SHOCK WAVE INTERACTIONS IN GENERAL RELATIVITY: THE GEOMETRY BEHIND METRIC SMOOTHING AND THE EXISTENCE OF LOCALLY INERTIAL FRAMES

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Abstract. We prove a necessary and sufficient condition for determining the essential smoothness of weak solutions of the Einstein equations at apparent singularities where the gravitational metric tensor is only Lipschitz continuous, but the curvature tensor is in L^∞, a regularity so low that locally inertial frames might not exist. Namely, we prove that the question whether there exists a coordinate transformation which smooths a metric from $C^{0,1}$ to $C^{1,1}$ in a neighborhood of a point is equivalent to the condition that the singular part of the metric connection can be extended to a Riemann flat connection in that neighborhood. This applies to shock-wave solutions in General Relativity, and the framework leads to a definition of the "curvature" of the singular part of a connection, (i.e., the curvature of the shock-set), and we prove that this vanishes if and only if the spacetime metric can be smoothed within the $C^{1,1}$ atlas. As an application of our method we prove that locally inertial frames always exist in a natural sense for shock wave metrics in spherically symmetric spacetimes, a new regularity result for the Glimm scheme, independent of whether the metric itself can be smoothed in a neighborhood.

1. Introduction

We prove a necessary and sufficient condition for determining the essential smoothness of gravitational metric tensors at apparent singularities where the the components are only Lipschitz continuous, but the curvature tensor is in L^∞. In particular, this applies at points of arbitrarily complex shock wave interaction in General Relativity (GR). The theory applies in n-dimensions, without assuming any spacetime symmetries. This establishes the space of L^∞ connections with (weak) Riemann curvature tensor also in L^∞, a space closed under $C^{1,1}$ coordinate transformations, as the natural framework for shock wave theory in GR. An application provides an explicit construction procedure and proof that locally inertial frames exist in

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a natural sense at points of arbitrary shock wave interaction in spherically symmetric spacetimes when the gravitational metric is only Lipschitz continuous. This establishes that the $C^{0,1}$ shock wave metrics generated by the Glimm scheme in [8], are locally inertial at every point, independent of the inherent regularity of the metric.[1]

It is well known that shock waves form in solutions of the Einstein-Euler equations, the equations that couple the spacetime geometry to perfect fluid sources, whenever the flow is sufficiently compressive [12, 17, 5]. But it is an open question as to the essential level of smoothness of the gravitational metric for general shock wave solutions admitting points of shock wave interaction. The existence theory [8] for shock waves in GR based on the Glimm scheme, (see also [2]), only yields Lipschitz continuity of the spacetime metric, a metric regularity too low to guarantee the existence of locally inertial coordinates within the atlas of *smooth* (C^2) coordinate transformations [16]. That spacetime is locally inertial at each point p, (i.e., there exist coordinate systems in which the metric is Minkowski at p, and all coordinate derivatives of the metric vanish at p), was Einstein's starting assumption for General Relativity, [6]. The requisite smoothness of the metric sufficient to guarantee the existence of locally inertial frames within the *smooth* atlas, is the metric regularity $C^{1,1}$, one degree smoother than the $C^{0,1}$ metrics constructed in [8]. In this case the Riemann normal coordinate construction generates a smooth transformation to locally inertial coordinates. In [15], the authors proposed the possibility that shock wave interaction might create a new kind of spacetime singularity which we named *regularity singularities*, a point in spacetime where the Lorentzian metric cannot be smoothed to $C^{1,1}$, and hence fails to admit locally inertial coordinates within the smooth atlas.

However, like other singularities in GR, such as the event horizon of the Schwarzschild spacetime, a singularity requires a *singular* coordinate transformation to regularize it. Thus the possibility remains that the spacetime metric at shock waves might be smoothed from $C^{0,1}$ to $C^{1,1}$ within the larger atlas of less regular $C^{1,1}$ coordinate transformations, because these transformations introduce jumps in the derivatives of the Jacobian which hold the potential to eliminate the jumps in metric derivatives. It remains an outstanding open problem as to whether such transformations exist to smooth the metric to $C^{1,1}$ at points of shock wave interaction in GR. If such smoothing transformations do not exist, then regularity singularities can be created by shock wave interaction alone. In particular, this would imply new scattering effects in gravitational radiation, [16].

The starting point for addressing this basic regularity question for GR shock waves is Israel's celebrated 1966 paper [10], which proves that, for

[1]The space $C^{0,1}$ denotes the space of Lipschitz continuous functions, and $C^{1,1}$ the space of functions with Lipschitz continuous derivatives. A function is bounded in $C^{0,1}$ if and only if the function and its weak derivatives are bounded in L^∞, c.f. [7], Chapter 5.8.

any smooth co-dimension one shock surface in n-dimensions, the gravitational metric can always be smoothed from $C^{0,1}$ to $C^{1,1}$ by transformation to Gaussian normal coordinates adjusted to the shock surface. This transformation was identified as an element of the $C^{1,1}$ atlas in [18]. However, these coordinates are only defined for single, non-interacting shock surfaces and do not exist for the more complicated $C^{0,1}$ metrics constructed in the Groah-Temple framework [8]. The only result going beyond Israel's result was accomplished in [16, 14], where authors proved that the gravitational metric *can* always be smoothed from $C^{0,1}$ to $C^{1,1}$ at a point of regular shock wave interaction between shocks from different characteristic families, in spherically symmetric spacetimes. The proof is based on a surprisingly complicated new constructive method based on analyzing non-local PDE's tailored to the structure of the shock-wave interaction. It is not clear whether or how this proof could be extended to more complicated interactions. For more complicated shock wave interactions in spherically symmetric spacetimes, and general asymmetric shock interactions in $(3+1)$-dimensions, the question as to the locally flat nature of space-time, or whether regularity singularities can be created by shock wave interactions, remains an open problem.

The atlas of $C^{1,1}$ coordinate transformations was introduced in [18] as the natural atlas for shock wave metrics with $C^{0,1}$ regularity in GR, because $C^{1,1}$ coordinate transformations preserve the Lipschitz continuity of the metric, and map bounded discontinuous curvature tensors to bounded discontinuous curvature tensors. For perfect fluids, shock waves are weak solutions of the Einstein-Euler equations, $G = \kappa T$ coupled with $Div\, T = 0$, where G is the Einstein tensor, T is the energy-momentum tensor for a perfect fluid, and κ is the coupling constant, c.f. [3, 20]. At shock waves, T is discontinuous and contains no delta function sources, the latter distinguishing shock waves from surface layers. The Einstein equations then imply that the curvature tensor G must also be free of delta function sources at shock waves, [10]. Since G contains second derivatives of the gravitational metric g, it follows that all delta function sources in the second derivatives of g must cancel out to make G bounded and discontinuous at the shocks. The results in [18] prove that this cancellation of delta function sources is a covariant property within the $C^{1,1}$ atlas, and is imposed by requiring that the gravitational metric satisfy $G = \kappa T$, in the weak sense, as L^∞ functions, c.f. Theorem 3.1. To rule out delta function sources in G, it is sufficient to assume the Riemann curvature tensor is bounded in L^∞.

In this paper we address the question of regularity singularities at the more general level of L^∞ symmetric connections with L^∞ bounded Riemann curvature tensors on n-dimensional manifolds. We consider the general problem of when the components of such connections can be raised from L^∞ to $C0,1$ within the $C^{1,1}$ atlas. This is a general covariant framework in which to address regularity singularities in GR, including the case of connections of bounded variation and metric connections, and by Corollary 2.2 below, there is no loss of generality in assuming the entire Riemann curvature tensor, not just the Einstein tensor G, is bounded in L^∞. For metric connections the problem is equivalent to raising the metric regularity from $C^{0,1}$ to $C^{1,1}$ within the $C^{1,1}$ atlas, because for metrics, Γ^k_{ij} are given by the Christoffel symbols which are homogeneous in first order derivatives of the metric. The authors' work in [16] indicated that the problem of the existence of locally inertial frames might be independent from the problem of the essential $C^{0,1}$ regularity of the connection. To make this distinction precise, we defined in [16] a *regularity singularity* to be a point p where the connection is essentially less smooth than $C^{0,1}$ in the sense that there does not exist a $C^{1,1}$ coordinate transformation in a neighborhood of p that smooths the connection to $C^{0,1}$ in that neighborhood. Independently, we say the L^∞ connection is *locally inertial* at p, if there exists a coordinate system within the $C^{1,1}$ atlas in which the connection vanishes at p, and is Lipschitz continuous just at p. Thus, locally inertial coordinates could exist at p even though the essential smoothness of the metric is less than $C^{0,1}$. Based on this, we say a regularity singularity at a point p is *weak* if there exists a locally inertial coordinate system at p, and *strong* if the connection does not admit locally inertial coordinates at p. For example, at a weak regularity singularity in GR, locally inertial coordinates would exist at p, but the metric smoothness remains below $C^{1,1}$, too low for many desirable properties to hold, (e.g. the Penrose-Hawking-Ellis Singularity theorems [9]).[2]

Our first main result, Theorem 2.1, provides a necessary and sufficient condition for a point p to be a regularity singularity of an L^∞ connection. It states that an L^∞ connection Γ can be transformed to a $C^{0,1}$ connection within the $C^{1,1}$ atlas in a neighborhood of p if and only if there exists a flat connection in a neighborhood of p with the same singular (discontinuous) part as Γ. This reduces the problem of smoothing a connection to the problem of extending the singular part of the connection to a flat connection in the spirit of the Nash embedding theorem, c.f. [13].

As an application of our method of proof, we show that strong regularity singularities do not exist in spherically symmetric spacetimes, recorded below in Theorem 2.6. This establishes a sense in which the GR shock wave solutions constructed by Glimm's method in [8] are always regular enough to admit locally inertial frames, but at points of shock wave interaction more complicated than those in [16, 14], we still do not know whether solutions constructed by the Glimm scheme have the requisite $C^{1,1}$ regularity that Israel established in general for smooth shock surfaces. In summary, it remains an open question as to whether weak regularity singularities exist in

[2]See [1, 4, 11] for results on lower regularity solutions of the vacuum Einstein equations, a setting that rules out shock-waves.

spherically symmetric spacetimes, and whether strong regularity singularities exist in general spacetimes.

Theorem 2.1 leads us to a natural definition of the *curvature of the singular part* of an L^∞ connection. Essentially, we define this to be the minimum of the L^∞-norm of the Riemann curvature of all connections $\hat{\Gamma}$ that differ from Γ by a Lipschitz tensor, so that the jumps of $\hat{\Gamma}$ agree with the jumps of Γ on the singular set. Theorem 2.5 below then states that the singular set of an L^∞ connection has zero curvature in a neighborhood of a point if and only if the connection can be smoothed to $C^{0,1}$ in a neighborhood of the point.

The main step in proving Theorem 2.1 is to extend the classical result, that Riemann flat connections are Euclidean, to connections having only L^∞ regularity, c.f. Theorem 2.3 below. To prove the theorem we need to define n-independent 1-forms starting with their restrictions to the coordinate axes, sets too small in measure for L^∞ functions to have a meaningful restriction. To handle these restrictions under the zero mollification limit, we introduce the "peeling property"in Lemma 4.6. We conclude that space of L^∞ connections with L^∞ Riemann curvature tensor provides a natural general setting in which to investigate the existence of regularity singularities.

2. Statement of Results

We now introduce the framework and state our main results. Let $\mathcal{M}$ be an n-dimensional manifold endowed with a symmetric connection Γ bounded in L^∞, so that in each coordinate system of the $C^{1,1}$ atlas, the components of Γ are L^∞ functions. Since we are interested in a local theory, assume Γ is given in a fixed coordinate system x^i defined in a neighborhood $\mathcal{U}$ of a point p, and assume that in x-coordinates the connection components Γ^k_{ij} satisfy

$$\|\Gamma\|_\infty \equiv \max_{k,i,j} \|\Gamma^k_{ij}\|_{L^\infty(\mathcal{U})} \leq M_0 \tag{2.1}$$

for some constant $M_0 > 0$. Our main theorem gives necessary and sufficient conditions for when there exists a $C^{1,1}$ coordinate transformation that lifts the regularity of Γ from L^∞ to $C^{0,1}$, where $C^{0,1}$ is a regularity not preserved in general by the $C^{1,1}$ atlas. (We use the standard convention that components in x-coordinates use indices $i, j, k, ...$ while components in y-coordinates use $\alpha, \beta, \gamma, ...$)

Theorem 2.1. *Assume Γ^k_{ij} is a symmetric L^∞ connection satisfying (2.1) in x-coordinates defined in neighborhood $\mathcal{U}$ of a point $p \in \mathcal{M}$. Then there exists a $C^{1,1}$ transformation $y \circ x^{-1}$ such that in y-coordinates*

$$\Gamma^\alpha_{\beta\gamma} \in C^{0,1}$$

if and only if there exists a symmetric Lipschitz continuous $(1,2)$*-tensor* $\tilde{\Gamma}^k_{ij}$ *such that* $\hat{\Gamma}^k_{ij} \equiv \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}$ *is an* L^∞ *connection which satisfies*

$$\mathrm{Riem}(\hat{\Gamma}) = 0$$

in the L^∞ *weak sense, in a neighborhood of* p*, (c.f.* (3.3) *below). Moreover, if such a tensor* $\tilde{\Gamma}^k_{ij}$ *exists, then* $\mathrm{Riem}(\Gamma)$ *is bounded in* L^∞*, and the smoothing transformation* $y \circ x^{-1}$ *satisfies*

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \frac{\partial y^\alpha}{\partial x^k} \hat{\Gamma}^k_{ij}.$$

Because the addition of a Lipschitz tensor cannot cancel a delta function in the curvature, Theorem 2.1 immediately gives a sufficient condition for a shock wave solution of the Einstein equations to have a strong regularity singularity at p.

