and increases along 1-shocks, as needed to ensure that entropy increases in time. Differentiating (3.23) yields

1-shock curves:

$$\frac{d(s - s_L)}{d(r - r_L)} = \left(\frac{\sqrt{\alpha} - 1}{\sqrt{\alpha} + 1}\right)^2 > 0, \qquad 0 < \alpha = \frac{v}{v_L} < 1,$$

$$\frac{d(S - S_L)}{d(r - r_L)} = -\frac{R}{a}\left(\frac{\alpha - 1}{\sqrt{\alpha} + 1}\right)^2 \cdot \frac{1}{\sqrt{\alpha}} < 0;$$

(3.24)

3-shock curves:

$$\frac{d(r - r_L)}{d(s - s_L)} = \left(\frac{\sqrt{\alpha} - 1}{\sqrt{\alpha} + 1}\right)^2 > 0, \qquad 0 < \alpha = \frac{v_L}{v} < 1.$$

$$\frac{d(S - S_L)}{d(s - s_L)} = \frac{R}{a}\left(\frac{\alpha - 1}{\sqrt{\alpha} + 1}\right)^2 \cdot \frac{1}{\sqrt{\alpha}} > 0,$$

Thus 1-shock curves can be parameterized with respect to $r - r_L$, and 3-shock curves with respect to $s - s_L$. Note also that the change in entropy along a 1-shock [respectively 3-shock] is monotone increasing [respectively decreasing], and on compact sets the change in entropy is uniformly bounded by some constant times $r - r_L$ [respectively $s - s_L$].

Since in the rs-plane, the shock curves reduce to those for the isothermal system [6], Riemann problems can be globally solved here by solving them in the rs-plane, and then connecting in the middle by a vertical contact discontinuity. Thus for the energy function $e_0(v, S)$, the functions

$$w = H_i(t_i, w_L, 0)$$

are everywhere defined as in Section 1, with $t_1 = r - r_L$, $t_2 = S - S_L$, $t_3 = s - s_L$. Because H_1 and H_3 are C^3 away from $t_i = 0$, with C^2 contact at $t_i = 0$, each H_i is C^2 with locally Lipschitz second derivatives, and hence so are the functions

$$w = H(t, w_L, 0),$$
$$w = G(a, b, w_L, 0),$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ as in Section 2. Moreover

$$\left| \frac{\partial H}{\partial t} \right| \neq 0 \tag{3.25}$$

holds everywhere for $e_0(v, S)$, since by (3.24) the shock–rarefaction curves in rsS-space are nowhere parallel, and vertical vectors are never in the span of two tangent vectors from 1, 3-shock-rarefaction curves. Thus the columns

$$\left(\frac{\partial H}{\partial t_i} \right)^t$$

of $\partial H / \partial t$ are everywhere independent, so that (3.25) holds. Since H is 1–1 at fixed w_L, (3.25) and the Inverse Function Theorem imply that

$$t = B'(w, w_L, 0)$$

as well as the interaction function

$$c = t = B(a, b, w_L, 0) \tag{3.26}$$

are everywhere defined, and everywhere C^2 with locally Lipschitz second derivatives. We wish to distinguish i-shocks from i-rarefaction waves among the waves in the interaction function (3.26), so let us adopt the following notation meaningful at any ε where B is defined:

$$
\begin{aligned}
\alpha_1 &= \text{the 1-shock in } a_1 = 0 && \text{if} \quad a_1 \geqslant 0 \\
&= |a_1| && \text{if} \quad a_1 \leqslant 0, \\
\mu_1 &= \text{the 1-rarefaction wave in } a_1 = a_1 && \text{if} \quad a_1 \geqslant 0 \\
&= 0 && \text{if} \quad a_1 \leqslant 0.
\end{aligned} \tag{3.27}
$$

Similarly let

$$
\begin{aligned}
&\alpha_2 = \text{1-shock in } b_1, && \mu_2 = \text{1-rarefaction wave in } b_1, \\
&\beta_1 = \text{3-shock in } a_3, && \eta_1 = \text{3-rarefaction wave in } a_3, \\
&\beta_2 = \text{3-shock in } b_3, && \eta_2 = \text{3-rarefaction wave in } b_3, \\
&|\delta_1| = |a_2|, && |\delta_2| = |b_2|, \\
&\alpha' = \text{1-shock in } c_1, && \mu' = \text{1-rarefaction wave in } c_1, \\
&\beta' = \text{3-shock in } c_3, && \eta' = \text{3-rarefaction wave in } c_3, \\
&|\delta'| = |c_2| = \text{contact wave in } c_2.
\end{aligned} \tag{3.27}
$$

We let

$$
\begin{aligned}
&\alpha, \beta && \text{denote arbitrary 1, 3-shocks,} \\
&\mu, \eta && \text{arbitrary 1, 3-rarefaction waves} \\
&\gamma_i && \text{arbitrary shock waves,} \\
&R_i && \text{arbitrary rarefaction waves,} \\
&|\delta_i| && \text{arbitrary contact waves,} \\
&q_i && \text{arbitrary shock or rarefaction waves,} \\
&p_i && \text{arbitrary waves,}
\end{aligned} \tag{3.27}
$$

in some solution. The symbols in (3.27) are used to denote both the names as well as the strengths of the corresponding waves (δ_i denotes signed strength, so that $|\delta_i|$ denotes strength). Let $D =$ the sum of the strengths of the approaching waves among a, b as defined in (1.0), and write

$$D = D_1 + D_2 + D_3,$$

where

$$
\begin{aligned}
D_1 &= \sum_{APP} |\delta_i| p_j, \\
D_2 &= \sum_{APP} R_i q_j, \\
D_3 &= \sum_{APP} \alpha_i \beta_j.
\end{aligned} \tag{3.28}
$$

Finally, letting

$$w = H_i(t_i, w_L, 0) = (r_i(t_i, w_L, 0), s_i(t_i, w_L, 0), S_i(t_i, w_L, 0)),$$

define

$$\delta_\alpha = S_1(t_1, w_L, 0) - S_L \geqslant 0,$$
$$\delta_\beta = S_L - S_3(t_3, w_L, 0) \geqslant 0.$$

PROPOSITION 3.1. *For every compact set U_{rs} in rs-space, there exists a constant $\frac{1}{2} \leqslant C_0 < 1$ such that every interaction at $\varepsilon = 0$ with w_L, w_M, w_R in U_{rs} (at any S) satisfies:*

$$\delta' - \delta_1 - \delta_2 = 0 \quad \text{when} \quad D_2 + D_3 = 0,$$
$$\mu' - \mu_1 - \mu_2 \leqslant 0 \quad \text{when} \quad D_3 = 0,$$
$$\eta' - \eta_1 - \eta_1 \leqslant 0 \quad \text{when} \quad D_3 = 0,$$
$$\alpha' - \alpha_1 - \alpha_2 = A,\ \beta' - \beta_1 - \beta_2 = B,$$

where A, B satisfy (I) *or* (II):

(I) $A = -\xi$ *and* $0 \leqslant B \leqslant C_0\xi$ *or* $B = -\xi$ *and* $0 \leqslant A \leqslant C_0\xi$;

(II) $A \leqslant 0$ *and* $B \leqslant 0$.

The constant C_0 is determined as follows: Since U_{rs} is compact and shock curves look the same at every entropy level, every Riemann problem in U_{rs} is solvable with shock wave strengths uniformly bounded by a constant P. Set

$$C_0 = \max \left\{ \frac{1}{2}, \frac{d(s - s_0)}{d(r - r_0)} \Big|_{r - r_0 = P} \text{along a 1-shock} \right\};$$

i.e., C_0 is a constant, $\frac{1}{2} \leqslant C_0 < 1$, which dominates the slopes of 1-shock curves and reciprocal slopes of 2-shock curves which occur in waves which solve interactions in U_{rs}.

Proof. The results above involving shocks and rarefaction waves are proven via a case by case study of interactions in the rs-plane where strengths are defined. This is given in Appendix II. That $\delta' - \delta_1 - \delta_2 = 0$ when $D_2 + D_3 = 0$ is a consequence of the fact that when there are no approaching waves among the shocks and rarefaction waves in $\langle w_L, w_M \rangle$, $\langle w_M, w_R \rangle$, the Riemann problems for $\langle w_L, w_R \rangle$ has the same solution waves in the rs-plane. Thus the only change across the interaction is in the contact waves, and hence

$$\delta_{\alpha'} - \delta_{\beta'} = \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\beta_1} - \delta_{\beta_2}.$$

However, since the net change in entropy is conserved across interactions, we have

$$\delta' + \delta_{\alpha'} - \delta_{\beta'} = \delta_1 + \delta_2 + \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\beta_1} - \delta_{\beta_2}$$

and hence

$$\delta' = \delta_1 + \delta_2.$$

PROPOSITION 3.2. *For every compact set U_{rs} in the rs-plane there exists a constant $M > 0$ such that every interaction at $\varepsilon = 0$ with W_L, W_M, W_R in U_{rs} (at any S) satisfies*:

$$|\delta'| - |\delta_1| - |\delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant -M(A + B).$$

Proof. Choose C_0 so that Proposition 3.1 holds. Since across any interaction the net change in entropy is conserved, and since rarefaction waves lie at constant entropy, we have

$$\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\beta_1} - \delta_{\beta_2} + \delta_1 + \delta_2 = \delta_{\alpha'} - \delta_{\beta'} + \delta'. \tag{3.29}$$

Assume, first, that case I of Proposition 3.1 occurs with $A = -\xi$. Then

$$\alpha' - \alpha_1 - \alpha_2 = A = -\xi, \qquad 0 \leqslant \beta' - \beta_1 - \beta_2 = B \leqslant C_0 \xi.$$

Because U_{rs} is a compact set, we can choose

$$\alpha_0 = \min \left\{ \inf_{U_{rs}} \frac{v}{v_L}, \inf_{U_{rs}} \frac{v_L}{v} \right\}$$

for v and v_L on the same shock curve in U_{rs}. Also, let

$$M = (1 - C_0)\tilde{M},$$

where

$$\tilde{M} = \frac{2R}{a} \left(\frac{\alpha_0 - 1}{\sqrt{\alpha_0} + 1} \right)^2 \frac{1}{\sqrt{\alpha_0}},$$

which is "twice the $\sup_{U_{rs}} |d(S - S_L)/dt_i|$ among 1- and 3-shocks that begin and end in U_{rs}" by (3.24). (3.24) also implies that along 1, 3-shocks, the derivative of entropy with respect to the strength of a shock along a shock curve, has magnitude

$$\frac{R}{a} \left(\frac{\alpha - 1}{\sqrt{\alpha} + 1} \right)^2 \frac{1}{\sqrt{\alpha}} \qquad 0 < \alpha \leqslant 1. \tag{3.30}$$

Since α decreases along the shock curve as the strength increases, we see that (3.30) and hence the magnitude of this derivative is monotone increasing along shocks. Thus, $\beta' \geqslant \beta_1 + \beta_2$ implies that

$$\delta_{\beta'} \geqslant \delta_{\beta_1} + \delta_{\beta_2}. \tag{3.31}$$

Moreover, for interactions in U_{rs}, our choice of M implies that if $\alpha' - \alpha_1 - \alpha_2 = A = -\xi$ then

$$\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leqslant \tfrac{1}{2}\tilde{M}\xi. \tag{3.32}$$

Hence, using (3.29) together with (3.32) we obtain

$$(\delta' - \delta_1 - \delta_2) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) = \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leqslant \tfrac{1}{2}\tilde{M}\xi$$

and so

$$\begin{aligned}(\delta' - \delta_1 - \delta_2) + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'})\\ \leqslant \tfrac{1}{2}\tilde{M}\xi + \tfrac{1}{2}\tilde{M}\xi = \tilde{M}\xi.\end{aligned} \tag{3.33}$$

Also using (3.29) together with (3.31) we obtain

$$-(\delta' - \delta_1 - \delta_2) + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta'} - \delta_{\beta_1} - \delta_{\beta_2}) = 0$$

and

$$-(\delta' - \delta_1 - \delta_2) + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant 0. \tag{3.34}$$

Putting (3.33) and (3.34) together yields

$$\begin{aligned}|\delta' - \delta_1 - \delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} + \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'})\\ \leqslant \tilde{M}\xi = M(1 - C_0)\xi \leqslant -M(A + B)\end{aligned}$$

and hence

$$|\delta'| - |\delta_1| - |\delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant -M(A + B).$$

This proves case (I).