Corollary 2.2. *Assume* g *is a Lipschitz continuous metric which solves the Einstein equations* $G = \kappa T$ *with* $T \in L^\infty$*. If the Riemann curvature of* g *contains delta function sources at point* p*, then* g *cannot be smoothed to* $C^{1,1}$ *within the* $C^{1,1}$ *atlas in a neighborhood of* p*.*

The main step in proving Theorem 2.1 is to establish the following theorem which asserts that Riemann flat L^∞ connections are Euclidean.

Theorem 2.3. *Assume* $\hat{\Gamma}^k_{ij}$ *are the components of an* L^∞ *symmetric connection* $\hat{\Gamma}$ *in* x*-coordinates defined in a neighborhood of* $p \in \mathcal{M}$*. Then*

$$\mathrm{Riem}(\hat{\Gamma}) = 0$$

in the L^∞ *weak sense in a neighborhood of* p *if and only if there exists a* $C^{1,1}$ *transformation* $y \circ x^{-1}$ *such that, in* y*-coordinates,*

$$\hat{\Gamma}^\alpha_{\beta\gamma} \;=\; 0 \quad a.e.$$

in a neighborhood of p*. Moreover, if such a transformation exists, then there exists a constant* $C > 0$*, independent of* $\hat{\Gamma}$*, such that, in a sufficiently small neighborhood of* p*, we have*

$$\|y \circ x^{-1}\|_{C^{1,1}} \leq C \max_{k,j,i} \|\hat{\Gamma}^k_{ij}\|_\infty. \tag{2.2}$$

We interpret Theorem 2.1 in the spirit of the Nash embedding theorems [13]. Namely, since the addition of a Lipschitz tensor would not alter the discontinuous jumps across shocks which form the singular set of Γ, Theorem 2.1 states that one can smooth the connection if and only if there exists a Riemann flat L^∞ connection $\hat{\Gamma}$ which has the same jump discontinuities as the original connection Γ on the same singular set, (because $\Gamma - \hat{\Gamma} = \tilde{\Gamma}$ is a continuous function). Since $\hat{\Gamma}$ is flat, it can be interpreted as an extension of the singular part of Γ into flat space, so the open question of regularity singularities can be thought of as whether one can embed the singular part of Γ into ambient flat space without changing the jumps.

Based on Theorem 2.1, we propose a natural definition for a curvature scalar of the singular part of an L^∞ connection Γ, so that the existence of regularity singularities is equivalent to this curvature scalar being non-zero. To define a positive curvature scalar associated with the singular set of an L^∞ connection Γ, we first introduce an auxilliary positive definite Riemannian metric h on $\mathcal{M}$, (which for definiteness can be taken to be the identity in the original x-coordinates), and define the invariant scalar

$$\alpha\big[\Gamma\big] \equiv (h^{-1})^{\nu\bar{\nu}}(h^{-1})^{\lambda\bar{\lambda}}(h^{-1})^{\sigma\bar{\sigma}} h_{\mu\bar{\mu}} (R_\Gamma)^\mu{}_{\nu\lambda\sigma}(R_\Gamma)^{\bar{\mu}}{}_{\bar{\nu}\bar{\lambda}\bar{\sigma}}, \tag{2.3}$$

where R_Γ denotes the Riemann curvature tensor of Γ. If R_Γ is in L^∞, the positivity of h implies $\alpha\big[\hat{\Gamma}\big]$ is a non-negative L^∞ function which vanishes if and only if the Riemann curvature of $\hat{\Gamma}$ vanishes. Thus α is a natural covariant measure of the L^∞ norm of the curvature tensor and motivates the following definition.

Definition 2.4. *Let Γ^k_{ij} be a symmetric L^∞ connection defined in a coordinate system x in a neighborhood $\mathcal{U}$ of $p \in \mathcal{M}$ such that the Riemann curvature tensor of Γ is also in L^∞. For each $M > 0$, we define the curvature of the singular part of Γ in $\mathcal{U}$ to be*

$$\alpha_M(\Gamma) \equiv \inf\left\{ \|\alpha\big[\Gamma - \tilde{\Gamma}\big]\|_{L^\infty(\mathcal{U})} \,\middle|\, \tilde{\Gamma}^k_{ij} \in C^{0,1}(\mathcal{U}) \text{ with } \max_{i,j,k} \|\tilde{\Gamma}^k_{ij}\|_{C^{0,1}(\mathcal{U})} \leq M \right\}, \tag{2.4}$$

where $\|\tilde{\Gamma}^k_{ij}\|_{C^{0,1}(\mathcal{U})}$ denotes the $C^{0,1}$-norm of the components of $\tilde{\Gamma}$ in x- coordinates. The curvature of the singular part of Γ is then defined to be zero in $\mathcal{U}$ if there exists $M > 0$ such that $\alpha_M = 0$ in $\mathcal{U}$.

Note that $\alpha[\cdot]$ is invariant but α_M depends on the $C^{0,1}$ norm of $\tilde{\Gamma}$ in $\mathcal{U}$, and this in turn depends on the $C^{0,1}$ norm of $\tilde{\Gamma}$ in the original coordinate system x in which the connection Γ is given. The point is that, if $\alpha_M(\Gamma) = 0$, then the compactness of the ball of radius $M > 0$ in $C^{0,1}$ implies the existence of a $\tilde{\Gamma} \in C^{0,1}$ such that $\hat{\Gamma} = \Gamma - \tilde{\Gamma}$ is Riemann flat, $\mathrm{Riem}(\hat{\Gamma}) = 0$, and Theorem (2.1) now implies that there exists y-coordinates which smooth the connection. Thus, according to Theorem 2.1, if the curvature scalar $\alpha_M(\Gamma) > 0$, then it measures the obstacle to smoothing an L^∞ connection Γ by addition of a tensor $\tilde{\Gamma}$ with Lipschitz norm bounded by M. In the case $\alpha_M(\Gamma) = 0$, (2.2) implies that the Lipschitz norm of $\tilde{\Gamma}$, the $C^{1,1}$ norm of the coordinate transformation $y \circ x^{-1}$ that smooths the connection, as well as the L^∞ norm of $\hat{\Gamma}$ are all bounded by a constant times $M + \|\Gamma\|_\infty$. In other words, the non-vanishing of the curvature of the singular part of an L^∞ connection Γ is necessary and sufficient for Γ to contain a regularity singularity. This is recorded in the following theorem (see section 6 for details):

Theorem 2.5. *Let Γ^k_{ij} be an L^∞ connection, with L^∞ curvature tensor, defined in x-coordinate in a neighborhood of a point $p \in \mathcal{M}$. Then there* *exists a $C^{1,1}$ transformation $y \circ x^{-1}$ which smooths Γ to $C^{0,1}$ in a neighborhood of p if and only if the curvature of the singular part of Γ is zero in a neighborhood of p.*

Our second main result gives a constructive proof that locally inertial frames always exist in a natural sense for the $C^{0,1}$ shock wave metrics generated by the Glimm scheme in spherically symmetric spacetimes in [8], independent of whether the metric itself can be smoothed to $C^{1,1}$.

Theorem 2.6. *Let $\mathcal{M}$ be a spherically symmetric Lorentz manifold with an L^∞ metric connection Γ and Riemann curvature tensor bounded in L^∞. Then, for any point $p \in \mathcal{M}$, there exists locally inertial coordinates y at p, that can be reached within the atlas of $C^{1,1}$ coordinate transformations, in the sense that a representation of the L^∞ equivalence class of the connection $\Gamma^\alpha_{\beta\gamma}$ in y-coordinates vanishes at p and is Lipschitz continuous at p.*

In particular, the Lipschitz continuity of $\Gamma^\alpha_{\beta\gamma}$ at p is necessary and sufficient to remove the Coriolis terms introduced in [16]. Theorem 2.6 thus proves that Coriolis terms are removable and that no strong regularity singularities exist in spherically symmetric spacetimes, but it remains open as to whether the metric can always be smoothed to $C^{1,1}$ at points of shock wave interaction. Thus, the problem of whether (weak) regularity singularities can be created by the Glimm scheme is still an open question.

3. Preliminaries

We establish that the class of L^∞ connections with L^∞ curvature tensors is preserved by the atlas of $C^{1,1}$ coordinate transformations $y \circ x^{-1}$. To start, assume the components Γ^k_{ij} are given L^∞ functions in x-coordinates, and introduce the components $R^k{}_{lij}$ of the Riemann curvature tensor $\mathrm{Riem}(\Gamma)$ as distributions on the space C^∞_0 of smooth test functions with compact support. For a smooth connection Γ, the coefficients of $\mathrm{Riem}(\Gamma)$ are

$$R^k{}_{lij} \equiv Curl(\Gamma)^k{}_{lij} + [\Gamma_i, \Gamma_j]^k_l, \tag{3.1}$$

where Γ_i denotes the matrix $\Gamma_i \equiv \big(\Gamma^k_{ij}\big)_{k,j=1,\dots,n}$ and

$$Curl(\Gamma)^k{}_{lij} \equiv \Gamma^k_{l[j,i]} \equiv \Gamma^k_{lj,i} - \Gamma^k_{li,j} \quad \text{and} \quad [\Gamma_i, \Gamma_j]^k_l \equiv \Gamma^k_{i\sigma}\Gamma^\sigma_{jl} - \Gamma^k_{j\sigma}\Gamma^\sigma_{il} \tag{3.2}$$

give the "curl" and "commutator" terms, respectively, where a comma denotes differentiation with respect to x. For an L^∞ connection Γ^k_{ij}, the components of $\mathrm{Riem}(\Gamma)$ are linear functionals defined as

$$\begin{aligned} R^k{}_{lij}[\psi] &\equiv -Curl(\Gamma)^k{}_{lij}[\psi] + \int [\Gamma_i, \Gamma_j]^k_l \psi \, dx \\ &\equiv -\int \Big(\Gamma^k_{lj}\psi_{,i} - \Gamma^k_{li}\psi_{,j}\Big)\, dx + \int [\Gamma_i, \Gamma_j]^k_l \psi \, dx, \end{aligned} \tag{3.3}$$

where $\psi \in C^\infty_0(\mathcal{U})$ are test functions on some open set $\mathcal{U} \subset \mathbb{R}^n$ and dx is standard Lebesgue measure. $\mathrm{Riem}(\Gamma)$ is bounded in L^∞ if there exists L^∞

functions ${R^k}_{lij}$ such that

$$ {R^k}_{lij}[\psi] = \int {R^k}_{lij} \psi dx, $$

and in this case ${R^k}_{lij}$ denotes the L^∞ function as well as the distribution. Thus $\mathrm{Riem}(\Gamma) = 0$ if ${R^k}_{lij} = 0$ as an L^∞ function.

Recall that the Riemann curvature tensor gives the commutator for the covariant derivative ∇_j by

$$ \nabla_i \omega_j - \nabla_j \omega_j = {R^\sigma}_{lij} \omega_\sigma, $$

where the covariant derivative is given by

$$ \nabla_j \omega_k \equiv \omega_{k;j} = \frac{\partial \omega_k}{\partial x^j} - \Gamma^\sigma_{kj} \omega_\sigma. $$

The transformation law for connections

$$ \Gamma^k_{ij} = J^k_\alpha \, \Gamma^\alpha_{\beta\gamma} J^\beta_i J^\gamma_j + J^k_\alpha \, \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}, \tag{3.4} $$

shows that L^∞ connections are preserved by $C^{1,1}$ coordinate transformations $y \circ x^{-1}$, because the Jacobian $J^\alpha_i \equiv \frac{\partial y^\alpha}{\partial x^i}$ and its inverse J^i_α are then Lipschitz continuous. The next theorem, (first proven in [18]), states that the atlas of $C^{1,1}$ coordinate transformations also preserves the L^∞ property of the Riemann curvature tensor, and therefore maintains the condition that the curvature tensor be free of delta function sources:

Theorem 3.1. *Let Γ^k_{ij} be an L^∞ connection in x-coordinates with Riemann curvature tensor in L^∞ with its components represented by L^∞ functions R^i_{jkl}. Let $\Gamma^\gamma_{\alpha\beta}$ be the L^∞ connection resulting from* (3.4) *under a $C^{1,1}$ coordinate transformation $y \circ x^{-1}$. Then, the the weak Riemann curvature tensor* (3.3) *of $\Gamma^\gamma_{\alpha\beta}$ in y-coordinates is in L^∞ and its components are represented by the L^∞ functions*

$$ R^\alpha_{\beta\gamma\delta} = R^i_{jkl} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \frac{\partial x^l}{\partial y^\delta} \frac{\partial y^\alpha}{\partial x^i}. \tag{3.5} $$

To prove Theorem 3.1, observe that the Riemann curvature tensor in x-coordinates is in L^∞ if and only if its curl part is represented by L^∞ functions, which we denote by $Curl_x(\Gamma)$. Taking the weak curl of the transformed connection (3.4), we find that weak derivatives of $\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}$ all cancel out. As a consequence $Curl_x(\Gamma)$ equals $Curl_y(\Gamma)$ contracted with the Jacobian, plus terms in L^∞ resulting from derivatives of the Lipschitz Jacobian. A straightforward computation then verifies (3.5).

4. A Geometric Equivalence for Smoothing L^∞ Connections

We first give the proof of Theorem 2.1 assuming Theorem 2.3, and then give a careful proof of Theorem 2.3.

4.1. **Proof of Theorem 2.1:** Note first that the splitting of Γ_{ij}^k into a connection and a $(1,2)$-tensor is consistent with the covariant transformation law (3.4), because the difference between two connections is always a tensor, c.f. [9]. That is, assuming $\hat{\Gamma}$ transforms under a coordinate transformation $y^\alpha \circ x^{-1}$ by the transformation rule of a connection,

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \left\{ \hat{\Gamma}^i_{jk} J^j_\beta J^k_\gamma J^\alpha_i + J^\alpha_i \frac{\partial^2 x^i}{\partial y^\beta y^\gamma} \right\}, \tag{4.1}$$

and $\tilde{\Gamma}$ transforms by the transformation law of a tensor,

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \tilde{\Gamma}^i_{jk} J^j_\beta J^k_\gamma J^\alpha_i,$$

where $J^\alpha_k \equiv \frac{\partial y^\alpha}{\partial x^k}$ is the Jacobian and $J^k_\alpha \equiv \frac{\partial x^k}{\partial y^\alpha}$ its inverse. It follows that $\Gamma \equiv \tilde{\Gamma} + \hat{\Gamma}$ transforms as a connection,

$$\Gamma^\alpha_{\beta\gamma} \equiv \tilde{\Gamma}^\alpha_{\beta\gamma} + \hat{\Gamma}^\alpha_{\beta\gamma} = \tilde{\Gamma}^i_{jk} J^j_\beta J^k_\gamma J^\alpha_i + \left\{ \hat{\Gamma}^i_{jk} J^j_\beta J^k_\gamma J^\alpha_i + J^\alpha_i \frac{\partial^2 x^i}{\partial y^\beta y^\gamma} \right\}. \tag{4.2}$$

To prove the backward implication, assume there exists a splitting $\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} + \hat{\Gamma}^k_{ij}$ in a neighborhood of p with $\tilde{\Gamma}^k_{ij} \in C^{0,1}$ a $(1,2)$-tensor and with $\hat{\Gamma}^k_{ij} \in L^\infty$ a connection such that $\mathrm{Riem}(\hat{\Gamma}) = 0$. Thus assuming Theorem 2.3, $\mathrm{Riem}(\hat{\Gamma}) = 0$ implies that there exists a coordinate transformation $y \circ x^{-1}$ within the atlas of $C^{1,1}$ transformations such that in y-coordinates

$$\hat{\Gamma}^\alpha_{\beta\gamma} = 0$$

in an L^∞ almost everywhere sense in a neighborhood of p, and hence can be assumed to vanish everywhere. Thus, by (4.1) and (4.2), we have in y-coordinates that

$$\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^i_{jk} J^j_\beta J^k_\gamma J^\alpha_i \ \in C^{0,1}$$

which proves the reverse implication.

For the forward implication, assume there exists a transformation $y \circ x^{-1} \in C^{1,1}$ such that $\Gamma^\alpha_{\beta\gamma} \in C^{0,1}$ in y-coordinates in some neighborhood of p. In this case, considering

$$\Gamma^k_{ij} = \Gamma^\alpha_{\beta\gamma} J^\beta_i J^\gamma_j J^k_\alpha + J^k_\alpha \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}, \tag{4.3}$$

we define

$$\tilde{\Gamma}^k_{ij} \equiv \Gamma^\alpha_{\beta\gamma} J^\beta_j J^\gamma_i J^k_\alpha \ \in C^{0,1}$$

as the Lipschitz continuous tensor part of Γ^k_{ij} and

$$\hat{\Gamma}^k_{ij} \equiv J^k_\alpha \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} \tag{4.4}$$

as the L^∞ connection part. We now claim the right hand side of (4.4) is flat, satisfying $\mathrm{Riem}(\hat{\Gamma}) = 0$ in a neighborhood of p, because it is the y-coordinate representation of the zero connection in x-coordinates. This follows from Theorem 3.1 because weak L^∞ curvature tensors transform as tensors. To see this explicitly, take the curl of (4.4) in the weak sense (3.3) and observe that the third order (weak) derivatives cancel because

$$\begin{aligned} Curl(\hat{\Gamma})^k{}_{lij}[\psi] &= -\int J^k_\alpha \left(y^\alpha_{,lj}\psi_{,i} - y^\alpha_{,li}\psi_{,j} \right) dx \\ &= \int \left(J^k_{\alpha,i} y^\alpha_{,lj} - J^k_{\alpha,j} y^\alpha_{,li} \right) \psi \, dx, \end{aligned}$$

due to the symmetry in i and j. Thus, the components of the Riemann curvature tensor of (4.4) are in fact given by the L^∞ functions

$$R^k_{lij} = J^k_{\alpha,i} y^\alpha_{,lj} - J^k_{\alpha,j} y^\alpha_{,li} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}.$$

Using now that $J^\gamma_k J^k_{\alpha,i} = -J^\gamma_{k,i} J^k_\alpha = -y^\gamma_{,ki} J^k_\alpha$, it follows that

$$J^\gamma_k R^k_{lij} = -y^\gamma_{,ki} J^k_\alpha y^\alpha_{,lj} + y^\gamma_{,kj} J^k_\alpha y^\alpha_{,li} + J^\gamma_k \Gamma^k_{i\sigma}\Gamma^\sigma_{jl} - J^\gamma_k \Gamma^k_{j\sigma}\Gamma^\sigma_{il},$$

and substituting (4.4) for the remaining Γ's, the above terms mutually cancel to give $J^\gamma_k R^k_{lij} = 0$. This completes the proof of Theorem 2.1. □

4.2. **Proof of Theorem 2.3:** Assume $\hat{\Gamma}^k_{ij}$ are the components of an L^∞ connection defined in a neighborhood of $p \in \mathcal{M}$ in x-coordinates and assume $\hat{\Gamma}^k_{ij}$ satisfies the L^∞ bound

$$\|\hat{\Gamma}\|_\infty \equiv \max_{k,i,j} \|\hat{\Gamma}^k_{ij}(x)\|_{L^\infty(\mathcal{U})} < \infty. \tag{4.5}$$

The backward implication of Theorem 2.3 follows immediately from Theorem 3.1. That is, assuming there exists a $C^{1,1}$ transformation $y \circ x^{-1}$ such that

$$\hat{\Gamma}^\alpha_{\beta\gamma} \;=\; 0 \quad a.e.$$

in y-coordinates, we have $\mathrm{Riem}(\hat{\Gamma}) = 0$ in y-coordinates in the weak sense of (3.3). Since weak L^∞ curvature tensors transform as tensors by Theorem 3.1, we must have $\mathrm{Riem}(\hat{\Gamma}) = 0$ in the weak sense in all coordinates.

The remainder of this section is devoted to the proof of the forward implication of Theorem 2.3. For this, assume $\hat{\Gamma}^k_{ij}(x)$ is an L^∞ connection given on some neighborhood $\mathcal{U}$ in x-coordinates, such that $\mathrm{Riem}(\hat{\Gamma}) = 0$ in the weak L^∞ sense. For the proof we establish a framework in which the argument in [19] can be extended to the weaker setting of connections in L^∞. The argument in [19] can be summarized as follows: The zero curvature condition is used to construct four independent 1-forms $\omega^\alpha = \omega^\alpha_j dx^j$, $(\alpha = 1, ..., n)$, which are *parallel* in every direction in $x(\mathcal{U})$, i.e., $\nabla_j \omega^\alpha = 0$, $j = 1, ..., n$, where now ∇_j denotes the covariant derivative for $\hat{\Gamma}$. The parallel condition is then used to construct coordinates $y^\alpha \circ x^{-1}$ with the property that

$$\frac{\partial y^\alpha}{\partial x^j} = \omega^\alpha_j$$

The 1-forms being parallel in every direction then further implies

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \frac{\partial \omega_j^\alpha}{\partial x^i} = \hat{\Gamma}^k_{ij} \frac{\partial y^\alpha}{\partial x^k}.$$

Substituting the previous equation into the transformation law (3.4) yields

$$\hat{\Gamma}^k_{ij} = \hat{\Gamma}^k_{ij} + \hat{\Gamma}^\alpha_{\beta\gamma} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \frac{\partial x^k}{\partial y^\alpha}, \tag{4.6}$$

which implies that $\hat{\Gamma}^\alpha_{\beta\gamma} = 0$. The problem in applying this argument to low regularity L^∞ connections is that such connections do not have meaningful restrictions to low dimensional curves and surfaces along which the parallel 1-forms can be solved for. Thus the main point is that derivatives of mollified L^∞ connections do not have a meaningful zero mollification limit in general, but can be controlled in the presence of an L^∞ bound on the Riemann curvature tensor.

The main step in the proof of the forward implication of 2.3 concerns the existence of parallel 1-forms. We state this in Proposition 4.1 below, and then show how to construct the sought after coordinates y. For this, without loss of generality, we assume from here on that the coordinate neighborhood $x(\mathcal{U})$ is an n-cube given by the direct product of n intervals,

$$x(\mathcal{U}) = \mathcal{I}_1 \times \cdots \times \mathcal{I}_n, \qquad \mathcal{I}_k = (a_k, b_k), \qquad a_k < 0 < b_k.$$

Proposition 4.1. *Assume $\hat{\Gamma}$ is a symmetric connection with* $\mathrm{Riem}(\hat{\Gamma}) = 0$ *with x-components $\hat{\Gamma}^k_{ij}(x)$, $x \in x(\mathcal{U})$, satisfying* (4.5). *Then there exists n linearly independent 1-forms $\omega^\alpha = \omega^\alpha_i dx^i$, $\alpha = 1, ..., n$, with components $\omega^\alpha_i(x)$ Lipschitz continuous in x, such that the 1-forms are parallel in the L^1 sense*

$$\|\nabla_j \omega^\alpha\|_{L^1(x(\mathcal{U}))} = 0, \qquad \forall j = 1, ..., n, \tag{4.7}$$

for ∇_j the covariant derivative of $\hat{\Gamma}$. Moreover, there exists a constant $C > 0$, independent of $\hat{\Gamma}$, such that, in a sufficiently small neighborhood of p, we have

$$\|\omega^\alpha\|_{C^{0,1}} \leq C \, \|\hat{\Gamma}\|_\infty. \tag{4.8}$$

We first assume Proposition 4.1 and prove that Proposition 4.1 implies Theorem 2.3. We then give the proof of Proposition 4.1 in Section 4.3. Assuming Proposition 4.1 holds, a simple computation shows that the 1-forms ω^α being L^1-parallel in the sense of (4.7) implies they are exact almost everywhere. Namely, (4.7) implies

$$\frac{\partial}{\partial x^i} \omega^\alpha_j = \frac{\partial}{\partial x^j} \omega^\alpha_i, \qquad \text{a.e. in } x(\mathcal{U}), \tag{4.9}$$

so that

$$d\omega^\alpha = \omega^\alpha_{i,j} \, dx^i \wedge dx^j = 0,$$

point-wise *almost* everywhere in $x(\mathcal{U})$. To prove the 1-forms ω^α are closed, we need to establish Poincare's Lemma for Lipschitz continuous 1-forms.