For case (II) we have $\alpha' - \alpha_1 - \alpha_2 = A \leqslant 0$ and $\beta' - \beta_1 - \beta_2 = B \leqslant 0$. Thus the estimate (3.32) applies in both cases to yield

$$\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'} \leqslant -\tfrac{1}{2}\tilde{M}B \leqslant -\tfrac{1}{2}MB, \tag{3.31$'$}$$

$$\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leqslant -\tfrac{1}{2}\tilde{M}A \leqslant -\tfrac{1}{2}MA. \tag{3.32$'$}$$

The argument in case (I) now goes through here, by replacing (3.33) and (3.34) with

$$(\delta' - \delta_1 - \delta_2) + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant -MA, \quad (3.33)'$$

$$-(\delta' - \delta_1 - \delta_2) + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant -MB, \quad (3.34)'$$

respectively. Together these yield

$$|\delta'| - |\delta_1| - |\delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{'}) \leqslant -M(A+B),$$

which proves Proposition 3.2.

PROPOSITION 3.3. *For every compact convex open set U_{rs} in the rs-plane, there exists a constant $G > 1$ such that every interaction at $\varepsilon = 0$ with W_L, W_M, W_R in U_{rs} (at any S) satisfies*:

(i) $|\delta'| - |\delta_1| - |\delta_2| \leqslant G(D_2 + D_3)$;

(ii) $\mu' - \mu_1 - \mu_2 \leqslant GD_3, \qquad \eta' - \eta_1 - \eta_2 \leqslant GD_3.$

Proof. Choose $\tau > 0$ so that Riemann problems between states in U_{rs} have solutions with shock–rarefaction wave strengths less than τ. We have $c = B(a, b, w_L, 0)$ is defined everywhere.

Case (i). Across any interaction in rsS-space, the net change in entropy is conserved, and hence

$$\delta' + \delta_{\alpha'} - \delta_{\beta'} = \delta_1 + \delta_2 + \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\beta_2} - \delta_{\beta_2}$$

and so

$$\delta' - \delta_1 - \delta_2 = \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} + \delta_{\beta'} - \delta_{\beta_1} - \delta_{\beta_2}.$$

The right-hand side of this equation is a function dependent only on the Riemann problem interaction in the rs-plane, and so we write

$$\begin{aligned}\delta' - \delta_1 - \delta_2 = c_2 - a_2 - b_2 &= B_2(a, b, w_L, 0) - a_2 - b_2 \\ &= f(a_1, a_3, b_1, b_3, r_L, s_L, 0).\end{aligned}$$

By Proposition 3.1, $\delta' - \delta_1 - \delta_2 = 0$ when $0 = D_2 + D_3 =$ "the approaching waves among $a_1 a_3, b_1 b_3$." Hence Theorem 1.1 applies with f defined on $|a_i| \leqslant \tau$, $|b_i| \leqslant \tau$, $(r_L, s_L) \in U_{rs}$ and so

$$|\delta'| - |\delta_1| - |\delta_2| \leqslant |\delta' - \delta_1 - \delta_2| \leqslant G(D_2 + D_3).$$

This proves Case (i).

Case (ii). Again $\mu' - \mu_1 - \mu_2$ and $\eta' - \eta_1 - \eta_2$ are dependent only on the interaction in the rs-plane. Let

$$\Gamma^0 = \{(a_1, a_3, b_1, b_3, r_L, s_L, 0): |a_i| \leqslant \tau, |b_i| \leqslant \tau, \qquad r_L, s_L \text{ in } U_{rs}\}.$$

A quick check of cases shows that in any given Γ_s^0, D_3 is a sum of form (2.1) among some subset of $\{a_1 a_3 b_1 b_3\}$. Thus in the notation of Theorem 2.3, Proposition 3.1 yields

$$\mu' - \mu_1 - \mu_2 = R(c_1) - R(a_1) - R(b_1) \leqslant 0 \qquad \text{when} \quad D_3 = 0$$

and

$$\eta' - \eta_1 - \eta_2 = R(c_3) - R(a_3) - R(b_3) \leqslant 0 \qquad \text{when} \quad D_3 = 0.$$

Hence the conditions of Theorem 2.3 are satisfied, implying

$$\mu' - \mu_1 - \mu_2 \leqslant GD_3,$$
$$\eta' - \eta_1 - \eta_2 \leqslant GD_3,$$

throughout Γ^0. This completes the proof of Proposition 3.3.

Collecting the results in Propositions 3.1 to 3.3, we have

LEMMA 3.1. *For every compact convex open set U_{rs} in the rs-plane, there exist constants $M > 0$, $\frac{1}{2} \leqslant C_0 < 1$, and $G > 1$ such that, at $\varepsilon = 0$, the following estimates hold across any interaction $\langle w_L, w_M \rangle + \langle w_M, w_R \rangle \to \langle w_L, w_R \rangle$ of states whose projections onto the rs-plane lie in U_{rs}:*

$$\alpha' - \alpha_1 - \alpha_2 = A, \qquad \beta' - \beta_1 - \beta_2 = B,$$

where $A + B \leqslant (C_0 - 1)\xi$ and where $A = -\xi$ or $B = -\xi$. Moreover

$$|\delta'| - |\delta_1| - |\delta_2| + (\delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'}) + (\delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}) \leqslant -M(A + B),$$
$$|\delta'| - |\delta_1| - |\delta_2| \leqslant G(D_2 + D_3),$$
$$\mu' - \mu_1 - \mu_2 \leqslant GD_2,$$
$$\eta' - \eta_1 - \eta_2 \leqslant GD_3.$$

We now consider any smooth parameterization of energy functions $e_\varepsilon(v, S) = e(v, S, \varepsilon)$, $0 \leqslant \varepsilon \leqslant 1$, such that $e(v, S, 0) = e_0(v, S)$ and such that conditions (3.7) are satisfied at each ε. By smooth parameterization we mean that e has sufficiently many derivatives with respect to (v, S, ε) (actually five derivatives is sufficient). We now study properties of the shock–rarefaction and contact curves for system (3) at ε near zero. We use the same notation for shock, rarefaction, and contact waves. For example we write

$$\delta_\alpha = S_1(t_1, w_L, \varepsilon) - S_L,$$
$$\delta_\beta = S_L - S_3(t_3(t_3, w_L, \varepsilon),$$

PROPOSITION 3.4. *Let U_1 be any compact convex open set in rsS-space. Let*

$$\underline{r} = \inf_{U_1} r, \qquad \bar{r} = \sup_{U_1} r, \qquad \underline{s} = \inf_{U_1} s, \qquad \bar{s} = \sup_{U_1} s.$$

Then there exists an $\varepsilon_1 > 0$ such that, for $0 \leqslant \varepsilon \leqslant \varepsilon_1$, 1- and 3-shock-rarefaction curves starting in U_1, can be parameterized with respect to $r - r_L$, $s - s_L$, respectively, in a neighborhood of U_1 as folows:

1-*shock–rarefaction curves are defined for $|r - r_L| \leqslant \bar{r} - \underline{r}$, $w_L \in U_1$ by*

$$\begin{aligned} s - s_L &= f_1(r - r_L, w_L, \varepsilon) \\ S - S_L &= g_1(r - r_L, w_L, \varepsilon) \end{aligned} \tag{3.35A}$$

3-*shock–rarefaction curves are defined for $|s - s_L| \leqslant \bar{s} - \underline{s}$, $w_L \in U_1$ by*

$$\begin{aligned} r - r_L &= f_3(s - s_L, w_L, \varepsilon) \\ S - S_L &= g_3(s - s_L, w_L, \varepsilon) \end{aligned} \tag{3.35B}$$

Moreover, (3.35A) *and* (3.36A) *define the functions*

$$\begin{aligned} w = (r, s, S) &= (r, s_L + f_1(r - r_L, w_L, \varepsilon), S_L + g_1(r - r_L, w_L, \varepsilon)) \\ &\equiv H_1'(r - r_L, w_L, \varepsilon), \end{aligned} \tag{3.36A}$$

$$\begin{aligned} w = (r, s, S) &= (r_L + f_3(s - s_L, w_L, \varepsilon), s, S_L + g_3(s - s_L, w_L, \varepsilon)) \\ &\equiv H_3'(s - s_L, w_L, \varepsilon), \end{aligned} \tag{3.36B}$$

where H_1' [resp. H_3'] is defined, is C^2, and has locally Lipschitz second order derivatives in $[-|\bar{r} - \underline{r}|, |\bar{r} - \underline{r}|] \times C \,|\, U_1 \times [0, \varepsilon_1]$ [resp. $[-|\bar{s} - \underline{s}|, |\bar{s} - \underline{s}|] \times C \,|\, U_1 \times [0, \varepsilon_1]$].

Proof. The shock conditions (3.15) and (3.16) for system (3) are

$$u - u_L = -\sqrt{(p_L - p)(v - v_L)}, \tag{3.15}$$

$$0 = e - e_L + \tfrac{1}{2}(p + p_L)(v - v_L). \tag{3.16}$$

Since S_L and v_L are smooth functions of $w_L = (r_L, s_L, S_L)$ it follows that (3.16) can be written as

$$0 = e - e_L + \tfrac{1}{2}(p + p_L)(v - v_L) = h(S - S_L, \alpha, w_L, \varepsilon), \tag{3.37}$$

where $\alpha = v/v_L$. At $\varepsilon = 0$ we have by (3.22) that $\partial h / \partial (S - S_L) \neq 0$ and h is

1–1 at fixed α, w_L, ε. Let V be the compact set in $(S-S_L, \alpha, w_L)$ space defined by

$$V = \mathscr{S} \times \Lambda \times \mathrm{Cl}\, U_1,$$

where

$$\mathscr{S} = [-S_M, S_M], \qquad \Lambda = \left[\frac{1}{\alpha_M}, \alpha_M\right],$$

where

$$S_M = \sup_{\mathrm{Cl}\, U_1} |S - S_L|, \qquad \alpha_M = \max\left\{\sup_{\mathrm{Cl}\, U_1} \alpha, \sup_{\mathrm{Cl}\, U_1} \frac{1}{\alpha}\right\}.$$

Then by the implicit function theorem, if (3.37) is satisfied at some value of the arguments of h at $\varepsilon = 0$, then locally (3.37) defines

$$S - S_L = K_3(\alpha, w_L, \varepsilon). \tag{3.38}$$

Say $S - S_L$ is in X, α is in Y, w_L is in Z, and ε is in $[0, \varepsilon']$. A finite number of $\{X_k \times Y_k \times Z_k\}_{k=1}^n$ cover the zero set of h at $\varepsilon = 0$. By the continuity of h and compactness of V, there exists an $\varepsilon_2 > 0$ such that for $\varepsilon \leqslant \varepsilon_2$, h is 1–1 at fixed α, w_L, ε, and

$$\{X_n \times Y_n \times Z_n\}_{k=1}^n$$

cover the zero set of h in V. Choose

$$\varepsilon_3 = \min\{\varepsilon_1', ..., \varepsilon_n', \varepsilon_2\}.$$

Then for $\varepsilon \leqslant \varepsilon_3$, (3.37) holds in V and only if (3.38) holds in V. This means that, for points in V, (3.15) can be written

$$u - u_L = K_4(\alpha, w_L, \varepsilon), \tag{3.39}$$

where K_4 is smooth away from $\alpha = 1$, since we can solve for S in terms of $(\alpha, w_L, \varepsilon)$. Hence we obtain from (.317)

$$\begin{aligned} r - r_L &= K_1(\alpha, w_L, \varepsilon), \\ s - s_L &= K_2(\alpha, w_L, \varepsilon), \qquad 0 = K_i(1, w_L, \varepsilon), \\ S - S_L &= K_3(\alpha, w_L, \varepsilon) \end{aligned} \tag{3.40}$$

defined for $w_L \in \mathrm{Cl}\, U_1$, $\alpha \in [1/\alpha_M, \alpha_M]$, $\varepsilon \in [0, \varepsilon_1]$ (and so in particular, (3.40) defines a parameterization of the shock curves in $\mathrm{Cl}\, U_1$, which start in $\mathrm{Cl}\, U_1$) and smooth for $\alpha \neq 1$. By the continuity of the eigenvectors of df, the 1-shock must be defined for $1/\alpha_M \leqslant \alpha \leqslant 1$, the 2-shocks for $1 \leqslant \alpha \leqslant \alpha_M$. To complete the proof of Proposition 3.4, we do the case for 1-shock–rarefaction curves.