For this, we first prove the following corollary of Proposition 4.1 stating that (4.9) make sense on lower dimensional subspaces, which is required to integrate the 1-forms up to a full coordinate system y^α. We introduce for each $l = 1, ..., n$ the subsets

$$\Omega_l \equiv \mathcal{I}_1 \times \cdots \times \mathcal{I}_l.$$

Lemma 4.2. *For each of the* 1*-forms* w^α *of Proposition 4.1 there exists numbers* $x_0^j \in \mathcal{I}_j$ *such that*

$$\int_{\Omega_l} \big|\partial_i \omega_j^\alpha - \partial_j \omega_i^\alpha\big|(x^1, ..., x^l, x_0^{l+1}, ..., x_0^n) dx^1 \cdots dx^l \;=\; 0, \qquad \forall i,j \leq l, \tag{4.10}$$

for each $l = 1, ..., n$*, where* $d\mu_l = dx^1 \cdots dx^l$ *denotes Lebesgue measure on* Ω_l *and where* ω^α *are the restrictions of the original* ω^α *to the* l*-dimensional cube* Ω_l*.*

Note that the derivatives of the Lipschitz continuous 1-forms ω^α in (4.10) are only taken in directions parallel to the restrictions Ω_l and are thus well-defined in $L^\infty(\Omega_l)$, since restrictions of $C^{0,1}$ functions are still Lipschitz continuous.

Proof. Consider a fixed $\alpha \in \{1, ..., n\}$. Equation (4.7) tells us that $\partial_j \omega_i^\alpha - \hat{\Gamma}^k_{ij} w_k^\alpha$ vanishes in $L^1(x(\mathcal{U}))$, so that the symmetry of the connection $\hat{\Gamma}^k_{ij} = \hat{\Gamma}^k_{ji}$ implies

$$\big\|\partial_j \omega_i^\alpha - \partial_i \omega_j^\alpha\big\|_{L^1(x(\mathcal{U}))} = 0. \tag{4.11}$$

It follows from Fubini's Theorem that there exists some $x_0^n \in \mathcal{I}_n$ such that

$$\int_{\Omega_{n-1}} \big|\partial_j \omega_i^\alpha - \partial_i \omega_j^\alpha\big| (x^1, ..., x^{n-1}, x_0^n) dx^1 \cdots dx^{n-1} = 0, \qquad \forall i,j \leq n-1,$$

because, if the previous integral were non-zero for all $x_0^n \in \mathcal{I}_n$, then (4.11) would be non-zero. Since the restrictions of the ω^α's are again Lipschitz continuous, we can apply this argument iteratively to obtain $x_0^l \in \mathcal{I}_l$ such that (4.10) holds for each $l = 1, ..., n$. □

From here on we assume, without loss of generality (by shifting the origin of the x-coordinates suitably), that

$$x_0^j = 0, \qquad j = 1, ..., n.$$

We now prove Poincare's Lemma for Lipschitz continuous 1-forms.

Lemma 4.3. *Let* ω^α*,* $\alpha = 1, ..., n$*, be* n *independent* 1*-forms with Lipschitz continuous components in* x*-coordinates which are exact in the sense that (4.10) holds. Then there exist functions* $y^\alpha \circ x^{-1}$ *in the atlas of* $C^{1,1}$ *coordinate transformations which satisfy*

$$\frac{\partial y^\alpha}{\partial x^j} = \omega_j^\alpha, \tag{4.12}$$

for each $j = 1, ..., n$*,* $\alpha = 1, ..., n$*.*

Proof. To prove Lemma 4.3, define the coordinates

$$y^\alpha(x^1,...,x^n) \;\equiv\; \sum_{i=1}^n \int_0^{x^i} \omega_i^\alpha(x^1,...,x^{i-1},t,0,...,0)dt \qquad (4.13)$$
$$= \int_0^{x^n} \omega_n^\alpha(x^1,...,x^{n-1},t)dt \;+ ... + \int_0^{x^1} \omega_1^\alpha(t,0,...,0)dt.$$

It follows immediately that $y^\alpha \in C^{0,1}(x(\mathcal{U}))$, and thus the sought after identity (4.12) holds for differentiation in $j=n$ direction.

We first give the argument for verifying (4.12) in cases $j \neq n$, when the ω^α are assumed smooth. In this case, differentiating (4.13) gives

$$\frac{\partial y^\alpha}{\partial x^j}(x^1,...,x^n) = \sum_{i=j+1}^n \int_0^{x^i} \frac{\partial \omega_i^\alpha}{\partial x^j}(x^1,...,x^{i-1},t,0,...,0)dt + \omega_j^\alpha(x^1,...,x^j,0,...,0) \qquad (4.14)$$

and using that for smooth ω^α (4.10) implies $\partial_j\omega_i^\alpha = \partial_i\omega_j^\alpha$ point-wise, we obtain

$$\frac{\partial y^\alpha}{\partial x^j}(x^1,...,x^n) = \sum_{i=j+1}^n \int_0^{x^i} \frac{\partial \omega_j^\alpha}{\partial x^i}(x^1,...,x^{i-1},t,0,...,0)dt + \omega_j^\alpha(x^1,...,x^j,0,...,0). \qquad (4.15)$$

Then from the Fundamental Theorem of Calculus, observing that boundary terms of the integrals mutually cancel, we find the sought after identity

$$\frac{\partial y^\alpha}{\partial x^j}(x^1,...,x^n) = \omega_j^\alpha(x^1,...,x^n). \qquad (4.16)$$

To adapt the argument (4.14) - (4.16) to the Lipschitz continuity of ω^α, we have to overcome the obstacle that $\partial_j\omega_i^\alpha = \partial_i\omega_j^\alpha$ only holds on the special surfaces identified in Lemma 4.2. For this, we first integrate equation (4.14) and then use that (4.10) of Lemma 4.2 implies

$$\partial_j\omega_i^\alpha(x^1,...,x^l,0,...,0) = \partial_i\omega_j^\alpha(x^1,...,x^l,0,...,0) \qquad \forall i,j \leq l, \qquad (4.17)$$

almost everywhere in the l-dimensional spaces Ω_l, for each $l=1,...,n$. We introduce for notational convenience

$$[f]_{L^1(\Omega_l)} \equiv \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_l} f(x^1,...,x^l)dx^1\cdots dx^l$$

for $f \in L^1(\Omega_l)$. Differentiating (4.13), we obtain again (4.14) because the Lebesgue dominated convergence Theorem applies to our Lipschitz continuous 1-forms. Integrating (4.14) now yields

$$\left[\frac{\partial y^\alpha}{\partial x^j}\right]_{L^1(\Omega_n)} = \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_n} \sum_{i=j+1}^n \int_0^{x^i} \frac{\partial \omega_i^\alpha}{\partial x^j}(x^1,...,x^{i-1},t,0,...,0)dt\, dx^1\cdots dx^n$$
$$+ \left[\omega_j^\alpha(\cdot,0,...,0)\right]_{L^1(\Omega_j)} \cdot vol(\mathcal{I}_{j+1}\times ... \times \mathcal{I}_n).$$

Defining for each $i = j+1,...,n$

$$I_i \equiv \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_{i-1}} \int_0^{x^i} \frac{\partial \omega_i^\alpha}{\partial x^j}(x^1,...,x^{i-1},t,0,...,0)dt dx^1...dx^{i-1}$$

and applying Fubini's Theorem, we get

$$\left[\frac{\partial y^\alpha}{\partial x^j}\right]_{L^1(\Omega_n)} = \sum_{i=j+1}^{n} \cdot vol(\mathcal{I}_{i+1} \times ... \times \mathcal{I}_n) \int_{\mathcal{I}_i} I_i \, dx^i + \big[\omega_j^\alpha(\cdot,0,...,0)\big]_{L^1(\Omega_j)} \cdot vol(\mathcal{I}_{j+1} \times ... \times \mathcal{I}_n). \quad (4.18)$$

We use now (4.17) to commute derivatives with indices of ω^α on each of the subspaces Ω_i, and this yields for each $i = j+1,...,n$

$$I_i = \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_{i-1}} \int_0^{x^i} \frac{\partial \omega_j^\alpha}{\partial x^i}(x^1,...,x^{i-1},t,0,...,0)dt dx^1...dx^{i-1}.$$

Apply the Fundamental Theorem of Calculus, we get

$$I_i = \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_{i-1}} \big[\omega_j^\alpha(x^1,...,x^{i-1}, \ \cdot \ ,0,...,0)\big]_0^{x^i} dx^1...dx^{i-1}.$$

Substituting this expression for I_i back into (4.18), we obtain for each $i = j+1,...,n$

$$\left[\frac{\partial y^\alpha}{\partial x^j}\right]_{L^1(\Omega_n)} = \sum_{i=j+1}^{n} \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_n} \big[\omega_j^\alpha(x^1,...,x^{i-1}, \ \cdot \ ,0,...,0)\big]_0^{x^i} dx^1...dx^n + \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_n} \omega_j^\alpha(x^1,...,x^j,0,...,0)dx^1...dx^n.$$

As in the step from (4.15) - (4.16), the boundary terms cancel in the above summation, leading to

$$\int_{\mathcal{I}_1} ... \int_{\mathcal{I}_n} \frac{\partial y^\alpha}{\partial x^j} dx^1 \cdots dx^n = \int_{\mathcal{I}_1} ... \int_{\mathcal{I}_n} \omega_j^\alpha(x^1,...,x^n)dx^1...dx^n. \quad (4.19)$$

Now, the argument (4.18) - (4.19) also holds for integration over subsets of the form $\tilde{\mathcal{I}}_1 \times ... \times \tilde{\mathcal{I}}_n$ where $\tilde{\mathcal{I}}_j \subset \mathcal{I}_j$, for each $j = 1,...,n$, so that (4.19) implies

$$\left\| \frac{\partial y^\alpha}{\partial x^j} - \omega_j^\alpha \right\|_{L^1(\mathcal{I}_1 \times ... \times \mathcal{I}_n)} = 0.$$

It follows that (4.12) holds almost everywhere in $x(\mathcal{U})$, that is,

$$\frac{\partial y^\alpha}{\partial x^j} = \omega_j^\alpha \qquad \text{a.e. in } \ x(\mathcal{U}).$$

Since the j-th weak derivative of y^α is given by the $C^{0,1}$ function ω_j^α for each $j = 1,...,n$, we conclude that $y^\alpha \in C^{1,1}\big(x(\mathcal{U})\big)$ and $\frac{\partial y^\alpha}{\partial x^j} = \omega_j^\alpha$ pointwise *everywhere* in $\mathcal{U}$, which proves (4.12). The independence of the 1-forms ω^α, for $\alpha = 1,...,n$, implies $x \mapsto y$ to be an invertible map on $x(\mathcal{U})$, with non-singular Jacobian $\frac{\partial y^\alpha}{\partial x^i} \in C^{0,1}(\mathcal{U})$. Thus the coordinate transformation $y \circ x^{-1}$

lies within the atlas of $C^{1,1}$ coordinate transformations. This completes the proof of Lemma 4.3. □

Proof of Theorem 2.3: We now complete the forward implication in the proof of Theorem 2.3 assuming Proposition 4.1 is true. Observe that the $C^{1,1}$ estimate (2.2) of Theorem 2.3 is an immediate consequence of definition (4.13) together with the Lipschitz estimate (4.8).

In the y-coordinates defined by (4.13), the components of $\hat\Gamma$ are given by

$$\hat\Gamma^k_{ij} = \frac{\partial^2 y^\sigma}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial y^\sigma} + \hat\Gamma^\gamma_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial x^k}{\partial y^\gamma} \tag{4.20}$$

almost everywhere in $x(\mathcal{U})$. But by (4.12) we have $\frac{\partial y^\alpha}{\partial x^j} = \omega^\alpha_j$ almost everywhere in $x(\mathcal{U})$, so that the ω^α being parallel in the sense of (4.7) now implies that

$$\frac{\partial^2 y^\sigma}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i}\omega^\sigma_j = \hat\Gamma^l_{ij}\omega^\sigma_l = \hat\Gamma^l_{ij}\frac{\partial y^\sigma}{\partial x^l} \tag{4.21}$$

almost everywhere in $x(\mathcal{U})$. Substituting (4.21) into (4.20) gives

$$\hat\Gamma^k_{ij} = \hat\Gamma^k_{ij} + \hat\Gamma^\gamma_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial x^k}{\partial y^\gamma}, \tag{4.22}$$

which together with the Jacobian $\frac{\partial y^\alpha}{\partial x^i}$ being non-singular implies

$$\hat\Gamma^\gamma_{\alpha\beta} = 0, \qquad \text{a.e. in } x(\mathcal{U}).$$

This completes the proof of Theorem 2.3 once we prove Proposition 4.1. □

4.3. **Proof of Proposition 4.1:** Assume $\hat\Gamma^k_{ij}(x)$ is an L^∞ connection given on some neighborhood $\mathcal{U}$ in x-coordinates, such that $\mathrm{Riem}(\hat\Gamma) = 0$ in the weak L^∞ sense. We construct n linearly independent 1-forms $\omega^\alpha = \omega^\alpha_i\, dx^i$ which are Lipschitz continuous and parallel in the sense of (4.7). Our strategy is to mollify the connection, and modify the standard argument for constructing parallel 1-forms when the curvature is zero and the connection is smooth. The mollified connection, however, has nonzero curvature, so we must keep track of errors in ϵ sufficiently to prove the curvatures tend to zero in L^1 when taking the zero mollification limit at the end. The basic L^1 estimates for this are established in Lemmas 4.4 and 4.5 below. The main technicality is that, as in the proof of Lemma 4.3, the construction requires integrating on lower dimensional surfaces, and the boundary terms arising on these surfaces must also cancel due to zero curvature in the zero mollification limit. In order to achieve this, we need a peeling property analogous to Lemma 4.2 to ensure that the curvature actually vanishes in the zero mollification limit on these lower dimensional sets. This is accomplished in Lemma 4.6 below. Unlike the case of Poincare Lemma, we need to mollify in order to apply uniqueness theorems for the ODE's arising from parallel transport.

To start, consider a standard mollification of $\hat{\Gamma}^k_{ij}(x)$,

$$(\hat{\Gamma}_\epsilon)^k_{ij}(x) = \int_{x(\mathcal{U})} \hat{\Gamma}^k_{ij}(\tilde{x})\phi_\epsilon(x-\tilde{x})d\tilde{x}, \tag{4.23}$$

with a mollifier $\phi \in C^\infty_0\big(x(\mathcal{U})\big)$, again assuming fixed coordinates x^i on $x(\mathcal{U})$. Then $\hat{\Gamma}_\epsilon \in C^\infty$ and $\|\hat{\Gamma}_\epsilon - \hat{\Gamma}\|_{L^1(x(\mathcal{U}))}$ converges to zero as $\epsilon \to 0$. Moreover,

$$|\hat{\Gamma}_\epsilon(x)| \leq \|\hat{\Gamma}\|_{L^\infty} \int |\phi_\epsilon(x-\tilde{x})|\, d\tilde{x} = \|\hat{\Gamma}\|_{L^\infty},$$

so that the L^∞ bound on $\hat{\Gamma}$, (4.5), implies

$$\|(\hat{\Gamma}_\epsilon)^k_{ij}\|_{L^\infty} \leq \|\hat{\Gamma}^k_{ij}\|_{L^\infty} \leq \|\hat{\Gamma}\|_\infty, \tag{4.24}$$

for $\|\hat{\Gamma}\|_\infty$ the ϵ independent constant in (4.5). To construct the 1-forms $\omega^\alpha = \omega^\alpha_i\, dx^i$, we establish the following three lemmas regarding the curvature $\text{Riem}(\hat{\Gamma}_\epsilon)$, assuming $\hat{\Gamma} \in L^\infty$.

Lemma 4.4. *Assume* $\text{Riem}(\hat{\Gamma})$ *is bounded in* L^∞. *Then the mollified curvature satisfies the* ϵ*-independent bound*

$$\big\|\text{Riem}(\hat{\Gamma}_\epsilon)\big\|_{L^\infty(x(\mathcal{U}))} \leq c\big\|\hat{\Gamma}\big\|^2_{L^\infty(x(\mathcal{U}))} + \big\|\text{Riem}(\hat{\Gamma})\big\|_{L^\infty(x(\mathcal{U}))}, \tag{4.25}$$

where c *is a combinatorial constant depending only on* n.

Proof. Recall that the Riemann curvature tensor can be written as a curl plus a commutator,

$$\text{Riem}(\hat{\Gamma}) \;\; \equiv \;\; Curl(\hat{\Gamma}) + [\hat{\Gamma}, \hat{\Gamma}], \tag{4.26}$$

c.f. (3.1) - (3.2). For the mollified "curl-part" of the curvature, observe that

$$\begin{aligned} Curl_x(\hat{\Gamma}_\epsilon)^k{}_{lij} &= \frac{\partial}{\partial x^j}(\hat{\Gamma}_\epsilon)^k_{li}(x) - \frac{\partial}{\partial x^i}(\hat{\Gamma}_\epsilon)^k_{lj}(x) \\ &= \int \Big(\hat{\Gamma}^k_{li}(\tilde{x})\frac{\partial}{\partial x^j}\phi_\epsilon(x-\tilde{x}) - \hat{\Gamma}^k_{lj}(\tilde{x})\frac{\partial}{\partial x^i}\phi_\epsilon(x-\tilde{x})\Big)d\tilde{x} \\ &= -\int \Big(\hat{\Gamma}^k_{li}(\tilde{x})\frac{\partial}{\partial \tilde{x}^j}\phi_\epsilon(x-\tilde{x}) - \hat{\Gamma}^k_{lj}(\tilde{x})\frac{\partial}{\partial \tilde{x}^i}\phi_\epsilon(x-\tilde{x})\Big)d\tilde{x} \end{aligned}$$

which is the weak curl of $\hat{\Gamma}$. Because $\text{Riem}(\hat{\Gamma})$ is assumed to be in L^∞, we conclude that there exists L^∞ functions that represent the curl of $\hat{\Gamma}$, since the commutator part in (3.1) contains no derivatives of $\hat{\Gamma}$. Denoting this L^∞ function by $Curl(\hat{\Gamma}) \in L^\infty$, the previous equations imply

$$Curl_x(\hat{\Gamma}_\epsilon) = \int Curl_{\tilde{x}}(\hat{\Gamma})\phi_\epsilon(x-\tilde{x})\, d\tilde{x}. \tag{4.27}$$

Using now the splitting (4.26) we write (4.27) as

$$Curl_x(\hat{\Gamma}_\epsilon)(x) \;\; = \;\; -\int \big([\hat{\Gamma},\hat{\Gamma}] - \text{Riem}(\hat{\Gamma})\big)(\tilde{x})\; \phi_\epsilon(x-\tilde{x})d\tilde{x}$$

from which we conclude that

$$\begin{aligned}\big\|Curl(\hat\Gamma_\epsilon)\big\|_{L^\infty} &\le \big\|[\hat\Gamma,\hat\Gamma]\big\|_{L^\infty} + \big\|\mathrm{Riem}(\hat\Gamma)\big\|_{L^\infty} \\ &\le c\big\|\hat\Gamma\big\|^2_{L^\infty} + \big\|\mathrm{Riem}(\hat\Gamma)\big\|_{L^\infty}, \end{aligned} \tag{4.28}$$

for some combinatorial constant c. Using the splitting (4.26) for $\hat\Gamma_\epsilon$, (4.28) implies that

$$\begin{aligned}\big\|\mathrm{Riem}(\hat\Gamma_\epsilon)\big\|_{L^\infty} &\le \big\|Curl(\hat\Gamma_\epsilon)\big\|_{L^\infty} + \big\|[\hat\Gamma,\hat\Gamma]\big\|_{L^\infty} \\ &\le c\big\|\hat\Gamma\big\|^2_{L^\infty} + \big\|\mathrm{Riem}(\hat\Gamma)\big\|_{L^\infty}, \end{aligned}$$

which gives the sought after ϵ independent bound (4.25) and proves Lemma 4.4. □

Lemma 4.5. *Assume* $\mathrm{Riem}(\hat\Gamma)$ *is bounded in* L^∞. *Then* $\mathrm{Riem}(\hat\Gamma_\epsilon)$ *converges to* $\mathrm{Riem}(\hat\Gamma)$ *in* $L^1(x(\mathcal{U}))$ *as* $\epsilon \to 0$.

Proof. Multiplying each component of $\mathrm{Riem}(\hat\Gamma_\epsilon)$, $(R_\epsilon)^k{}_{lij}$, by a test-function $\psi \in C^\infty_0\big(x(\mathcal{U})\big)$ and integrating over $x(\mathcal{U})$, we find

$$\begin{aligned}\int_{x(\mathcal{U})} (R_\epsilon)^k{}_{lij}\psi dx &= \int (\hat\Gamma_\epsilon)^k{}_{l[j,i]}\psi dx + \int_{x(\mathcal{U})} [(\hat\Gamma_\epsilon)_i,(\hat\Gamma_\epsilon)_j]^k_l\, \psi\, dx \\ &= -\int_{x(\mathcal{U})} \Big((\hat\Gamma_\epsilon)^k{}_{lj}\psi_{,i} - (\hat\Gamma_\epsilon)^k{}_{li}\psi_{,j}\Big)\, dx + \int_{x(\mathcal{U})} [(\hat\Gamma_\epsilon)_i,(\hat\Gamma_\epsilon)_j]^k_l\, \psi\, dx.\end{aligned}$$

Now, as ϵ approaches zero, $(\hat\Gamma_\epsilon)^k{}_{ij} \to \hat\Gamma^k{}_{ij}$ in L^1, so taking this limit in the last line of the previous equation and using that the test functions and their derivatives are bounded, we conclude

$$\lim_{\epsilon\to 0}\int (R_\epsilon)^k{}_{lij}\psi dx \;= -\int \Big(\hat\Gamma^k{}_{lj}\psi_{,i} - \hat\Gamma^k{}_{li}\psi_{,j}\Big)\, dx + \int [\hat\Gamma_i,\hat\Gamma_j]^k_l\, \psi\, dx \;= R^k{}_{lij}[\psi],$$

where we used the weak form of the Riemann curvature (3.3) in the last line. This proves Lemma 4.5. □

The following lemma establishes the L^1-*peeling property* which is crucial for the construction of parallel one-forms (4.7), because it allows us to assign initial conditions consistently, analogous to the initial conditions in Lemma 4.2.

Lemma 4.6. *Assume* $\mathrm{Riem}(\hat\Gamma) = 0$. *For every sequence* $\epsilon \to 0$ *there exists a subsequence* ϵ_k *(with* $\epsilon_k \to 0$ *as* $k \to \infty$*) and some point* $(\bar x^1,...,\bar x^n) \in \mathcal{I}_1\times...\times\mathcal{I}_n$ *such that the mollified curvature satisfies the* L^1 peeling property *at* $\bar x \equiv (\bar x^1,...,\bar x^n)$*, by which we mean that for each* $m = 1,...,n$*,*

$$\lim_{\epsilon_k\to 0}\int_{\mathcal{I}_1}...\int_{\mathcal{I}_m} (R_{\epsilon_k})^k{}_{lij}(x^1,...,x^m,\bar x^{m+1},...,\bar x^n)dx^1\cdots dx^m = 0, \tag{4.29}$$

that is,

$$\big\|(R_{\epsilon_k})^k{}_{lij}(\cdot,...,\cdot,\bar x^{m+1},...,\bar x^n)\big\|_{L^1(\mathcal{I}_1\times...\times\mathcal{I}_m)} \longrightarrow 0 \qquad as\;\; \epsilon_k \to 0.$$

Proof. Define $\tilde{x} \equiv (x^1, ..., x^m) \in \mathcal{I}_1 \times ... \times \mathcal{I}_m$ and $\bar{x} \equiv (x^{m+1}, ..., x^n) \in \mathcal{I}_{m+1} \times ... \times \mathcal{I}_n$. Fubini's Theorem implies that

$$(\bar{R}_\epsilon)^k{}_{lij}(\bar{x}) \equiv \int_{\mathcal{I}_1 \times ... \times \mathcal{I}_m} (R_\epsilon)^k{}_{lij}(\tilde{x}, \bar{x}) d\tilde{x}$$

is an integrable function over $\mathcal{I}_{m+1} \times ... \times \mathcal{I}_n$. Since, $\mathrm{Riem}(\hat{\Gamma}_\epsilon)$ converges to zero in $L^1(x(\mathcal{U}))$ by Lemma 4.5, it follows that $(\bar{R}_\epsilon)^k{}_{lij}$ converges to zero in $L^1(\mathcal{I}_{m+1} \times ... \times \mathcal{I}_n)$, namely

$$\int_{\mathcal{I}_{m+1} \times ... \times \mathcal{I}_n} (\bar{R}_\epsilon)^k{}_{lij}(\bar{x}) d\bar{x} = \int_{\mathcal{U}} (R_\epsilon)^k{}_{lij}\, dx \longrightarrow 0, \quad \text{as } \ \epsilon \to 0.$$

Therefore, there exists a subsequence ϵ_k^m (with $\epsilon_k^m \to 0$ as $k \to \infty$) and some point $\bar{x} \in \mathcal{I}_{m+1} \times ... \times \mathcal{I}_n$ at which $(\bar{R}_{\epsilon_k^m})^k{}_{lij}(\cdot)$ converges to zero as $k \to \infty$. For this point $\bar{x}$, it follows that $(R_{\epsilon_k^m})^k{}_{lij}(\cdot, \bar{x})$ converges to 0 in $L^1(\mathcal{I}_1 \times ... \times \mathcal{I}_m \times \{\bar{x}\})$ as $k \to \infty$.