For 1-shocks, $\partial(r - r_L)/\partial\alpha > 0$ and $r - r_L \leqslant 0$ in (3.40) when $\varepsilon = 0$, and hence the inverse function theorem implies that globally, K_1 defines

$$\alpha = K_5(r - r_L, w_L, 0) \qquad \text{in } V. \tag{3.41}$$

Thus the continuity of K_1 together with the compactness of V implies that in some neighborhood $[0, \varepsilon_1]$ of $\varepsilon = 0$, $\alpha = K_5\ (r - r_L, w_L, \varepsilon)$ with $r - r_L \leqslant 0$ holds if and only of (3.40) holds in V with $r - r_L \leqslant 0$. Hence we have

$$\begin{aligned} s - s_L &= f_1(r - r_L, w_L, \varepsilon), \\ S - S_L &= g_1(r - r_L, w_L, \varepsilon), \end{aligned} \qquad r \leqslant r_L, \tag{3.42}$$

where at each $w_L \in \mathrm{Cl}\, U_1$, and ε in $[0, \varepsilon_1]$, (3.42) is defined on some interval $\underline{r} \leqslant r \leqslant r_L$ which parameterizes the 1-shock curve in U_1 starting from w_L at this ε. Therefore, without loss of generality, we can assume that (3.42) is defined for $r - r_L \leqslant \underline{r} - \bar{r}$, w_L in $\mathrm{Cl}\, U_1$ and ε in $[0, \varepsilon_1]$. Note also that f_1 and g_1 are as smooth as $e_v(v, S, \varepsilon)$ and hence are C^4 functions.

The 1-rarefaction wave starting at w_L for the system at ε is the positive portion of the integral curve of the vector field of eigenvectors

$$R_1 = \left(1, \frac{1}{\sqrt{e_{vv}}}, u + \frac{e_v}{\sqrt{e_{vv}}}\right), \tag{3.43}$$

where R_1 has at least two less derivatives than e and so is C^3. At $\varepsilon = 0$ these curves lie along the lines parallel to the positive r-axis, since S and s are 1-Riemann invariants at $\varepsilon = 0$. Let $\gamma(\xi, w_L, \varepsilon)$ be the 1-rarefaction wave starting at w_L for the system at ε:

$$\frac{d\gamma(\xi)}{d\xi} = R_1(\gamma(\xi)), \qquad \xi \geqslant 0. \tag{3.44}$$

By scaling R_1 if necessary, we can assume that ξ parameterizes γ with respect to arclength. This defines

$$\begin{aligned} r - r_L &= P_1(\xi, w_L, \varepsilon), \\ s - s_L &= P_2(\xi, w_L, \varepsilon), \\ S - S_L &= P_3(\xi, w_L, \varepsilon). \end{aligned} \tag{3.45}$$

Now at $\varepsilon = 0$, $\xi = r - r_L \geqslant 0$ and hence $(\partial r - r_L)/\partial\xi = 1$. Let U_2 be any compact set in rsS-space. Then by the Inverse Function Theorem, there exists an $\varepsilon_1 > 0$ such that for $\varepsilon \leqslant \varepsilon_1$, $\xi = P_4(r - r_L, w_L, \varepsilon)$ throughout U_2. Choosing U_2 large enough, we can write

$$\begin{aligned} s - s_L &= f_1(r - r_L, w_L, \varepsilon), \\ S - S_L &= g_1(r - r_L, w_L, \varepsilon), \end{aligned} \tag{3.46}$$

where f_1 and g_1 are defined for w_L in $\mathrm{Cl}\, U_1$, ε in $[0, \varepsilon_1]$, and $0 \leqslant r - r_L \leqslant \bar{r} - \underline{r}$. Hence (3.46) defines a parameterization of the 1-rarefaction waves in U_1 that start in U_1, where f_1 and g_1 are again at least C^3.

Letting ε_1 be the minimum of this ε_1 and the one found above, we can put (3.42) and (3.46) together to write the 1-shock–rarefaction curves in U_1 for $0 \leqslant \varepsilon \leqslant \varepsilon_1$ as

$$\begin{aligned} s - s_L &= f_1(r - r_L, w_L, \varepsilon), \\ S - S_L &= g_L(r - r_L, w_L, \varepsilon), \end{aligned} \tag{3.47}$$

defined for $|r - r_L| \leqslant \bar{r} - \underline{r}$, w_L in $\mathrm{Cl}\, U_1$ and ε in $[0, \varepsilon_1]$, where f_1 and g_1 are C^3 functions for $r - r_L \neq 0$.

We show that f_1 and g_1 are C^2 at $r = r_L$, and since one sided third derivatives exist at $r = r_L$, this implies that f_1 and g_1 are C^2 with locally Lipschitz second derivatives. Shock curves in U_1 between 0 and ε_1 are functions of $r - r_L$ at a fixed w_L in U_1, and by Lax [4], the curves defined by (3.47) have C^2 contact at w_L. This implies that first and second derivatives of f_1 and g_1 with respect to $r - r_L$ exist, and are continuous, at $r = r_L$. Since derivatives of f_1 and g_1 with respect to w_L and ε are zero from the left and right of $r - r_L = 0$, and $R_i(r - r_L, w_L, \varepsilon)$ is differentiable in w_L and ε, f_1 and g_1 are C^2 at $r - r_L = 0$. Hence f_1 and g_1 are C^2 with locally Lipschitz second derivatives in their domains. This completes the proof of Proposition 3.4.

Let U_1 and ε_1 satisfy Proposition 3.4. We wish to parametrize the 1, 3-shock–rarefaction curves in U_1 with respect to $r - r_L + \varepsilon \cdot$ arclength [respectively $s - s_L + \varepsilon \cdot$ arclength]. Thus let

$$t_1 = r - r_L + \varepsilon \int_{r_L}^{r} \left| \frac{\partial}{\partial r} H_1'(r - r_L, w_L, \varepsilon) \right| dr,$$

$$t_3 = s - s_L + \varepsilon \int_{s_L}^{s} \left| \frac{\partial}{\partial s} H_3'(r - r_L, w_L, \varepsilon) \right| ds.$$

It is immediate that

$$\frac{\partial t_1}{\partial (r - r_L)} > 0 \qquad \text{and} \qquad \frac{\partial t_3}{\partial (s - s_L)} > 0$$

and t_1 [resp. t_3] is a function of $(r - r_L, w_L, \varepsilon)$ [resp. $(s - s_L, w_L, \varepsilon)$] which is as smooth as H_i'. This implies that t_i can be taken to parameterize the i-shock–rarefaction curve; i.e., the Inverse Function Theorem implies that (3.47) defines

$$r - r_L = A_1(t_1, w_L, \varepsilon), \qquad s - s_L = A_3(t_3, w_L, \varepsilon),$$

where A_i are C^2 with locally Lipschitz second derivatives. Therefore, substituting into H_i' gives the shock–rarefaction curves in U_1 parametrized with respect to t_i at each $w_L \in U_1$, $\varepsilon \in [0, \varepsilon_1]$:

$$\begin{aligned} w &= H_1'(r - r_L, w_L, \varepsilon) = H_1(t_1, w_L, \varepsilon), \\ w &= H_3'(s - s_L, w_L, \varepsilon) = H_3(t_3, w_L, \varepsilon), \end{aligned} \tag{3.48}$$

where H_i are C^2 with locally Lipschitz second derivatives. Moreover, since u and p are 2-Riemann invariants at every ε, r and s are constant along 2-contact discontinuity curves. Thus contact discontinuity curves can be smoothly parameterized with respect to $t_2 = S - S_L$, and we can write

$$w = H_2(t_2, w_L, \varepsilon) = (r_L, s_L, t_2 + S_L).$$

This enables us to define

$$w = H(t, w_L, \varepsilon) = H_3(t_3, H_2(t_2, H_1(t_1, w_L, \varepsilon), \varepsilon), \varepsilon)$$

and

$$w = G(a, b, w_L, \varepsilon) = H(b, H(a, w_L, \varepsilon), \varepsilon),$$

where H, G are C^2 with locally Lipschitz second derivatives throughout their domains. Again, the domain and range variables of G determine an interaction with $w = w_R$, $w_M = H(a, w_L, \varepsilon)$.

Consistent with the notation in (3.27), we let $|t_i|$ be the strength of a wave, and hence in regions where shock–rarefaction curves are parameterized with respect to t_i we have

$$\begin{aligned} &\text{For 1-waves } \alpha = \mathrm{Var}_r^-(\alpha) + \varepsilon\,\mathrm{Var}(\alpha), \qquad \mu = \mathrm{Var}_r^+(\mu) + \varepsilon\,\mathrm{Var}(\mu), \\ &\text{For 3-waves } \beta = \mathrm{Var}_s^-(\beta) + \varepsilon\,\mathrm{Var}(\beta), \qquad \eta = \mathrm{Var}_s^+(\eta) + \varepsilon\,\mathrm{Var}(\eta), \\ &\text{For 2-waves } \delta = \Delta S. \end{aligned} \tag{3.49}$$

(Here, e.g., $\mathrm{Var}_r^-(\alpha)$ = the variation of α in the minus r direction.)

PROPOSITION 3.5. *For every compact, convex open set U_1 in rsS-space, there exists $U_2 \supset U_1$, $\tau_2 > \tau > 0$, and $\varepsilon_1 > 0$ such that*

(i) *For each fixed w_L in* Cl U_1, *ε in $[0, \varepsilon_1]$*

$$\mathrm{Cl}\, U_1 \subset \operatorname*{Range}_{|t_i| \leqslant \tau} H(t, w_L, \varepsilon).$$

(ii) *For each (a, b, w_L, ε) in $\Gamma = \{t \in R^3 : |t_i| \leqslant \tau\}^2 \times \mathrm{Cl}\, U_1 \times [0, \varepsilon_1]$,*

$$G(a, b, w_L, \varepsilon) \subset U_2.$$

(iii) *For each fixed w_L in* Cl U_2 *and ε in $[0, \varepsilon_1]$, H is defined for $|t_i| \leqslant \tau_2$ and* Cl $U_2 \subset \mathrm{Range}_{|t_i| \leqslant \tau_2} H(t, w_L, \varepsilon)$.

Proof. Choose $U_1' \supset \mathrm{Cl}\, U_1$, and $\tau > 0$ such that at $\varepsilon = 0$,

$$U_1' \subset \operatorname*{Range}_{|t_i| \leqslant (1/2)\tau} H(t, w_L, 0) \qquad \text{for each fixed } w_L \text{ in } \mathrm{Cl}\, U_1.$$

Choose U_2 such that at $\varepsilon = 0$,

$$G(a, b, w_L, 0) \subset U_2 \qquad \text{for each } w_L \in U_1',\ |a_i| \leqslant 2\tau,\ |b_i| \leqslant 2\tau.$$

Choose $U_2' \supset \mathrm{Cl}\, U_2$ and $\tau_2 > 0$ such that at $\varepsilon = 0$

$$U_2' \subset \operatorname*{Range}_{|t_i| \leqslant (1/2)\tau_2} H(t, w_L, 0) \qquad \text{for each fixed } w_L \text{ in } \mathrm{Cl}\, U_2.$$

We claim that there exists an $\varepsilon_1 > 0$ such that (i), (ii), and (iii) hold with the above choices of U_2, τ, and τ_2.

First we show that there exists an $\varepsilon_1 > 0$ such that, if ε is in $[0, \varepsilon_1]$, then (i) holds. For every w, w_L in $\mathrm{Cl}\, U_1$, by choice of τ we have $w = H(t, w_L, 0)$ for some t with $|t_i| \leqslant \tau/2$. Since

$$\left| \frac{\partial H}{\partial t} \right| \neq 0 \qquad \text{at} \qquad \varepsilon = 0,$$

and since by Proposition 3.4 H is defined and continuous in a neighborhood of $(t, w_L, 0)$, the Inverse Function Theorem implies that $t = B'(w, w_L, \varepsilon)$ is defined in some neighborhood $W \times V \times [0, \varepsilon]$ such that $W \subset U_1'$, $V \subset U_1'$ and $|t_i| \leqslant \tau$. Fixing w_L, such a W, V, and ε exist for every w in $\mathrm{Cl}\, U_1$. A finite number $\{W_1, ..., W_n\}$ cover $\mathrm{Cl}\, U_1$ since it is compact. Choose $\tilde{V} = \bigcap_{k=1}^{n} V_k$, $\tilde{\varepsilon} = \min\{\varepsilon_1, ..., \varepsilon_n\}$. Then for each fixed $\tilde{w}$ in $\tilde{V}$ and ε in $[0, \tilde{\varepsilon}]$, $\mathrm{Cl}\, U_1 \subset \mathrm{Range}_{|t_i| \leqslant \tau} H(t, \tilde{w}, \varepsilon)$. Now such a $\tilde{V}$ exists about each w_L in $\mathrm{Cl}\, U_1$, so a finite number $\{\tilde{V}_1, ..., \tilde{V}_m\}$ cover $\mathrm{Cl}\, U_1$. Choose $\varepsilon_1 = \min\{\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_m\}$. Then for ε in $[0, \varepsilon_1]$ and fixed w_L in $\mathrm{Cl}\, U_1$, we have

$$\mathrm{Cl}\, U_1 \subset \operatorname*{Range}_{|t_i| \leqslant \tau} H(t, w_L, \varepsilon),$$

which proves (i). Condition (iii) now follows by the same argument.