Now, applying this construction with respect to $\mathcal{I}_1 \times ... \times \mathcal{I}_n$, we first find a point $\bar{x}^n \in \mathcal{I}_n$ together with a subsequence ϵ_k^{n-1} of ϵ such that

$$(R_{\epsilon_k^{n-1}})^k{}_{lij}(\,\cdot\,, \bar{x}^n) \longrightarrow 0, \qquad \text{in } L^1(\mathcal{I}_1 \times ... \times \mathcal{I}_{n-1}), \quad \text{as } \ k \to \infty.$$

Given this convergence on the $n-1$ sub-cube $\mathcal{I}_1 \times ... \times \mathcal{I}_{n-1} \times \{\bar{x}^n\}$, we again apply the above construction (but now with respect to the sub-cube) to obtain a point $\bar{x}^{n-1} \in \mathcal{I}_{n-1}$ and a subsequence ϵ_k^{n-2} of ϵ_k^{n-1} such that

$$(R_{\epsilon_k^{n-2}})^k{}_{lij}(\,\cdot\,, \bar{x}^{n-1}, \bar{x}^n) \longrightarrow 0, \qquad \text{in } L^1(\mathcal{I}_1 \times ... \times \mathcal{I}_{n-2}), \quad \text{as } \ k \to \infty.$$

Continuing, we successively find a subsequence ϵ_k of ϵ and a point $(\bar{x}^1, ..., \bar{x}^n) \in x(\mathcal{U})$ at which the peeling property (4.29) holds. This proves Lemma 4.6. □

Our goal now is to construct n linearly independent 1-forms $\omega_\epsilon^\alpha = (\omega_\epsilon^\alpha)_i\, dx^i$, $\alpha = 1, ..., n$, of the mollified connections $\hat{\Gamma}_\epsilon$ by parallel translating in x-coordinate directions $\mathbf{e}_1, ..., \mathbf{e}_n$, one direction at a time, starting with initial data given at a point $\bar{x}$ where the peeling property holds to control the L^1-norms of the curvature on the initial data. The resulting 1-forms $\omega_\epsilon^\alpha = (\omega_\epsilon^\alpha)_i\, dx^i$, will not be parallel in every direction because the curvature of the mollified connections is in general nonzero. However, since the Riemann curvature converges to zero in L^1 as $\epsilon \to 0$, one can prove that the ω_ϵ^α tend to parallel 1-forms in the limit $\epsilon \to 0$, once their convergence in $C^{0,1}$ has been established. Concerning this convergence, the uniform L^∞ bound on the curvature alone will imply that the resulting 1-forms are Lipschitz continuous *uniformly* in ϵ, so that the Arzela-Ascoli Theorem yields a convergent subsequence of the 1-forms ω_ϵ^α that converge to Lipschitz continuous 1-forms $\omega_i^1 dx^i, ..., \omega_i^n dx^i$ as $\epsilon \to 0$.

To begin the construction of the parallel 1-forms, assume a sequence $\epsilon \to 0$ such that the curvature satisfies the peeling property (4.29) at the point $\bar{x} = (\bar{x}^1, ..., \bar{x}^n)$. Assume without loss of generality that $\bar{x} = (0, ..., 0)$, and $\mathcal{I}_k = (-1, 1) \equiv \mathcal{I}$ for each $k = 1, ..., n$. We begin with the construction of

1-forms on the two-surface $\mathcal{I}_1 \times \mathcal{I}_2 \times \{\bar{x}^3\} \times ... \times \{\bar{x}^n\}$ which are parallel in the x^2-direction, and then extend to $(x^1, ..., x^n) \in x(\mathcal{U})$ by induction, in the following four steps:

Step (i): First solve for 1-forms $\omega_\epsilon^\alpha = (\omega_\epsilon^\alpha)_j dx^j$, for $\alpha = 1, ..., n$, parallel along the x^1-axis. I.e., solve the ODE initial value problem

$$\nabla_1^\epsilon (\omega_\epsilon^\alpha)_j (x^1, 0) \equiv \frac{\partial (\omega_\epsilon^\alpha)_j}{\partial x^1}(x^1, 0) - \left(\hat{\Gamma}_\epsilon\right)^k_{1j} (\omega_\epsilon^\alpha)_k (x^1, 0) = 0, \quad (4.30)$$

$$(\omega_\epsilon^\alpha)(0, 0) = \mathbf{e}^\alpha, \quad (4.31)$$

where we suppress the dependence on $(\bar{x}^3, ..., \bar{x}^n) = (0, ..., 0)$, which are fixed. Since we seek independent 1-forms, we choose the initial data for the 1-forms at the point $(0, 0)$ to be the n-independent coordinate co-vectors $\mathbf{e}^\alpha \equiv dx^\alpha$. For the construction it suffices to keep α fixed and, for ease of notation in Steps (i)-(iv), we write $\omega^\epsilon \equiv \omega_i^\epsilon dx^i$ instead of $\omega_\epsilon^\alpha \equiv (\omega_\epsilon^\alpha)_i dx^i$.

Taking $t = x^1$, (4.30)-(4.31) is an initial value problem for an ODE of the form

$$\dot{u} + A_\epsilon u = 0,$$

where $u(t) = (\omega_1^\epsilon(t, 0), ..., \omega_n^\epsilon(t, 0)) \in \mathbb{R}^n$, and $(A_\epsilon)_j^k(t) = \left(\hat{\Gamma}_\epsilon\right)^k_{1j}(t, 0)$ is an $n \times n$ matrix which is smooth and bounded in the L^∞ norm, uniformly in ϵ, by (4.24). Thus the Picard-Lindelöff existence theorem for ODE's implies there exists a unique smooth solution $u(t) = \omega^\epsilon(t, 0)$. Moreover, the Grönwall inequality together with the L^∞ bound on A_ϵ imply the resulting 1-forms $\omega^\epsilon(x^1, 0) \equiv \omega^\epsilon(x^1, \bar{x}^2, ..., \bar{x}^n) = \omega^\epsilon(x^1, 0, ..., 0)$ are bounded, uniformly in ϵ, which then yields Lipschitz continuity in the x^1-direction, uniformly in ϵ.

Step (ii): Given the $\omega^\epsilon(x^1, 0)$ from Step (i), assume for simplicity $\omega^\epsilon(x^1, 0)$ exists for all $x^1 \in (-1, 1)$, use $\omega^\epsilon(x^1, 0)$ as initial data to solve for the parallel transport in the x^2-direction starting from $x^2 = 0$, by solving the ODE initial value problem

$$\nabla_2^\epsilon \omega_j^\epsilon(x^1, x^2) \equiv \frac{\partial \omega_j^\epsilon}{\partial x^2}(x^1, x^2) - \left(\hat{\Gamma}_\epsilon\right)^k_{2j} \omega_k^\epsilon(x^1, x^2) = 0, \quad (4.32)$$

$$\omega^\epsilon(x^1, x^2) = \omega^\epsilon(x^1, 0) \quad \text{at} \quad x^2 = 0. \quad (4.33)$$

For fixed $x^1 \in (-1, 1)$, taking $t = x^2$, (4.32)-(4.33) is an initial value problem for an ODE of the form

$$\dot{u} + A_\epsilon u = 0, \quad (4.34)$$

with $u(t) = \omega^\epsilon(x^1, t) \in \mathbb{R}^n$ and $(A_\epsilon)_j^k(t) = \left(\hat{\Gamma}_\epsilon\right)^k_{2j}(x^1, t)$ an $n \times n$ matrix which is smooth and bounded in the supnorm uniformly in ϵ, according to (4.24). The Picard-Lindelöff theorem implies that there exists a unique smooth solution $\omega^\epsilon(x^1, t)$, and for ease we again assume $\omega^\epsilon(x^1, t)$ to be defined throughout the interval $-1 < t < 1$ for each $x^1 \in (-1, 1)$. The

ϵ-independent supnorm bound $\|A_\epsilon\|_{L^\infty} \leq \|\hat{\Gamma}\|_\infty$ on $\mathcal{I}_1 \times \mathcal{I}_2$, together with the Grönwall inequality for (4.34), imply the supnorm bound

$$\|\omega^\epsilon\|_{L^\infty(\mathcal{I}_1\times\mathcal{I}_2)} \leq K_0, \tag{4.35}$$

where we use K_0 to denote a universal constant depending only on $\hat{\Gamma}$, independent of ϵ, with $K_0 = o(\|\hat{\Gamma}\|_\infty$ as $\|\hat{\Gamma} \to 0$. Moreover, $\|A_\epsilon\|_{L^\infty} \leq \|\hat{\Gamma}\|_\infty$ implies the $\omega^\epsilon(x^1,x^2) \equiv \omega^\epsilon(x^1,x^2,0,...,0)$ satisfies a Lipschitz bound in the x^2-direction,

$$\left\|\frac{\partial\omega^\epsilon}{\partial x^2}\right\|_{L^\infty(\mathcal{I}_1\times\mathcal{I}_2)} \leq K_0, \tag{4.36}$$

the right hand side of (4.36) again a constant depending on $\hat{\Gamma}$, independent of ϵ. However, we still need to control the Lipschitz bound (4.36) in the x^1-direction. This is accomplished in Step (iii).

Step (iii): In (ii) we constructed $\omega^\epsilon(x^1,x^2)$ parallel in x^2 for each fixed $x^1 \in \mathcal{I}_1$, (i.e., $\nabla^\epsilon_2\omega^\epsilon \equiv \omega^\epsilon_{;2} = 0$), which gives Lipschitz continuity in x^2-direction uniform in ϵ and x^1. To get Lipschitz continuity in x^1-direction uniform in ϵ and x^2, we estimate the change of $u \equiv \nabla^\epsilon_1\omega^\epsilon$ in x^2-direction, starting from $x^2 = 0$ where $u \equiv \nabla^\epsilon_1\omega^\epsilon(x^1,0) = 0$ by construction.

To obtain the equation for the change in x^2-direction, use the definition of curvature to write

$$\nabla^\epsilon_2\left[\omega^\epsilon_{k;1}\right] \;=\; \nabla^\epsilon_1\nabla^\epsilon_2\omega^\epsilon_k + (R_\epsilon)^\sigma_{k21}\,\omega^\epsilon_\sigma, \tag{4.37}$$

so the definition of covariant derivative gives

$$\nabla^\epsilon_1\nabla^\epsilon_2\omega^\epsilon_k = \frac{\partial}{\partial x^1}\left[\omega^\epsilon_{k;2}\right] - (\hat{\Gamma}_\epsilon)^\sigma_{1k}\omega_{\sigma;2} - (\hat{\Gamma}_\epsilon)^\sigma_{12}\omega^\epsilon_{k;\sigma}\ .$$

Substituting this into (4.37) using $\omega_{\sigma;2} = 0$, we conclude that (4.37) is equivalent to

$$\nabla^\epsilon_2\left[\omega^\epsilon_{k;1}\right] + (\hat{\Gamma}_\epsilon)^\sigma_{12}\omega^\epsilon_{k;\sigma} - (R_\epsilon)^\sigma_{k21}\,\omega^\epsilon_\sigma = 0. \tag{4.38}$$

On the other hand, the definition of ∇^ϵ_2 gives

$$\nabla^\epsilon_2\left[\omega^\epsilon_{k;1}\right] = \frac{\partial}{\partial x^2}\left[\omega^\epsilon_{k;1}\right] - (\hat{\Gamma}_\epsilon)^\sigma_{2k}\omega^\epsilon_{\sigma;1} - (\hat{\Gamma}_\epsilon)^\sigma_{21}\omega^\epsilon_{k;\sigma}\ . \tag{4.39}$$

Putting (4.39) into (4.38) and canceling the common term gives the ODE for $u \equiv \omega^\epsilon_{k;1}$,

$$\frac{\partial}{\partial x^2}\left[\omega^\epsilon_{k;1}\right] - (\hat{\Gamma}_\epsilon)^\sigma_{2k}\omega^\epsilon_{\sigma;1} - (R_\epsilon)^\sigma_{k21}\,\omega^\epsilon_\sigma = 0. \tag{4.40}$$

Thus, for fixed x^1, letting $t = x^2$ and

$$u_k(t) \equiv \omega^\epsilon_{k;1}(x^1,t), \quad k = 1,...,n,$$

the x^2-directional change of u is determined by the system of ODE's

$$\dot{u} + A_\epsilon u + B_\epsilon = 0, \tag{4.41}$$

where $u = (u_1, ..., u_n)$, and the $n \times n$-matrix A_ϵ as well as the n-vector B_ϵ are given by

$$(A_\epsilon)_k^\sigma = -\left(\hat{\Gamma}_\epsilon\right)_{2k}^\sigma \qquad \text{and} \qquad (B_\epsilon)_k = -\left(R_\epsilon\right)_{k21}^\sigma \omega_\sigma^\epsilon.$$

In addition to the ϵ-independent bound on A_ϵ, we have by Lemma 4.4 and (4.35) that

$$\|B_\epsilon\|_{L^\infty} \le \|R_\epsilon\|_{L^\infty} \|\omega_\sigma^\epsilon\|_{L^\infty} \le K_0, \tag{4.42}$$

where again K_0 denotes a constant independent of ϵ, and the L^∞-norm is taken on $\mathcal{I}_1 \times \mathcal{I}_2$. To apply the Grönwall inequality, write (4.41) equivalently as the integral equation

$$u(t) = -\int_0^t B_\epsilon(s) ds - \int_0^t A_\epsilon(s) u(s) ds,$$

where we use that the initial condition is zero at $x^2 = 0$, because parallel translation at $x^2 = 0$ holds for all $\epsilon > 0$. Taking the point-wise Euclidean norm $|\cdot|$ gives

$$|u|(t) \le \varphi(t) + \int_0^t |A_\epsilon|(s)|u|(s) ds, \qquad \text{for} \qquad \varphi(t) \equiv \int_0^t |B_\epsilon|(s) ds,$$

where $|A_\epsilon|$ denotes the corresponding operator norm. Thus, the Grönwall inequality gives the estimate

$$|\omega_{k;1}^\epsilon|(t) \equiv \ |u|(t) \ \le \varphi(t) + \int_0^t \varphi(s) \ |A_\epsilon|(s) \ \exp\Big(\int_s^t |A_\epsilon|(\tilde{s}) d\tilde{s}\Big) ds,$$

which together with the bound on A_ϵ yields

$$|\omega_{k;1}^\epsilon|(t) \le K_0 \int_0^t |B_\epsilon| dt, \tag{4.43}$$

for $K_0 > 0$ independent of ϵ. Estimate (4.43) and (4.42) together with the definition of the covariant derivative, $\omega_{i;1}^\epsilon = \omega_{i,1}^\epsilon - \left(\hat{\Gamma}_\epsilon\right)_{i1}^\sigma \omega_\sigma^\epsilon$, implies the supnorm of the derivative $\frac{\partial}{\partial x^1}\omega^\epsilon$ is bounded uniformly in ϵ by

$$\Big|\frac{\partial \omega_k^\epsilon}{\partial x^1}\Big|(t) \le K \int_0^t |B_\epsilon| dt + \big|(\hat{\Gamma}_\epsilon)_{k1}^\sigma w_\sigma^\epsilon\big| \le K_0. \tag{4.44}$$

We conclude that the components $\omega_i^\epsilon(x^1, x^2, 0, ..., 0)$ are Lipschitz continuous in (x^1, x^2), uniformly in ϵ.

Step (iv): We now implement the induction step from $m-1$ to m with $m \le n$, which now requires controlling $m-1$ commutators of covariant derivative. The step $m = 3$ is essentially different from $m = 2$ because it is at this step that, for example, $\nabla_1 \omega$ does not vanish on the initial data surface $\mathcal{I}_1 \times \mathcal{I}_2$. This is the obstacle to constructing locally inertial frames for $n \ge 3$ in the next section.

For the induction assumption, let $\omega^\epsilon(x^1,...,x^{m-1},0,...,0)$ be the 1-form in $C^\infty(\mathcal{I}_1\times...\times\mathcal{I}_{m-1})$ which generalizes the construction in Steps (i) - (ii) as follows: We assume the parallel transport condition,

$$\nabla^\epsilon_k\omega^\epsilon(x^1,...,x^k,0,...,0)=0,\qquad \forall k\le m-1, \tag{4.45}$$

and we assume the Lipschitz norm of ω^ϵ to be bounded uniformly in ϵ analogously to (4.43). That is, for each $l\le m-1$ we assume

$$\big\|\omega^\epsilon\big\|_{L^\infty(\Omega_l)}\le K_0,$$

where again K_0 denotes a constant $K_0=o(\|\hat\Gamma\|_\infty)$, independent of ϵ, and we assume that

$$\begin{aligned}|\omega^\epsilon_{k;j}|&(x^1,...,x^{m-1},0,...,0)\\ &\le K_0\sum_{l=1}^{m-1}\left|\int_0^{x^l}\big|\,(R_\epsilon)^\sigma_{klj}\,\omega^\epsilon_\sigma\big|(x^1,...,x^{l-1},t,0,...,0)\,dt\right|.\end{aligned}\tag{4.46}$$

Note that (4.46) together with the curvature bound from Lemma 4.4 imply the ϵ-independent bound

$$\|\omega^\epsilon\|_{C^{0,1}(\Omega_{m-1})}\le K_0,$$

where $\Omega_l\equiv\mathcal{I}_1\times...\times\mathcal{I}_l\times\{0\}\times...\times\{0\}$ for $l=1,...,n$ and

$$\|\omega^\epsilon\|_{C^{0,1}(\Omega_l)}\equiv\big\|\omega^\epsilon\big\|_{L^\infty(\Omega_l)}+\sum_{l=1}^{m-1}\big\|\partial_l\omega^\epsilon\big\|_{L^\infty(\Omega_l)}.$$

The induction step now is to prove that there exists a 1-form $\omega^\epsilon\in C^\infty(\Omega_m)$ which agrees with ω^ϵ when $x^m=0$ and satisfies the Lipschitz bound (4.46) on Ω_m for some constant $K_0>0$ independent of ϵ, such that for each $k\le m$ the parallel transport condition (4.45) holds. (For ease, we assume that $\mathcal{I}_l=(-1,1)$ for each $l=1,...,n$.) As in Step (ii), we extend ω^ϵ from Ω_{m-1} to Ω_m by solving the ODE for parallel transport in x^m-direction,

$$\nabla^\epsilon_m\hat\omega^\epsilon(x^1...,x^m,0,...,0)=0 \tag{4.47}$$

for fixed $x^1,...,x^{m-1}$ and with initial data

$$\hat\omega^\epsilon(x^1,...,x^{m-1},0,...,0)=\omega^\epsilon(x^1,...,x^{m-1},0,...,0).$$

We denote the solution of (4.47) again by $\omega^\epsilon\equiv\hat\omega^\epsilon$. Analogous to Step (ii), $\omega^\epsilon\in C^\infty(\Omega_m)$ and the parallel condition (4.45) is satisfied by construction for each $k\le m$. Moreover, the Grönwall inequality implies that ω^ϵ is sup-norm bounded over Ω_m and by (4.24) this bound is independent of ϵ. The ϵ-independent bound on $\|\partial_m\omega^\epsilon\|_{L^\infty(\Omega_m)}$ now follows from (4.47).

It remains to prove ϵ-independent bounds on $\|\partial_j\omega^\epsilon\|_{L^\infty(\Omega_m)}$ for each $j<m$ to prove the Lipschitz bound analogous to (4.46) on Ω_m. For this we prove the following Lemma.

Lemma 4.7. *The 1-forms solving (4.47) satisfy*

$$|\omega^{\epsilon}_{k;j}|(x^1,...,x^m,0,...,0) \le K_0 \sum_{l=1}^{m} \left| \int_0^{x^l} \left|(R_\epsilon)^{\sigma}_{klj}\, \omega^{\epsilon}_{\sigma}\right|(x^1,...,x^{l-1},t,0,...,0)\, dt \right|, \tag{4.48}$$

for some constant $K_0 > 0$ depending only on $\hat\Gamma$, independent of ϵ.

Proof. We proceed similarly to Step (iii) and use the definition of the curvature tensor to write for each $j < m$

$$\nabla^{\epsilon}_m \nabla^{\epsilon}_j \omega^{\epsilon}_k = \nabla^{\epsilon}_j \nabla^{\epsilon}_m \omega^{\epsilon}_k + (R_\epsilon)^{\sigma}_{kmj}\omega^{\epsilon}_{\sigma}. \tag{4.49}$$

Computing the components of the covariant derivatives in (4.49) in terms of their connection coefficients, using that $\omega^{\epsilon}_{j;m} = 0$ for all $j = 1,..,n$, we find that

$$\begin{aligned} \nabla^{\epsilon}_j \nabla^{\epsilon}_m \omega^{\epsilon}_k &= \partial_j[\omega^{\epsilon}_{k;m}] - (\hat\Gamma_\epsilon)^{\sigma}_{jk}[\omega^{\epsilon}_{\sigma;m}] - (\hat\Gamma_\epsilon)^{\sigma}_{jm}[\omega^{\epsilon}_{k;\sigma}] \\ &= -(\hat\Gamma_\epsilon)^{\sigma}_{jm}[\omega^{\epsilon}_{k;\sigma}] \end{aligned}$$

and

$$\nabla^{\epsilon}_m \nabla^{\epsilon}_j \omega^{\epsilon}_k = \partial_m[\omega^{\epsilon}_{k;j}] - (\hat\Gamma_\epsilon)^{\sigma}_{mk}[\omega^{\epsilon}_{\sigma;j}] - (\hat\Gamma_\epsilon)^{\sigma}_{mj}[\omega^{\epsilon}_{k;\sigma}].$$

Substituting the previous two identities into (4.49), we find that (4.49) is equivalent to the system of ODE's

$$\partial_m[\omega^{\epsilon}_{k;j}] - (\hat\Gamma_\epsilon)^{\sigma}_{mk}[\omega^{\epsilon}_{\sigma;j}] - (R_\epsilon)^{\sigma}_{kmj}\omega^{\epsilon}_{\sigma} = 0. \tag{4.50}$$

Applying the Grönwall inequality to the ODE (4.50) leads to the estimate

$$\begin{aligned} |\omega^{\epsilon}_{k;j}|(x^1,...,x^m,0,...,0) \;\le\; & K_0 \int_0^{x^m} |(R_\epsilon)^{\sigma}_{kmj}\omega^{\epsilon}_{\sigma}|(x^1,...,x^{m-1},t,0,...,0)dt \\ & + |\omega^{\epsilon}_{k;j}|(x^1,...,x^{m-1},0,...,0), \end{aligned} \tag{4.51}$$

where $K_0 > 0$ is independent of ϵ because of (4.24).[3] Using the induction assumption (4.46) to replace the initial data term $|\omega^{\epsilon}_{k;j}|(x^1,...,x^{m-1},0,...,0)$ on the right hand side of (4.51) gives us the sought after estimate (4.48). □

The ϵ-independent Lipschitz bound for ω^ϵ on $x(\mathcal{U}) = \mathcal{I}_1 \times ... \times \mathcal{I}_n$ now follows directly from (4.48). Namely, $\|(R_\epsilon)^{\sigma}_{kmj}\omega^{\epsilon}_{\sigma}\|_{L^\infty} \le K_0 \|\hat\Gamma\|^2_{L^\infty}$ according to Lemma 4.4 and the boundedness of $\|\omega^{\epsilon}_{\sigma}\|_{L^\infty}$ derived above. We conclude with the ϵ independent Lipschitz bound

$$\|w^\epsilon\|_{C^{0,1}(x(\mathcal{U}))} \equiv \|w^\epsilon\|_{L^\infty(x(\mathcal{U}))} + \sum_{j=1}^{n} \left\| \frac{\partial w^\epsilon}{\partial x^j} \right\|_{L^\infty(x(\mathcal{U}))} \le K_0, \tag{4.52}$$

for some constant $K_0 > 0$ depending on $\|\hat\Gamma^k_{ij}\|_\infty$, independent of ϵ. This completes the induction step and proves that the 1-forms $\omega^\epsilon(x^1,...,x^n)$ are Lipschitz continuous, uniformly in ϵ. This completes Step (iv).