We now show that there exists an $\varepsilon_1 > 0$ such that if ε is in $[0, \varepsilon_1]$, then (ii) holds.

Let $(a, b, w_L, 0)$ be any point such that $|a_i| \leqslant \tau$, $|b_i| \leqslant \tau$, and w_L is in $\mathrm{Cl}\, U_1$. Then $w = G(a, b, w_L, 0)$ is in U_2. Hence, by continuity of G, there exists a neighborhood V of (a, b, w_L) and an $\varepsilon > 0$ such that

$$G(V \times [0, \varepsilon]) \subset U_2.$$

A finite number $\{V_1, ..., V_n\}$ cover

$$\{a \in R^3 \colon |a_i| \leqslant \tau\} \times \{b \in R^3 \colon |b_i| \leqslant \tau\} \times \mathrm{Cl}\, U_1.$$

Choose $\varepsilon_1 = \min\{\varepsilon_1, ..., \varepsilon_n\}$. ε_1 clearly satisfied condition (ii). Choosing ε_1 to be the smallest ε_1 from the three cases, Proposition 3.5 follows.

PROPOSITION 3.6. *For every compact convex open set U_1 in rsS-space, there exists an $\varepsilon_1 > 0$ such that Conditions* (H) *of Section* 1 *hold.*

Proof. Choose ε_1', U_2, τ, τ_2 so that Proposition 3.5 holds. At $\varepsilon = 0$, $|\partial H/\partial t| \neq 0$ and H is globally 1–1 at each w_L, and so by the continuity of H, these conditions also hold in a neighborhood of $\varepsilon = 0$ on compact sets. Since Γ_2 is compact, Proposition 3.6 follows for some $0 < \varepsilon_1 \leqslant \varepsilon_1'$. This proves Proposition 3.6.

Hence, using Lemma 1.1 we can define the interaction function $c = B(a, b, w_L, \varepsilon)$ so that B is C^2 with locally Lipschitz second derivatives on Γ, and such that every interaction that occurs in U_1, occurs in this domain of B. For convenience, we now assume that U_{rs} and $\tilde{U}_{rs}$ are arbitrary compact convex open sets in rs-space, and that U_1 and $\tilde{U}_1$ are arbitrary sets of the form $U_{rs} \times [\underline{S}, \bar{S}]$ and $\tilde{U}_{rs} \times [\underset{\sim}{S}, \tilde{S}]$, respectively. We prove the main interaction lemma for energy functions near $e_0(v, S)$.

LEMMA 3.2. *For every set $U_1 = U_{rs} \times [\underline{S}, \bar{S}]$ in rsS-space there exists $\varepsilon_1 > 0$ and $G > 1$ such that interactions are defined for every w_L, w_M and w_R in U_1, and such that the following estimates hold for these interactions*:

(Change in strength at ε) $\leqslant$ (Change in strength at $\varepsilon = 0$) + $G\varepsilon D$;

i.e.

$$\Delta\alpha = \alpha' - \alpha_1 - \alpha_2 \leqslant \Delta\alpha^0 + G\varepsilon D,$$

$$\Delta\beta = \beta' - \beta_1 - \beta_2 \leqslant \Delta\beta^0 + G\varepsilon D,$$

$$\Delta\,|\delta| = |\delta'| - |\delta_1| - |\delta_2| \leqslant \Delta\,|\delta|^0 + G\varepsilon D,$$

$$\Delta\mu = \mu' - \mu_1 - \mu_2 \leqslant \Delta\mu^0 + G\varepsilon D,$$

$$\Delta\eta = \eta' - \eta_1 - \eta_2 \leqslant \Delta\eta^0 + G\varepsilon D,$$

$$\Delta\delta_\alpha = \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} \leqslant \Delta\delta_\alpha^0 + G\varepsilon D,$$

$$\Delta\delta_\beta = \delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'} \leqslant \Delta\delta_\beta^0 + G\varepsilon D.$$

Proof. We do the proofs for $\Delta\,|\delta|$, $\Delta\mu$, and $\Delta\delta_\alpha$; the others are done similarly. First, let ε_1 be chosen for Proposition 3.6 so that U_1, ε_1 satisfy Conditions (H).

Case (i). $\Delta\,|\delta| \leqslant \Delta\,|\delta|^0 + G\varepsilon D.$

Since U_1, ε_1 satisfy Conditions (H), Theorem 1.1 applies, and thus across interactions in U_1 we have

$$|c_2(\varepsilon)-c_2(0)|=|\delta'(\varepsilon)-\delta'(0)|\leqslant G\varepsilon D$$

and thus

$$|\delta'(\varepsilon)|-|\delta'(0)|\leqslant G\varepsilon D$$

and so

$$|\delta'(\varepsilon)|-|\delta_1|-|\delta_2|\leqslant|\delta'(0)|-|\delta_1|-|\delta_2|+G\varepsilon D$$

or

$$\Delta\,|\delta|\leqslant\Delta\,|\delta|^0+G\varepsilon D.$$

This proves case (i).

Case (ii). $\Delta\mu\leqslant\Delta\mu^0+G\varepsilon D$.

Since U_1, ε_1 satisfy Condition (H), interactions in U_1 occur between the states of the interactions function $c=B(a,b,w_L,\varepsilon)$ in the domain Γ. With the notation of Theorem 2.2, we have

$$\mu(\varepsilon)-\mu(0)=R(c_1(\varepsilon))-R(c_1(0))\leqslant 0$$

and so Theorem 2.2 applies to yield

$$\mu(\varepsilon)-\mu(0)\leqslant G\varepsilon D,$$

which implies

$$\mu(\varepsilon)-\mu_1-\mu_2\leqslant\mu(0)-\mu_1-\mu_2+G\varepsilon D$$

or

$$\Delta\mu\leqslant\Delta\mu^0+G\varepsilon D.$$

This proves case (ii).

Case (iii). $\Delta\delta_\alpha\leqslant\Delta\delta_\alpha^0+G\varepsilon D$.

At each ε, δ_α is the change in entropy along a 1-shock curve. Since $w=(r,s,S)=H_1(t_1,w_L,\varepsilon)$ defines the 1-shock–rarefaction curve, we can define $\delta_\alpha(t_1,w_L,\varepsilon)=S-S_L$. Of course $\delta_\alpha(t_1,w_L,\varepsilon)=0$ when $t_1=0$ since $S=S_L$. Moreover, $\delta_\alpha(t_1,w_L,\varepsilon)$ is a C^2 function of (t_1,w_L,ε) with locally Lipschitz second derivatives. With this notation we can write

$$\delta_{\alpha'}(\varepsilon)=\delta_\alpha(c_1,w_L,\varepsilon)=\delta_\alpha(B_1(a,b,w_L,\varepsilon),w_L,\varepsilon),$$

defined on Γ. Thus

$$\begin{aligned}\Delta\delta_\alpha&=\delta_\alpha(a_1,w_L,\varepsilon)+\delta_\alpha(b_1,w_L,\varepsilon)-\delta_\alpha(c_1,w_L,\varepsilon)\\&=g_\alpha(a,b,w_L,\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\Delta\delta_\alpha - \Delta\delta_\alpha^0 &= g_\alpha(a, b, w_L, \varepsilon) - g_\alpha(a, b, w_L, 0)\\ &= f_\alpha(a, b, w_L, \varepsilon).\end{aligned}$$

But when $D = 0$, $\Delta\delta_\alpha = 0$ and $\Delta\delta_\alpha^0 = 0$, and hence

$$f_\alpha(a, b, w_L, \varepsilon) = 0.$$

Also when $\varepsilon = 0$, $\Delta\delta_\alpha^0 = 0$, and again

$$f_\alpha(a, b, w_L, \varepsilon) = 0.$$

Since f_α is C^2 with locally Lipschitz second derivatives, Corollary 1 of Theorem 2.1 applies to each Γ_s to yield

$$\Delta\delta_\alpha - \Delta\delta_\alpha^0 \leqslant |\Delta\delta_\alpha - \Delta\delta^0| \leqslant G\varepsilon D$$

and hence $\Delta\delta_\alpha \leqslant \Delta\delta_\alpha^0 + G\varepsilon D$. This completes the proof of Lemma 3.2. Without loss of generality, we let G be the same as the one in Lemma 3.1.

LEMMA 3.3. *For every set $U_1 = U_{rs} \times [\underline{s}, \bar{s}]$, there exists an $M > 0$ depending only on U_{rs}, and an $\varepsilon_1 > 0$ such that, if $w_L, w_R \in U_1$, the associated Riemann problem is solvable for each $\varepsilon \in [0, \varepsilon_1]$, and the waves in these Riemann problems satisfy the following estimates*:

$$\begin{aligned}
M \operatorname{Var}_r^-(\alpha) &> \operatorname{Var}(\alpha), & \operatorname{Var}_s^-(\alpha) &< \alpha,\\
M \operatorname{Var}_s^-(\beta) &> \operatorname{Var}(\beta), & \operatorname{Var}_r^-(\beta) &< \beta,\\
2 \operatorname{Var}_r^+(\mu) &> \operatorname{Var}(\mu), & \operatorname{Var}_r^-(\mu) &= 0,\\
2 \operatorname{Var}_s^+(\eta) &> \operatorname{Var}(\eta), & \operatorname{Var}_s^-(\eta) &= 0,\\
\operatorname{Var}_s(\mu) &< \tfrac{1}{4}\mu, & \operatorname{Var}_r(\eta) &< \tfrac{1}{4}\eta.
\end{aligned}$$

Proof. These estimates are immediate consequences of the fact that the estimates hold at $\varepsilon = 0$, together with the fact that, in some neighborhood of $\varepsilon = 0$, Riemann problems in U_1 are uniquely solvable, and the shock-rarefaction curves that give these solutions depend differentiably on ε.

LEMMA 3.4. *For every compact set E in rsS-space, there exists a constant $0 < C_1 < 1$ such that, for every $B_{C_1}(w)$ with w in E ($B_{C_1}(w)$ = ball of radius C_1, center w), interaction problems in $B_{C_1}(w)$ are solvable for each $\varepsilon \in [0, 1]$ with solution waves that satisfy the estimates of Lemma* 3.2.

Proof. By Theorem 1.1, there is a neighborhood $B_C(w)$ about every w in E such that Conditions (H) are satisfied in $B_C(w)$. This is all that is needed to obtain Lemma 3.2 for interactions that occurs in $B_C(w)$. E being compact implies that a finite number

$$\left\{\frac{B_{C_k}}{2}\right\}_{k=1}^{n}$$

cover E. Let

$$C_1 = \min \left\{\frac{C_k}{2}\right\}_{k=1}^{n}.$$

Then for every $w \in E$, $B_{C_1}(W) \subset B_{C_k}(w_k)$ for some k, and so Lemma 3.4 follows.

Let E be an arbitrary compact set in rsS-space. Let $\tilde{U}_1 = \tilde{U}_{rs} \times [\underset{\sim}{S}, \tilde{S}]$ be a compact set in rsS-space that contains the points within a distance C_1 of E, C_1 from Lemma 3.4. Choose $\tilde{\varepsilon}_1 > 0$ so that Lemmas 3.1 to 3.3 apply to $\tilde{U}_1$. Then Riemann problems $\langle w_L, w_R \rangle$ in $\tilde{U}_1$ are uniquely solvable if $\varepsilon \leqslant \tilde{\varepsilon}_1$, or if $w_L, w_R \in B_{C_1}(w)$ for w in E and $0 \leqslant \varepsilon \leqslant 1$. Let V_0 denote the variation in the solution of one of these Riemann problems at any time $t > 0$.

LEMMA 3.5. *With V_0 defined as above, there exists a constant $K_0 > 1$ such that*

$$V_0 \leqslant K_0 \, \|w_L - w_R\|.$$

Proof. Since the waves in the solutions of Riemann problems above have uniformly bounded total variation, the lemma is only interesting when $\|w_L - w_R\|$ is small. But the existence of a K_0 in a neighborhood of every $(w, \varepsilon) \in \tilde{U}_1 \times [0, 1]$ follows from the strict hyperbolicity of the equations, together with the continuity of the eigenvectors with respect to ε. The compactness of $\tilde{U}_1 \times [0, 1]$ then implies the existence of a uniform constant K_0, as desired.

4. EXISTENCE THEOREM USING GLIMM DIFFERENCE SCHEME

We first describe the Glimm difference approximation $U_h(x, t)$, $h > 0$ as described by Liu in [5]. Fix mesh lengths $h > 0$, $l > 0$ in the x, t directions, respectively. $U_h \equiv (r_h, s_h, S_h)$ is defined inductively by the following process:

Choose an equidistributed random number α_k in $(-1, 1)$ and consider mesh points $a_{m,n} = ((m + \alpha_n) h, nl)$, m an integer, $m + n$ even. Now if

$U_h(x, t)$ is defined for $0 < t < nl$, we get a piecewise constant function on $t = nl$, $-\infty < x < \infty$, by setting

$$U_h(x, nl) = U_h((m + a_n)h, nl - 0), \qquad (m-1)h < x < (m+1)h,$$

where $m + n$ is even, and then by solving the corresponding Riemann problems we can construct the approximate solution $U_h(x, t)$ in the strip $nl < t < (n+1)l$, $-\infty < x < \infty$.

In the above process, in order that in each strip $nl \leqslant t < (n+1)l$ the solutions of the Riemann problems do not interact, we impose the following (Courant–Friedricks–Lewy) condition:

$$\frac{l}{h} \leqslant \min_{\substack{-\infty < x < \infty \\ t \geqslant 0}} \left\{ 1 \Big/ \left(-\frac{\partial p(v, S)}{\partial v} \right)^{1/2} (x, t) \right\}. \tag{4.1}$$

We shall show later such an $l/h > 0$ can be chosen for the initial data we consider.

In order to obtain a subsequence of approximate solutions U_h which converges to a solution of system (3), we need to obtain a uniform bound on the total variation of $U_h(\cdot, t)$ on each line $t = \text{constant} > 0$. To this end, we need a functional F which measures the total variation of U_h along any "I-curve." A curve is an I-curve if it consists of line segments of the form $L_{m,n,m+1,n+1}$, $L_{m,n,m+1,n-1}$ joining $a_{m,n}$ to $a_{m+1,n+1}$, and joining $a_{m,n}$ to $a_{m+1,n-1}$, respectively, and if the mesh index m increases monotonically from $-\infty$ to $+\infty$ along such a curve. We can partially order the I-curves by saying that larger curves lie toward larger time. Let O denote the I-curve passing through the mesh points on $t = 0$ and $t = h$. In what follows, J, J_1 and J_2 are I-curves, and J_2 is the immediate successor of J_1 if J_i pass through the same mesh points except one with $J_1 < J_2$. We write $J \subset U_1 \subset rsS$-space if the states that cross J lie in U_1, and we let $\text{Var}(J)$ denote the total variation in r, s and S of all the waves that cross J.

Let E be an arbitrarily large compact set in rsS-space. We consider now regions $U_1 = U_{rs} \times [\underline{S}, \bar{S}]$ that satisfy the conditions in Lemmas 3.1 to 3.4 for some $\varepsilon_1 > 0$, $G > 1$, $M > 1$, $K_0 > 1$, $0 < C_1 < 1$. Let

$$0 < \frac{l}{h} < \min_{U_1 \times [0,1]} \left\{ 1 \Big/ \left(-\frac{\partial p(v, S, \quad)}{\partial v} \right)^{1/2} \right\}.$$

Since $U_1 \times [0, 1]$ is compact and $\partial p / \partial v$ is continuous, such a positive minimum exists. We now consider I-curves that evolve in U_1 from initial data in E of total variation V. We wish to choose U_1 and ε_1 so that, if $\varepsilon \cdot V$ is sufficiently small, then I-curves either remain in U_1 or in $B_{C_1}(w)$ some $w \in E$, and $\text{Var}(J)$ remains uniformly bounded for all I-curves J. We obtain this by showing that a certain functional F which dominates the total variation of the solution on J, decreases. Hence let

$$L(J) = \sum_J \{\alpha_i - M_0\delta_{\alpha_i} + \beta_i - M_0\delta_{\beta_i}\} + M_0 \sum_J |\delta_i| + \varepsilon \sum_J \{\mu_i + \eta_i\} + V,$$

$$L(J) = \sum_J \{\alpha_i = M_0\delta_{\alpha_i} + \beta_i - M_0\delta_{\beta_i}\} + M_0 \sum_J |\delta_i| + \varepsilon \sum_J \{\mu_i + \eta_i\} + V,$$

$$Q(J) = M_1 \sum_{\substack{APP \\ J}} p_i |\delta_i| + M_2 \sum_{\substack{APP \\ J}} q_i R_i + M_3 \sum_{\substack{APP \\ J}} \alpha_i \beta_i,$$

$$F(J) = L(J) + \varepsilon Q(J),$$

where M_0, M_1, M_2, and M_3 are to be chosen later.

Since near $\varepsilon = 0$, the derivative of the change in entropy with respect to shock strength goes to infinity along shock curves, we need a bound on the variation of I-curves in the rs-plane which is independent of M_0. To this end we define

$$\begin{aligned} L_0(J) &= \sum_J \{\alpha_i + \beta_i\} + V, \\ F_0(J) &= L_0(J) + \varepsilon Q(J). \end{aligned} \tag{4.3}$$

LEMMA 4.1. *Let U_1, ε_1, and M satisfy the conditions of Lemmas* 3.1 *to* 3.3 *as above. Let J_2 be an I-curve which evolves at $\varepsilon \leqslant \varepsilon_1$ from initial data $w_0(x)$ of variation V, through I-curves that lie in U_1; i.e., assume that for $J \leqslant J_1$, J is in U_1. Then the following estimates hold for any $M_0 \leqslant 1/2M$, and any J_2 an immediate successor of J_1:*

(i) $\mathrm{Var}_{rs}(J_2) \leqslant 20 L_0(J_2)$;

(ii) $\mathrm{Var}(J_2) \leqslant K L(J_2)$, where $K = \dfrac{20}{M_0^2}, M_0 \leqslant \dfrac{1}{2M}$

Proof. Since all I-curves go from w^- to w^+ ($w^\pm = \lim_{x \to \pm\infty} w_0(x)$) we have

$$|\mathrm{Var}_r^+(J_2) - \mathrm{Var}_r^-(J_2)| = |r^+ - r^-| \leqslant V.$$

So

$$\mathrm{Var}_r^+(J_2) \leqslant V + \mathrm{Var}_r^-(J_2). \tag{*}$$

Moreover, by Lemma 3.3

$$\mathrm{Var}_r^+(J_2) \geqslant \mathrm{Var}_r^+(\text{1-rarefaction waves}) \geqslant \frac{1}{2} \sum_{J_2} \mu_i,$$

$$\begin{aligned} \mathrm{Var}_r^-(J_2) &= \mathrm{Var}_r^-(\text{2-rarefaction waves}) + \mathrm{Var}_r^-(\text{shock waves}) \\ &\leqslant \frac{1}{4} \sum_{J_2} \eta_i + \sum_{J_2} \alpha_i + \beta_i, \end{aligned}$$

where the notation is clear. Hence (*) yields

$$\frac{1}{2}\sum_{J_2}\mu_i \leqslant V+\sum_{J_2}(\alpha_i+\beta_i)+\frac{1}{4}\sum_{J_2}\eta_i.$$

Similarly

$$\frac{1}{2}\sum_{J_2}\eta_i \leqslant V+\sum_{J_2}(\alpha_i+\beta_i)+\frac{1}{4}\sum_{J_2}\mu_i$$

and adding we obtain

$$\sum_{J_2}\mu_i+\eta_i \leqslant 8\left(V+\sum_{J_2}\alpha_i+\beta_i\right). \tag{**}$$

But again by Lemma 3.3 we have

$$\mathrm{Var}_{rs}(J_2) \leqslant 2\sum_{J_2}(\alpha_i+\beta_i)+2\sum_{J_2}(\mu_i+\eta_i)$$

and hence

$$\begin{aligned}\mathrm{Var}_{rs}(J_2) &\leqslant 2\sum_{J_2}(\alpha_i+\beta_i)+16\left(V+\sum_{J_2}\alpha_i+\beta_i\right)\\ &\leqslant 20L_0(J_2).\end{aligned}$$

This proves case (i). Next, by Lemma 3.3 we have

$$\begin{aligned}\sum_{J_2}\alpha_i-M_0\delta_{\alpha_i}+\beta_i-M_0\delta_{\beta_i} &\geqslant \sum_{J_2}\alpha_i-\frac{1}{2M}\delta_{\alpha_i}+\sum_{J_2}\beta_i-\frac{1}{2M}\delta_{\beta_i}\\ &\geqslant \sum_{J_2}\alpha_i-\tfrac{1}{2}\alpha_i+\sum_{J_2}\beta_i-\tfrac{1}{2}\beta_i\\ &\geqslant \frac{1}{2}\sum_{J_2}\alpha_i+\beta_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}(J_2) &\leqslant \mathrm{Var}(\text{shocks})+\mathrm{Var}(\text{rarefaction waves})+\mathrm{Var}(\text{contact waves})\\ &\leqslant M\sum_{J_2}(\alpha_i+\beta_i)+2\sum_{J_2}(\mu_i+\eta_i)+\sum_{J_2}|\delta_i|\end{aligned}$$

and by (**)

$$\begin{aligned}
&\geqslant M\sum_{J_2}(\alpha_i+\beta_i)+16\sum_{J_2}(\alpha_i+\beta_i)+16V+\sum_{J_2}|\delta_i|\\
&\leqslant 20M\left\{\sum_{J_2}(\alpha_i+\beta_i)+V\right\}+\sum_{J_2}|\delta_i|\\
&\leqslant 40M\left\{\sum_{J_2}(\alpha_i-M_0\delta_{\alpha_i}+\beta_i-M_0\delta_{\beta_i})+V\right\}+\sum_{J_2}|\delta_i|\\
&\leqslant \frac{40M}{M_0}\left\{\sum_{J_2}(\alpha_i-M_0\delta_{\alpha_i}+\beta_i-M_0\delta_{\beta_i})+M_0\sum_{J_2}|\delta_i|+V\right\}\\
&\leqslant 80M^2L(J_2)=\frac{20}{M_0^2}L(J_2).
\end{aligned}$$

This completes the proof of Lemma 4.1.

Again consider any compact set E in rsS-space from which our initial data are chosen. To show that F decreases we need to choose a set U_1, containing E, which is independent of the constant K_0 in Lemma 3.5. We also need α_i and β_i to dominate $M_0\delta_{\alpha_i}$ and $M_0\delta_{\beta_i}$ far away from $\varepsilon=0$. To this end we distinguish between U_1 and $\tilde{U}_1=\tilde{U}_{rs}\times[\underset{\sim}{S},\tilde{S}]$, and will choose $U_1\supset\tilde{U}_1$ depending on estimates we obtain in $\tilde{U}_1$. Thus let $\tilde{U}_1$, $\tilde{M}$, $\tilde{\varepsilon}_1$, C_1 satisfy the conditions of Lemmas 3.1 to 3.5. K_0 then depends on $\tilde{U}_1$. Moreover, we have the following lemma.

LEMMA 4.2. *Let* $\tilde{U}_1\supset E$, $\tilde{\varepsilon}_1$, $\tilde{M}$ *and* C_1 *satisfy the conditions of Lemmas* 3.1 *to* 3.5 *as above. Let* J_2 *be an I-curve which evolves at* ε *in* $[0,1]$ *from initial data of variation* V, *through I-curves that lie in* $B_{C_1}(w)$ *for some* $w\in E$; *i.e., assume that for* $J\leqslant J_1$, J *is in* $B_{C_1}(w)$. *Then the following estimate holds for any* J_2 *an immediate successor of* J_1, *so long as* M_0 *satisfies*:

$$M_0\leqslant\min\left\{\frac{\tilde{\varepsilon}_1^2}{2},\frac{1}{2\tilde{M}}\right\}=\tilde{M}_0:$$

$$\operatorname{Var}(J_2)\leqslant KL(J_2),\qquad K=\frac{20}{M_0^2}.$$

Proof. Since strengths of waves are defined as in (3.49), we have, for $\varepsilon\geqslant\tilde{\varepsilon}_1$,

$$\begin{aligned}
\alpha_i-M_0\delta_{\alpha_i}&\geqslant\operatorname{Var}_r(\alpha_i)+\tilde{\varepsilon}_1\operatorname{Var}(\alpha_i)-\frac{\tilde{\varepsilon}_1^2}{2}\operatorname{Var}(\alpha_i)\\
&\geqslant\frac{\tilde{\varepsilon}_1}{2}\operatorname{Var}(\alpha_i)
\end{aligned}$$

and likewise

$$\beta_i - M_0 \delta_{\beta_i} \geqslant \frac{\tilde{\varepsilon}_1}{2} \operatorname{Var}(\beta_i).$$

Therefore,

$$L(J_2) \geqslant \frac{\varepsilon_1}{2} \sum_{J_2} \{\operatorname{Var}(\alpha_i) + \operatorname{Var}(\beta_i)\} + M_0 \sum_{J_2} \operatorname{Var}(\delta_i)$$
$$+ \tilde{\varepsilon}_1^2 \sum_{J_2} \{\operatorname{Var}(\mu_i) + \operatorname{Var}(\eta_i)\} \geqslant \frac{1}{K} \operatorname{Var}(J_2).$$

If $\varepsilon \leqslant \tilde{\varepsilon}_1$, then Lemma 4.1 applies and we have

$$L(J_2) \geqslant \frac{1}{K} \operatorname{Var}(J_2).$$

This proves Lemma 4.2.

We now come to the main theorem of Section 4.

THEOREM 4.1. *Let E be any compact set in rsS-space, and let $N > 1$ be any positive constant. Then there exists a constant $C = C(E, N)$ such that, for every initial data $w_0(x) \subset E$ with $TV(w_0(x)) = V \leqslant N$, if $\varepsilon V < C$, then there exists a global weak solution to problem* (3).

Proof. We show that there exists constants M_i, C depending only on E and N, such that, if $\varepsilon V < C$, then F bounds the variation of all I-curves J, F decreases, and I-curves remain within a region where our estimates hold, and where Condition (4.1) is maintained. In order to be sure that $M_0 \delta_{\alpha_i}$ and $M_0 \delta_{\beta_i}$ are dominated by the respective shock strengths, we need to do an induction on F_0 as well as F. This yields a uniform bound on the variation of all I-curves that arise from $w_0(x)$, and hence by [2, 8], implies that there exists a global weak solution to problem (3). First, note that by the definition of wave strength, $p_i \leqslant 2 \operatorname{Var}(p_i)$ for any wave p_i, and hence, for any I-curve J,

$$\begin{aligned} \sum_J p_i \leqslant \sum_J 2 \operatorname{Var}(p_i) = 2 \operatorname{Var}(J), \\ L(O) \leqslant 2 \operatorname{Var}(O) + V \end{aligned} \tag{A}$$

We now define the constants M_0, M_1, M_2, M_3 and C. Let $\tilde{U}_1 \supset E$, $\tilde{\varepsilon}_1$, $\tilde{M}_0$, K_0, and C_1 be chosen so that Lemmas 3.4, 3.5, and 4.2 hold. Let U_{rs} be the set of points in the rs-plane within a distance of $20(6K_0N)$ of $\tilde{U}_{rs}$. This choice of U_{rs} determines the M and C_0 of Lemmas 3.1 to 3.3 and Lemma 4.1. Define

$$M_0 = \min\left\{\frac{1}{2M}, \tilde{M}_0\right\}, \qquad K = \max\left\{\frac{20}{M_0^2}, 1\right\}.$$

Let $U_1 = U_{rs} \times [\underset{\sim}{S}, \bar{S}]$, where

$$\min\{\operatorname{dist}(\tilde{U}_1, \underset{\sim}{S}), \operatorname{dist}(\tilde{U}_1, \bar{S})\} > 6K_0 KN.$$

Choose $0 < \varepsilon_1 \leqslant \tilde{\varepsilon}_1$ and $G > 1$ so that Lemmas 3.1 to 3.4 and Lemma 4.1 hold with this choice of U_1. Let

$$0 < \frac{l}{h} < \min_{U_1 \times [0,1]} \left\{ 1 \Big/ \left(-\frac{\partial p(v, S, \varepsilon)}{\partial v} \right)^{1/2} \right\}.$$

Let J and O denote I-curves that satisfy the conditions of either Lemma 4.1 or 4.2. We have

$$\begin{aligned} \operatorname{Var}(J) &\leqslant KL(J) \qquad \text{for} \quad 0 \leqslant \varepsilon \leqslant 1, \\ \operatorname{Var}_{rs}(J) &\leqslant 20 L_0(J) \qquad \text{for} \quad \varepsilon \leqslant \varepsilon_1. \end{aligned} \tag{B}$$

Now define

$$\begin{aligned} M_1 &= 8G, \\ M_2 &= 8G + 2M_1 K(6K_0 N)G, \\ M_3 &= 8G + 2M_1 K(6K_0 N)G + 4M_2 K(6K_0 N)G, \end{aligned} \tag{C}$$

and let

$$K_1 = 7K_0^2 M_3 N. \tag{D}$$

Note that Lemma 3.5 implies $\operatorname{Var}(O) \leqslant K_0 V$ if $\varepsilon \leqslant \varepsilon_1$ or if $V \leqslant C_1$. In this case we have by (A)

$$\begin{aligned} L(O) &\leqslant 2 \operatorname{Var}(O) + V \leqslant 3K_0 V, \\ L_0(O) &\leqslant 2 \operatorname{Var}(O) + V \leqslant 3K_0 V. \end{aligned} \tag{E}$$

Moreover, $Q(O) \leqslant M_3 (2 \operatorname{Var}(O))^2 \leqslant 4M_3 K_0^2 V^2$, hence

$$Q(O) \leqslant 4M_3 K_0^2 NV \leqslant K_1 V, \tag{F}$$

and so for $0 \leqslant \varepsilon \leqslant 1$,

$$F(O) \leqslant 3K_0 V + 4M_3 K_0^2 NV \leqslant 7K_0^2 M_3 NV = K_1 V. \tag{G}$$

Let $\varepsilon_0 = \min\{\varepsilon_1, 1/K_1\}$. Then if $\varepsilon \leqslant \varepsilon_0$, we have $\varepsilon \cdot Q(0) \leqslant V$ and hence

$$\begin{aligned} F(0) &\leqslant 2L(O) \leqslant 6K_0 V, \\ F_0(O) &\leqslant 2L_0(O) \leqslant 6K_0 V, \end{aligned} \qquad \varepsilon \leqslant \varepsilon_0. \tag{H}$$

Note here that ε_0, M_i depend only on E, N. Let $C_2 = \varepsilon_0 C_1/K_1 K$. Then if $\varepsilon \geqslant \varepsilon_0$ and $\varepsilon V < C_2$, we have $KK_1 V < C_1$, which by (G) implies

$$KF(O) < C_1 \qquad \text{for} \quad \varepsilon \geqslant \varepsilon_0, \tag{I}$$

so that $C_1/K \leqslant 1$, together with (H), implies that if $\varepsilon V < C_2$, then

$$F(O) \leqslant 6K_0 N, \qquad \text{for} \quad 0 \leqslant \varepsilon \leqslant 1. \tag{J}$$

Finally, let

$$C_3 = \min\left\{\frac{(12M_3GK)^{-1}}{K_1}, \frac{1-C_0}{4M_3KK_1}\right\},$$

in which case, if $\varepsilon V < C_3$, then $\varepsilon F(O) \leqslant \varepsilon K_1 V \leqslant K_1 C_3$ and so

$$\varepsilon F(O) < (12M_3GK)^{-1} \quad \text{and} \quad \varepsilon F(O) < \frac{1-C_0}{4M_3K}, \quad \text{for } 0 \leqslant \varepsilon \leqslant 1. \tag{K}$$

Let $C = \min\{C_2, C_3\}$.

We now prove by induction that with these choices of M_0, M_1, M_2, M_3 and C, if $\varepsilon V < C$, then F decreases across successive I-curves and I-curves remain within regions where our estimates for Lemmas 3.1 to 3.5, 4.1 and 4.2 hold (i.e., J remains within U_1 if $\varepsilon \leqslant \varepsilon_0$, and within $B_{C_1}(w)$ some $w \in E$ if $\varepsilon \geqslant \varepsilon_0$). Specifically, we prove by induction that the following inequalities hold at every I-curve J:

$$\begin{aligned} &\mathrm{Var}(J) \leqslant KL(J) \leqslant KF(J) \leqslant KF(O), \quad \text{for } 0 \leqslant \varepsilon \leqslant 1, \\ &\mathrm{Var}_{rs}(J) \leqslant 20L_0(J) \leqslant 20F_0(J) \leqslant 20F_0(O) \leqslant 20(6K_0N), \quad \text{for } 0 \leqslant \varepsilon \leqslant \varepsilon_0. \end{aligned}$$

Hence, by (I) and (J), all approximate solutions from $w_0(x)$ have uniformly bounded total variation at each ε in $[0, 1]$, which by Glimm [2] and Liu [8], implies that there exists a global weak solution to our problem (3). Now if $J = O$, then since $w_0(x) \subset \bar{U}_1$, and since $\varepsilon V < C$, we have from (B) and (H) that for $\varepsilon \leqslant \varepsilon_0$,

$$\mathrm{Var}_{rs}(O) \leqslant 20L_0(O) \leqslant 20F_0(O) \leqslant 20(6K_0N),$$

implying $O \subset U_{rs}$. Moreover

$$\operatorname{Var}(O) \leqslant KF(O) \leqslant 6KK_0N,$$

implying $O \subset U_1$.

For $\varepsilon \geqslant \varepsilon_0$ we have from (B) and (I)

$$\operatorname{Var}(O) \leqslant KF(O) < C_1$$

and hence $O \subset B_{C_1}(w)$ some $w \in E$. Thus if $J = O$, then the desired bounds on $\operatorname{Var}(J)$ obtain, and so O lies within the regions where our estimates hold. Now let J_2 be an immediate successor of J_1, and let Δ be the diamond shaped region between J_1 and J_2 in the xt-plane. The states on $J_2 \backslash J_1$ are obtained from the Riemann problem of states $\langle w_L, w_R \rangle$ which lie at the lateral vertices of Δ; and states on $J_1 \backslash J_2$ are obtained from the Riemann problems $\langle w_L, w_M \rangle$, $\langle w_M, w_R \rangle$, where w_M is the state at the lower vertice of Δ. Assume by induction that $F(J) \leqslant F(O)$ and $F_0(J) \leqslant F_0(O)$ for all $J \leqslant J_1$ (and hence that $J \subset U_1$ for $\varepsilon \leqslant \varepsilon_0$ or $J \subset B_{C_1}(w)$ for $\varepsilon \geqslant \varepsilon_0$, $\omega \in E$). Then across Δ the interaction $\langle w_L, w_M \rangle + \langle w_M, w_R \rangle \to \langle w_L, w_R \rangle$ occurs, and the estimates of Lemmas 3.1 to 3.3 apply to this interaction. The following estimates are now easily obtained by applying the interaction estimates of Lemma 3.1 to corresponding wave strength differences across Δ, and including an error of $G\varepsilon D$ in each case as required by Lemmas 3.2 and 3.4.

$$\begin{aligned} L(J_2) - L(J_1) &\leqslant \alpha' - \alpha_1 - \alpha_2 + \beta' - \beta_1 - \beta_2 \\ &\quad + M_0\{|\delta'| - |\delta_1| - |\delta_2| + \delta_{\alpha_1} + \delta_{\alpha_2} - \delta_{\alpha'} + \delta_{\beta_1} + \delta_{\beta_2} - \delta_{\beta'}\} \\ &\quad + \varepsilon\{\mu' - \mu_1 - \mu_2 + \eta' - \eta_1 - \eta_2\} \\ &\leqslant A + B + 2G\varepsilon D - M_0(A + B) + 3G\varepsilon D + 2\varepsilon\{GD_3 + G\varepsilon D\} \\ &\leqslant \tfrac{1}{2}(A + B) + 7G\varepsilon D \leqslant -\frac{(1 - C_0)}{2}\xi + 7G\varepsilon D \text{ since } M_0 \leqslant \frac{1}{M}. \end{aligned}$$

$$\begin{aligned} L_0(J_2) - L_0(J_1) &\leqslant \alpha' - \alpha_1 - \alpha_2 + \beta' - \beta_1 - \beta_2 \\ &\leqslant (A + B) + 2G\varepsilon D \leqslant -(1 - C_0)\xi + 2G\varepsilon D. \end{aligned}$$

$$\begin{aligned} Q(J_2) - Q(J_1) &\leqslant M_1 \sum_{J_1 \backslash \Delta} p_i(|\delta'| - |\delta_1| - |\delta_2|) - M_1 D_1 \\ &\quad + M_2 \sum_{J_1 \backslash \Delta} q_i(\mu' - \mu_1 - \mu_2) \\ &\quad + M_2 \sum_{J_1 \backslash \Delta} q_i(\eta' - \eta_1 - \eta_2) - M_2 D_2 \\ &\quad + M_3 \sum_{J_1 \backslash \Delta} \gamma_i(\alpha' - \alpha_1 - \alpha_2) \end{aligned}$$

$$+ M_3 \sum_{J_1 \setminus \Delta} \gamma_i(\beta' - \beta_1 - \beta_2) - M_3 D_3$$
$$\leqslant M_1 2 \operatorname{Var}(J_1)\{G(D_2 + D_3) + G\varepsilon D\} - M_1 D_1$$
$$+ M_2 2 \operatorname{Var}(J_1)\{GD_2 + G\varepsilon D\} - M_2 D_2$$
$$+ 2M_3 2 \operatorname{Var}(J_1)\{G\varepsilon D\} - M_3 D_3 + M_3 2 \operatorname{Var}(J_1)\xi.$$

By the induction hypothesis, $\operatorname{Var}(J_1) \leqslant KF(J_1) \leqslant KF(O)$, hence

$$Q(J_2) - Q(J_1) \leqslant 2M_1 KF(O)\{G(D_2 + D_3) + G\varepsilon D\} - M_1 D_1$$
$$+ 4M_2 KF(O)\{GD_3 + G\varepsilon D\} - M_2 D_2$$
$$+ 4M_3 KF(O)\{G\varepsilon D\} - M_3 D_3 + M_3 2KF(O)\xi.$$

The following estimate on $F(J_2) - F(J_1)$ is obtained by collecting terms with εD_i or ξ, and then factoring them out. Since our estimate for $L_0(J_2) - L_0(J_1)$ is stronger than the one for $L(J_2) - L(J_1)$, the following estimates will also hold for $F_0(J_2) - F_0(J_1)$.

$$F(J_2) - F(J_1) \leqslant \{7G + 4\{M_1 + M_2 + M_3\} KG\varepsilon F(0) - M_1\} \varepsilon D_1$$
$$+ \{7G + 2M_1 KF(O)G$$
$$+ 4\{M_1 + M_2 + M_3\} KG\varepsilon F(O) - M_2\} \varepsilon D_2$$
$$+ \{7G + 2M_1 KF(O)G + 4M_2 KF(O)G$$
$$+ 4\{M_1 + M_2 + M_3\} KG\varepsilon F(O) - M_3\} \varepsilon D_3$$
$$+ \left\{ 2M_3 K\varepsilon F(O) - \frac{1 - C_0}{2} \right\} \xi.$$

By (K), $\varepsilon F(O) < K_1 C_3$ which implies that

$$4\{M_1 + M_2 + M_3\} KG\varepsilon F(O) \leqslant 1 \text{ and } \left\{ 2M_3 K\varepsilon F(O) - \frac{1 - C_0}{2} \right\} \xi \leqslant 0.$$

Moreover, by (J), $F(O) \leqslant 6K_0 N$ and so

$$F(J_2) - F(J_1) \leqslant \{7G + 1 - M_1\} \varepsilon D_1$$
$$+ \{7G + 2M_1 K(6K_0 N)G + 1 - M_2\} \varepsilon D_2$$
$$+ \{7G + 2M_1 K(6K_0 N)G + 4M_2 K(6K_0 N)G + 1 - M_3\} \varepsilon D_3$$
$$\leqslant 0 \text{ by our choice of } M_1, M_2 \text{ and } M_3 \text{ in (C).}$$

By the note above we also have

$$F_0(J_2) - F_0(J_1) \leqslant 0.$$

We now can conclude that, if $\varepsilon \leqslant \varepsilon_0$,

$$\mathrm{Var}_{rs}(J_2) \leqslant 20L_0(J_2) \leqslant 20F_0(J_2) \leqslant 20F_0(J_1) \leqslant 20F_0(O) \leqslant 20(6K_0 N)$$

and hence $J_2 \subset U_{rs}$. Moreover

$$\mathrm{Var}(J_2) \leqslant KL(J_2) \leqslant KF(J_2) \leqslant KF(J_1) \leqslant KF(O) \leqslant 6KK_0 N$$

and hence $J_2 \subset U_1$. If $\varepsilon_0 \leqslant \varepsilon \leqslant 1$, then

$$\mathrm{Var}(J_2) \leqslant KL(J_2) \leqslant KF(J_2) \leqslant KF(J_1) \leqslant KF(O) \leqslant C_1$$

and hence $J_2 \subset B_{C_1}(w)$ some $w \in E$. This concludes the proof of Theorem 4.1.

5. An Existence Theorem for Polytropic Gases

In this section we use Theorem 4.1 to prove an existence theorem for polytropic gases, which is essentially the result obtained by Liu in [5]. The main difference here is that Liu does not view the polytropic gas equations as being near the energy function $e_0(v, S)$, and moreover, Liu measures the variation of solutions in coordinates that depends on the parameter ε which appears in the polytropic gas equations.

The equation for a polytropic gas is given in (3.10). It is important to note that for a given polytropic gas which satisfies (3.9) for some $c_v = R/\varepsilon$ with R and ε fixed, the equation for the energy function in (3.10) depends on our choice of ground states for specific entropy S and specific internal energy e. For example, if we fix ground states for S and e, and choose units for v, S, p, and e, then the equation for the energy of our polytropic gas will be

$$e(v, S) = h(v \exp(-S/R)),$$

where

$$h(x) = \exp\left(-\frac{\varepsilon S_0}{R}\right) x^{-\epsilon} + C_0$$

for some (now determined) constant S_0 and C_0. If we were now to change the ground states from which entropy and energy are measured, then the equation for the energy function of this polytropic gas would change, and this change would occur in the constants S_0 and C_0. Moreover, since any values for the ground states of entropy and energy can be chosen, we can, by an appropriate choice, obtain an equation for the energy function of this gas with preassigned values of S_0 and C_0. Let us choose these ground states so that

$$\exp\left(-\frac{\varepsilon S_0}{R}\right)=\frac{1}{\varepsilon} \qquad \text{and} \qquad C_0=-\frac{1}{\varepsilon}.$$

We now can take

$$e_\epsilon(v, S)=\frac{\left\{v\exp\left(-\frac{S}{R}\right)\right\}^{-\epsilon}-1}{\varepsilon} \tag{5.1}$$

as the energy function of a polytropic gas. But

$$\lim_{\epsilon\to 0} e_\epsilon(v, S)=-\ln(v\exp(-S/R))=e_0(v, S)$$

and the convergence is smooth in (v, S, ε). Thus Lemma 4.1 applies, and we have the following theorem: for any compact set E in rsS-space and any $N>0$, there exists a constant $C>0$ such that, if our initial data $w_0(x)$ are in E with total variation $\{w_0(x)\}=V\leqslant N$, then if $\varepsilon V<C$, there exists a global weak solution to problem (3) with energy function (5.1).

APPENDIX I

In this article we prove Lemma 1.2 of section 1, which addresses the local solvability of Riemann problems, as well as the local smoothness of the functions H and B for the general system (2).

LEMMA 1.2. *For any $\bar{u}$ in the domain of $f_\epsilon(u)$, there exists a neighborhood U_1 of $\bar{u}$ such that conditions* (H) *of Section* 1 *hold with* $\varepsilon_1=1$.

We prove this with the aid of the following propositions, and refer to the notation of system (2).

PROPOSITION 1. *For every $\bar{u}$ in the domain of $f_\epsilon(u)$, there exists a neighborhood U of $\bar{u}$, and a $\tau_2>0$, so that $u=H(t, u_L, \varepsilon)$ is defined, is C^2 with locally Lipschitz second derivatives, and satisfies $|\partial H/\partial t|\neq 0$ in*

$$T_2\times U\times[0, 1], \qquad \textit{where} \quad T_2=\{t\in R^n: |t_i|\leqslant\tau_2\}.$$

Proof. The eigenvalues of df_ϵ are real and distinct, and letting R_i denote the ith eigenvector of df_ϵ, we have that $R_i(u, \varepsilon)$ is a C^3 function of its arguments. Assume $|R_i| = 1$, and assume that R_i points in the direction of increasing λ_i if the ith characteristic field is genuinely nonlinear. Define

$$\begin{aligned} &u = H_i(t_i, u_L, \varepsilon) \qquad \text{so that} \qquad \frac{du}{dt_i} = R_i(u, \varepsilon), \\ &u = u_L \qquad \text{if} \quad t_i = 0, \end{aligned} \tag{A}$$

where $t_i \geqslant 0$ if the ith characteristic field is genuinely nonlinear. Then H_i is in C^3, and at fixed ε, H_i is the arclength parameterization of either the i-rarefaction curve or else the i-contact discontinuity curve starting at u_L, for the system at ε.

We now show that if the ith characteristic field is genuinely nonlinear, then for any ε, there is a neighborhood $U \times \Lambda$ of $(\bar{u}, \varepsilon)$ such that in $U \times \Lambda$, the ith shock curve exists and is defined by

$$u = H_i(t_i, u_L, \varepsilon) \qquad \text{for} \quad t_i \leqslant 0,$$

where H_i is in C^3, and at fixed u_L and ε, H_i is the arclength parameterization of the ith shock curve from u_L for the system at ε. The jump conditions for system (2) at fixed ε are

$$(u - u_L)\sigma = f_\varepsilon(u) - f_\varepsilon(u_L),$$

where σ is the shock speed. This can be written as

$$[G_\varepsilon(u, u_L) - \sigma](u - u_L) = 0, \tag{B}$$

where

$$G_\varepsilon(u, u_L) = G(u, u_L, \varepsilon) = \int_0^1 \frac{\partial f_\varepsilon(u_L + z(u - u_L))}{\partial u}\, dz.$$

$G(u, u_L, \varepsilon)$ is a C^3 function of its arguments, and $G(u, u_L, \varepsilon)$ approaches $\partial f(u)/\partial u\,|_{u=u_L}$ as u approaches u_L. Moreover, σ must be an eigenvalue $\bar{\lambda}_i(u, u_L, \varepsilon)$ of G, with $u - u_L$ a right eigenvector. Since $G(\bar{u}, \bar{u}, \varepsilon) = (\partial f/\partial u)(\bar{u}, \varepsilon)$, G has real and distinct eigenvalues in a neighborhood W of $(\bar{u}, \bar{u}, \varepsilon)$, and so σ exists in this neighborhood. Note also that $\sigma = \bar{\lambda}_i(u, u_L, \varepsilon)$ approaches $\lambda_i(u_L, \varepsilon)$ as u approaches u_L. In W, let $\bar{l}_i(u, u_L, \varepsilon)$ be the left eigenvector for G; i.e., let

$$\bar{l}_i(u, u_L, \varepsilon) \cdot G(u, u_L, \varepsilon) = \bar{\lambda}_i(u, u_L, \varepsilon)\, \bar{l}_i(u, u_L, \varepsilon)$$

in W. Then, when (B) holds with $\sigma = \bar{\lambda}_i$, we have $\bar{l}_j \cdot [G - \sigma] \cdot (u - u_L) = 0$, which implies that

$$(\lambda_j - \lambda_i)\, \bar{l}_j \cdot (u - u_L) = 0.$$

Since $\tilde{\lambda}_j \neq \tilde{\lambda}_i$ in W for $i \neq j$, (B) holds for $\sigma = \tilde{\lambda}_i$ if and only if the following system of $n-1$ equations are satisfied:

$$\tilde{l}_j \cdot (u - u_L) = 0, \qquad i \neq j. \tag{C}$$

Since $\tilde{l}_j$ are independent at $(\bar{u}, \bar{u}, \varepsilon)$, the implicit function theorem implies that there exists a neighborhood V of $(0, \bar{u}, \varepsilon)$ in which (C) defines a smooth one parameter family of states at each (u_L, ε), which is as smooth as the $\tilde{l}_j$. By invoking the usual entropy condition of Lax [4] (that $\lambda_i(u) < \tilde{\lambda}_i(u, u_L) < \lambda_i(u_L)$ on a shock curve), we can now write

$$u = H_i(t_i, u_L, \varepsilon) \qquad \text{for } t_i \leqslant 0, \tag{D}$$

where H is C^3 in V and where, at fixed u_L and ε, (D) defines the arclength parameterization of the i-shock curve from u_L for the system at ε.

Putting (A) and (D) together, we have the i-shock–rarefaction or contact discontinuity curves defined in a neighborhood of $(\bar{u}, \varepsilon)$ by

$$u = H_i(t_i, u_L, \varepsilon) \text{ for } u_L \in U, \ |t_i| \leqslant \tau_2, \qquad \varepsilon \in \Lambda. \tag{E}$$

By Lax [4], at fixed u_L and ε, H_i has second derivatives with respect to t_i at $t_i = 0$, and by the smoothness of H_i, one-sided third derivatives exist at $t_i = 0$. Since $u_L = H_i(0, U_L, \varepsilon)$, we also have that at $t_i = 0$, one sided limits of any three derivatives with respect to u_L and ε exist and are the same. Finally,

$$\frac{\partial H_i}{\partial t_i}(0, u_L, \varepsilon) = R_i(u_L, \varepsilon) \qquad \text{and} \qquad \frac{\partial^2 H_i}{\partial t_i^2}(0, u_L, \varepsilon) = \nabla R_i \cdot R_i(w_L, \varepsilon)$$

have at least three and two derivatives, respectively, with respect to (u_L, ε), and so $H_i(t_i, u_L, \varepsilon)$ is a C^2 function with locally Lipschitz second derivatives in V. By choosing τ_2 and U containing $\bar{u}$ sufficiently small, we can define

$$H(t, u_L, \varepsilon) = H_n(t_n, H_{n-1}(t_{n-1}, \ldots, H_1(t_1, u_L, \varepsilon), \ldots, \varepsilon), \varepsilon)$$

in $T_2 \times U \times \Lambda$, where H is C^2 with locally Lipschitz second derivatives. Moreover, since the eigenvectors of df_ε are linearly independent, $|\partial H/\partial t| \neq 0$ at $t = 0$, and so we can assume that $|\partial H/\partial t| \neq 0$ in $T_2 \times U \times \Lambda$ and that H is one to one for each fixed u_L, ε in $U \times \Lambda$. Now since $[0, 1]$ is compact, a finite number of Λ's, say $\{\Lambda_k\}_{k=1}^n$, cover $[0, 1]$. Therefore, by renaming τ_2 and U as

$$\tau_2 = \min\{\tau_k\}_{k=1}^n \qquad \text{and} \qquad U = \bigcap_{k=1}^n U_k,$$

we can conclude that H is defined on $T_2 \times U \times [0, 1]$, where $\bar{u}$ is in U and H

is C^2 with locally Lipschitz second derivatives. This completes the proof of Proposition 1.

PROPOSITION 2. *For every* $0 < \tau \leqslant \tau_2$, *there exists a neighborhood* U *containing* $\bar{u}$, *such that, for every fixed* (u_L, ε) *in* $U \times [0, 1]$, *we have*

$$U \subset \operatorname*{Range}_{|t_i| \leqslant \tau} H(t, u_L, \varepsilon).$$

Proof. Since $\bar{u} = H(0, \bar{u}, \varepsilon)$ and $|\partial H/\partial t| \neq 0$ at $(0, \bar{u}, \varepsilon)$, the inverse function theorem implies that there exists neighborhoods V, W, $T_1 = \{t \in R_n : |t_i| \leqslant \tau_1 \leqslant \tau_2\}$, and Λ, such that

$$u = H(t, u_L, \varepsilon) \tag{F}$$

if and only if

$$t = B'(u, u_L, \varepsilon) \tag{G}$$

for u in V, u_L in W, t in T_1 and ε in Λ. By the continuity of (G) together with the compactness of Λ, we can choose U containing $\bar{u}$ so that

$$B'(u, u_L, \varepsilon) \subset \{t \in R^n : |t_i| \leqslant \tau\}$$

for each fixed u_L and ε in $U \times \Lambda$. Now, again, a finite number of Λ's, say $\{\Lambda_k\}_{k=1}^n$, cover $[0, 1]$. Therefore, by renaming

$$U = \bigcap_{k=1}^{n} U_k,$$

we have that

$$U \subset \operatorname*{Range}_{|t_i| \leqslant \tau} H(t, u_L, \varepsilon)$$

for any fixed (u_L, ε) in $U \times [0, 1]$. This completes the proof of Proposition 2.

Proof of Lemma 2.1. By Proposition 2, we can choose U_2 containing $\bar{u}$ so that, for each fixed (u_L, ε) in $U_2 \times [0, 1]$, we have

$$U_2 \subset \operatorname*{Range}_{|t_i| \leqslant \tau_2} H(t, u_L, \varepsilon).$$

Now choose $\tau \leqslant \tau_2$ and U_1' so that, for (u_L, ε) in $U' \times [0, 1]$ we have

$$\operatorname*{Range}_{\substack{|a_i| \leqslant \tau \\ |b_i| \leqslant \tau}} G(a, b, u_L, \varepsilon) \subset U_2.$$

The existence of such a τ and U_1' is an immediate consequence of the continuity of G together with the compactness of $[0, 1]$. Finally, by Proposition 2 again, we can choose $\mathrm{Cl}\, U_1 \subset U_1'$ so that, for each fixed $(u_L, \varepsilon) \in U_1 \times [0, 1]$, we have

$$U_1 \subset \operatorname*{Range}_{|t_i| \leqslant \tau} H(t, u_L, \varepsilon).$$

Conditions (i) through (v) of Conditions (H) are now satisfied with τ, τ_2, U_2, U_1 defined above. This completes the proof of Lemma 2.1.

Appendix II

This is a case by case study of four wave interactions in the rs-plane for the system at $\varepsilon = 0$, needed for the proof of Proposition 3.1. Each ingoing wave can be a shock or rarefaction wave and this yields the sixteen cases listed in Table I which are treated separately. Each case here has between one and four sub-cases depending on the possible choices of shocks and rarefaction waves among the outgoing waves. All shock curves have the same shape, are convex and are assumed to occur in a compact set where 1-shock curves [resp. 3-shock curves] have slopes $\leqslant C_0$ [resp. $\geqslant C_0$] for some $0 < C_0 < 1$. 1, 3-rarefaction curves lie along the r-axis, s-axis, respectively. In the table, a_1 [resp. a_3] is the first incoming 1-wave [resp. 3-wave] and b_1 [resp. b_3] is the second incoming 1-wave [resp. 3-wave]. Since the interactions which occur here in the rs-plane are the same as those which occur in the 2×2 isothermal system [6], we let $S_{1,2} R_{1,2}$ denote 1, 3-shocks and 1, 3-rarefaction waves, respectively. In each case we show that (I) or (II) of Proposition 3.1 obtains, and that the change in rarefaction wave strengths is negative in each family when $D_3 = 0$. Here $\alpha' - \alpha_1 - \alpha_2 = A$ and $\beta' - \beta_1 - \beta_2 = B$ and conditions (I) and (II) are

$$\text{(I)} \quad A = -\xi, \qquad 0 \leqslant B \leqslant C_0 \xi$$

or

$$B = -\xi, \qquad 0 \leqslant A \leqslant C_0 \xi,$$

$$\text{(II)} \quad A \leqslant 0 \qquad \text{and} \qquad B \leqslant 0.$$

We use the geometric arguments which are diagrammed in Fig. 1. That is, in (i) of Fig. 1 the ratio $y/z \leqslant C_0$ by the mean value theorem. Part (ii) of Fig. 1 for 1-shocks simply states the the rigid motion of a curve with positive slopes smaller than C_0, determines a position which intersects the original position only if the translation is in a direction of slope smaller than C_0. To illustrate we include the most difficult cases, (1) and (9), listed in Table (I). A complete list of cases is provided in [9].

(1) (A) $(\alpha_1+\beta_1)+(\alpha_2+\beta_2)\to(a'+\beta')$. Two possible interactions are diagrammed in Fig. 2. In either case we have

$$\mu'-\mu_1-\mu_2\leqslant 0,$$
$$\eta'-\eta_1-\eta_2\leqslant 0.$$

For (i) of Fig. 2, we have that

$$\alpha'-\alpha_1-\alpha_2=y_1-z_2=A,$$
$$\beta'-\beta_1-\beta_2=y_2-z_1=A,$$

where $y_1\leqslant C_0 z_1$ and $y_2\leqslant C_0 z_2$. Now if $A\leqslant 0$ and $B\leqslant 0$ condition (II) applies. If not, assume $B>0$. Then

$$y_1<z_1<y_2<z_2$$

and so

$$A+B=(y_1-z_2)+(y_2-z_1)\leqslant -C_0(z_1+z_2)$$
$$\leqslant -C_0(z_2-y_1)$$

and so

$$B=y_2-z_1\leqslant(1-C_0)(z_0-y_1)\leqslant C_0(z_2-y_1)=C_0\xi.$$

By symmetry $A\leqslant C_0\xi$ when $B=-\xi\leqslant 0$, and so condition (I) is satisfied.

For (ii) of Fig. 2, we have that

$$\alpha'-\alpha_1-\alpha_2=y_1+y_2=A,$$
$$\beta'-\beta_1-\beta_2=-z_1-z_2=B,$$

where $y_1\leqslant C_0 z_1$ and $y_2\leqslant C_0 z_2$, and so condition (I) applies with $B=-\xi$, $A\leqslant C_0\xi$.

(1) (B) $(\alpha_1+\beta_1)+(\alpha_2+\beta_2)\to(\alpha'+\eta')$. This interaction is diagrammed in Fig. 3. Here we have

$$\alpha'-\alpha_1-\alpha_2=y=A,$$
$$\beta'-\beta_1-\beta_2=-z=B,$$

and since $y\leqslant C_0 z$, condition (I) is satisfied. Moreover, $\mu'-\mu_1-\mu_2=0$ always, and when the approaching shocks vanish (i.e., when $D_3=0$), all waves are shock waves and $\eta'-\eta_1-\eta_2=0$.

(9) (A) $(\mu_1+\beta_1)+(\alpha_2+\beta_2)\to(\alpha'+\beta')$. This interaction is diagrammed in Fig. 4. Here we have

$$\mu'-\mu_1-\mu_2\leqslant 0,$$
$$\eta'-\eta_1-\eta_2\leqslant 0,$$

and moreover

$$\alpha' - \alpha_1 - \alpha_2 = A = -z,$$
$$\beta' - \beta_1 - \beta_2 = B = y,$$

where $y \leqslant C_0 z$. Thus condition (I) is satisfied with $A = -\xi$.

(9) (B) $(\mu_1 + \beta_1) + (\alpha_2 + \beta_2) \to (\mu' + \beta')$. This interaction is diagrammed in Fig. 5. Here we have

$$\alpha' - \alpha_1 - \alpha_2 = A = -z,$$
$$\beta' - \beta_1 - \beta_2 = B = y,$$

where $y \leqslant C_0 z$, and so condition (I) is satisfied with $A = -\xi$. Moreover, $\eta' - \eta_1 - \eta_2 \leqslant 0$ always holds. When $D_3 = 0$, we have that either $(\alpha_2 = 0)$ and ($\beta_1 = 0$ or $\beta_2 = 0$), or $\alpha_2 \neq 0$ and $\beta_1 = 0$. In the first case it is clear that $\mu' = \mu_1$. In the second case, β_2 does not intersect β' since the slope of $\alpha_2 < C_0$. Hence, $\mu' \leqslant \mu_1$ again, and thus $\mu' - \mu_1 - \mu_2 \leqslant 0$ when $D_3 = 0$.

DIAGRAMS

TABLE II

Cases

Case Number	a_1	a_3	b_1	b_3
(1)	S_1	S_2	S_1	S_2
(2)	S_1	S_2	S_1	R_2
(3)	S_1	S_2	R_1	S_2
(4)	S_1	S_2	R_1	R_2
(5)	S_1	R_2	S_1	S_2
(6)	S_1	R_2	S_1	R_2
(7)	S_1	R_2	R_1	S_2
(8)	S_1	R_2	R_1	R_2
(9)	R_1	S_2	S_1	S_2
(10)	R_1	S_2	S_1	R_2
(11)	R_1	S_2	R_1	S_2
(12)	R_1	S_2	R_1	R_2
(13)	R_1	R_2	S_1	S_2
(14)	R_1	R_2	S_1	R_2
(15)	R_1	R_2	R_1	S_2
(16)	R_1	R_2	R_1	R_2

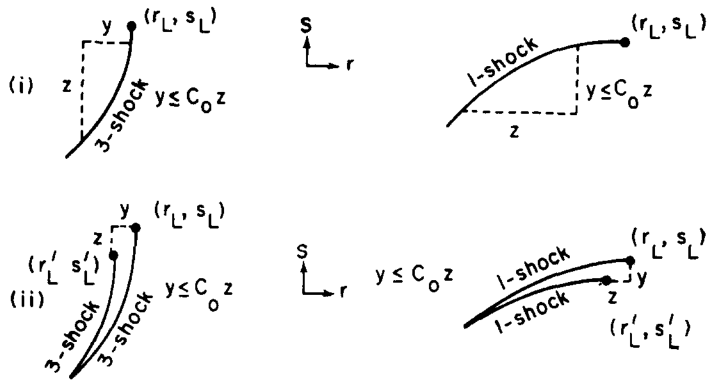

FIGURE 1

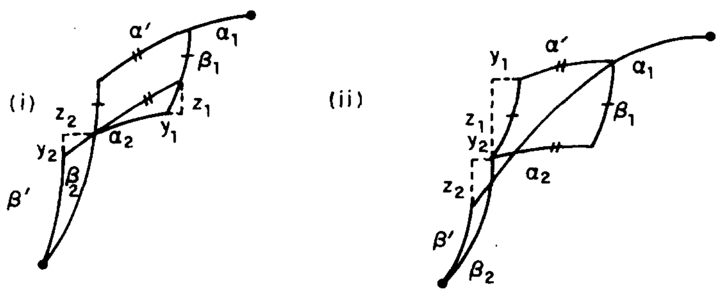

FIGURE 2

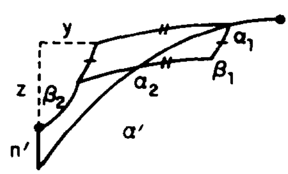

FIGURE 3

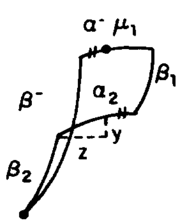

FIGURE 4

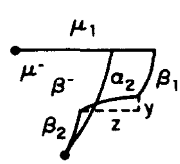

FIGURE 5

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