[3]The difference between the Grönwall estimate in (4.51) and the one in Step (iii) is the presence of the second term on the right hand side which is due to the initial data ω^ϵ being not parallel for $\epsilon > 0$ and $j \ge 2$.

To summarize, in Steps (i) - (iv) we constructed n families of smooth 1-forms $(w_\epsilon)^\alpha_i dx^i$, (with $\alpha = 1, ..., n$), such that each component satisfies the uniform Lipschitz bound (4.52). Thus, for each $\alpha = 1, ..., n$, the Arzela-Ascoli Theorem yields a subsequence of the 1-forms $(w_\epsilon)^\alpha_i dx^i$ that converges uniformly to a Lipschitz continuous 1-form

$$(w_\epsilon)^\alpha_i \longrightarrow \omega^\alpha_i \quad \text{as} \quad \epsilon \to 0.$$

Moreover, the Lipschitz estimate (4.8) follows from (4.52) by Taylor expanding K_0 in $\|\hat{\Gamma}^k_{ij}\|_\infty$ and restricting the neighborhood sufficiently. Since for each $\alpha = 1, ..., n$ the initial data in Step (i) was chosen such that each 1-form $(w_\epsilon)^\alpha_i dx^i$ agrees with the unit co-vector $\mathbf{e}^\alpha_k dx^k = dx^\alpha$ at the point $(\bar{x}^1, ..., \bar{x}^n) = (0, ..., 0)$ for any $\epsilon > 0$, the limit 1-form $\omega^\alpha_i dx^i$ is identical to dx^α at $(\bar{x}^1, ..., \bar{x}^n)$ as well. Thus, the 1-forms $(w_\epsilon)^\alpha_i dx^i$ are linearly independent and linear independence throughout $x(\mathcal{U})$ now follows from the uniqueness of solutions of ODE's, c.f. (4.31).

To complete the proof of Proposition 4.1, it remains to prove that the limit 1-forms are parallel in every direction with respect to $\hat{\Gamma}$ in the L^1 sense of (4.7). For this, integrate the ODE estimate (4.48) for $m = n$ over $x(\mathcal{U}) = \mathcal{I}_1 \times ... \times \mathcal{I}_n \equiv \Omega_n$, to get

$$\begin{aligned} \|\omega^\epsilon_{k;j}\|_{L^1(x(\mathcal{U}))} &\leq K_0 \sum_{l=1}^n \big\|(R_\epsilon)^\sigma_{klj} \omega^\epsilon_\sigma\big\|_{L^1(\Omega_l)} \\ &\leq K_0 \sum_{l=1}^n \big\|(R_\epsilon)^\sigma_{klj}\|_{L^1(\Omega_l)} \|\omega^\epsilon_\sigma\big\|_{L^\infty(\Omega_n)} \end{aligned} \tag{4.53}$$

where $\Omega_l \equiv \mathcal{I}_1 \times ... \times \mathcal{I}_l \times \{0\} \times ... \times \{0\} \subset x(\mathcal{U})$ for $l = 1, ..., n$ and $K_0 > 0$ a universal constant independent of ϵ. Since ω^ϵ is bounded in $L^\infty(\Omega_n)$ independent of ϵ, the L^1-peeling property of the curvature (4.29) now implies that the right hand side of (4.53) converges to zero for some subsequence $\epsilon_k \to 0$. Thus each of the 1-forms ω^α is parallel in $L^1(x(\mathcal{U}))$ in every direction, as claimed in (4.7). This completes the proof of Proposition 4.1. □

5. A Construction of Locally Inertial Frames

We begin by giving the definition of locally inertial coordinates for L^∞ connections in n-dimensions:

Definition 5.1. *Let Γ be an L^∞ connection. We say a coordinate system y is locally inertial for Γ at p if the components satisfy*

$$\left|\Gamma^\alpha_{\beta\gamma}(y)\right| \leq K|y - y(p)| \quad a.e., \tag{5.1}$$

for some constant K independent of y. We say Γ is locally inertial at p if there exists a locally inertial coordinate system at p.

Condition (5.1) is equivalent to the existence of an L^∞ representation of the components $\Gamma^\alpha_{\beta\gamma}(y)$ such that (5.1) holds in the pointwise everywhere sense, and $\Gamma^\alpha_{\beta\gamma}(y(p)) = 0$. In this section we use the coordinate construction of Section 4.3 to prove that locally inertial coordinates exist for L^∞ connections in 2-dimensional manifolds when the Riemann curvature tensor of the connection is assumed bounded in L^∞. Building on this construction in 2-dimensions, we prove that locally inertial frames always exist in 4-dimensional spherically symmetric spacetimes with Lipschitz continuous metric. Thus in particular, it is sufficient to apply to the GR shock wave solutions generated by the Glimm method, [8]. Interestingly, this argument does not extend to three or more dimensions essentially because the induction step in (iv) of the proof of Proposition 4.1 at $n > 2$ differs from the $n = 2$ step by boundary terms arising from the Gronwall estimate (4.51), and these terms would not vanish in the zero mollification limit when the analogue of the peeling property was used for nonzero curvature. To construct locally inertial coordinates in 2-dimensional spacetimes, we construct 1-forms as in Steps (i) and (ii) of Proposition 4.1, (the case $\mathrm{Riem}(\Gamma) = 0$), and define coordinates y^α by integrating over these 1-forms. These 1-forms are not in general parallel, but as a consequence of the L^∞ curvature bound, we prove the 1-forms are parallel within error of order $O(|x|)$ when curvature is non-zero. This then implies that the connection is order $O(|y|)$ in coordinates y^α, the condition that y^α be locally inertial. Theorem 2.6 of the introduction follows from Theorem 5.2 and 5.5 of this section.

5.1. **Locally Inertial Frames in 2-Dimensions.** The goal of this section is to prove the following theorem:

Theorem 5.2. *Assume $\mathcal{M}$ is a two dimensional manifold endowed with a symmetric L^∞-connection with Riemann curvature tensor bounded in L^∞, and let $p \in \mathcal{M}$. Then there exists locally inertial coordinates at p within the $C^{1,1}$ atlas.*

To prove Theorem 5.2, we first recall the construction of the 1-forms ω^α in Steps (i) and (ii) of Proposition 4.1. This is sufficient to establishe 1-forms $\omega^\alpha = \omega^\alpha_k dx^k$ parallel in the x^2-direction, $\alpha = 1, 2$, but not also in the x^1-direction when the curvature of Γ is non-zero. To sketch this, let Γ_ϵ denote the mollified connection (4.23), and solve the ODE for the mollified 1-forms $(\omega_\epsilon)^\alpha$

$$\begin{aligned} \nabla_1(\omega_\epsilon)^\alpha_j(x^1,0) &\equiv \frac{\partial(\omega_\epsilon)^\alpha_j}{\partial x^1}(x^1,0) - (\Gamma_\epsilon)^k_{1j}(\omega_\epsilon)^\alpha_k(x^1,0) = 0, \\ (\omega_\epsilon)^\alpha(0,0) &= \mathbf{e}^\alpha, \end{aligned} \tag{5.2}$$

taking as initial data $\mathbf{e}^\alpha = dx^\alpha$ at the point $(x^1, x^2) = (0, 0)$, $\alpha = 1, 2$. Next take the resulting 1-forms $\omega^\alpha(x^1, 0)$ as initial data and solve the ODE initial value problem

$$\nabla_2(\omega_\epsilon)^\alpha_j(x^1,x^2) \equiv \frac{\partial(\omega_\epsilon)^\alpha_j}{\partial x^2}(x^1,x^2) - (\Gamma_\epsilon)^k_{2j}(\omega_\epsilon)^\alpha_k(x^1,x^2) = 0,$$

$$(\omega_\epsilon)^\alpha(x^1,x^2) \;=\; (\omega_\epsilon)^\alpha(x^1,0) \quad \text{at} \;\; x^2=0, \tag{5.3}$$

for each $x^1 \in (-1,1)$. Without loss of generality, we can assume the resulting 1-forms are defined for all $(x^1,x^2) \in x(\mathcal{U}) = (-1,1)\times(-1,1)$. It is straightforward then to adapt the analysis in Steps (i) - (iii) of Section 4.3 to our setting here and obtain for each $\alpha = 1,2$ a subsequence of the family of 1-forms $(\omega_\epsilon)^\alpha$ which converges to a Lipschitz continuous 1-form ω^α as $\epsilon \to 0$. By (5.3), each ω^α is parallel in x^2-direction in the L^1 sense

$$\|\nabla_2 w^\alpha\|_{L^1} = 0. \tag{5.4}$$

However, in contrast to Section 4.3, we cannot expect ω^α to be parallel in x^1-direction when $Riem(\Gamma) \neq 0$. However, as a result of the L^∞ curvature bound, the ω^α are approximately parallel in the sense of the following lemma.

Lemma 5.3. *The* 1*-forms* ω^α*, obtained from the zero-mollification limit of* (5.3)*, satisfy*

$$\Big|\frac{\partial \omega^\alpha_j}{\partial x^i} - \Gamma^k_{ij}\omega^\alpha_k\Big|(x^1,x^2) \leq K_0\big(|x^1|+|x^2|\big) \equiv O(x) \tag{5.5}$$

almost everywhere, where $K_0 > 0$ *is some constant depending only on* $\|\Gamma\|_{L^\infty}$ *and* $\|\mathrm{Riem}(\Gamma)\|_{L^\infty}$*.*

Proof. Equation (5.4) immediately implies (5.5) for $i=2$ because the right hand side of(5.4) vanishes when $i=2$. It remains only to verify (5.5) for $i=1$. For the case $i=1$, observe that the computation (4.37) - (4.40) of Step (iii) in Section 4.3, again gives that the ODE (5.3) implies

$$\frac{\partial u_k}{\partial x^2} \;=\; (\Gamma_\epsilon)^\sigma_{2k}u_\sigma + (R_\epsilon)^\sigma_{k12}(\omega_\epsilon)^\alpha_\sigma \tag{5.6}$$

for $u_k \equiv \nabla_1(\omega_\epsilon)^\alpha_k$ and where $(R_\epsilon)^\sigma_{kij}$ denotes the components of $\mathrm{Riem}(\Gamma_\epsilon)$. Applying the Grönwall inequality to (5.6) and the fact that $\nabla_1\omega^\alpha_k(x^1,0)=0$ by (5.2), we obtain

$$|\nabla_1(\omega_\epsilon)^\alpha_k|(x^1,x^2) \leq K_0\int_0^{x^2} |(R_\epsilon)^\sigma_{k12}(\omega_\epsilon)^\alpha_\sigma|(x^1,t)dt, \tag{5.7}$$

where here $K_0>0$ always denotes a generic constant depending on $\|\Gamma\|_{L^\infty}$ and $\|\mathrm{Riem}(\Gamma)\|_{L^\infty}$, but independent of ϵ. Using that (5.3) implies $\|\omega^\alpha_\sigma\|_{L^\infty} < K_0\|\Gamma\|_{L^\infty}$ together with the curvature bound (4.25), we obtain from (5.7) the further estimate

$$|\nabla_1(\omega_\epsilon)^\alpha_k|(x^1,x^2) \;\leq\; K_0\|\Gamma\|_{L^\infty}\max_{\sigma=1,2}\big\|R^\sigma_{k12}\big\|_{L^\infty}\,\big|x^2\big| \;\equiv\; K_0\big|x^2\big|. \tag{5.8}$$

Now Γ_ϵ converges in $L^1(x(\mathcal{U}))$ as $\epsilon \to 0$, so there exists a subsequence converging pointwise almost everywhere. From this pointwise convergence and the fact that $\frac{\partial}{\partial x^1}(\omega_\epsilon)^\alpha$ converges in $L^\infty(x(\mathcal{U}))$, we conclude that

$$|\nabla_1\omega^\alpha_k|(x^1,x^2) \;\leq\; K_0\big|x^2\big| \qquad a.e., \tag{5.9}$$

which is the sought after error estimate (5.5) for $i=1$. $\square$

To prove Theorem 5.2, we define for each $\alpha = 1,2$ the coordinates y^α on $x(\mathcal{U})$ by

$$y^\alpha(x^1,x^2) \equiv \int_0^{x^1} \omega_1^\alpha(s,x^2)ds + \int_0^{x^2} \omega_2^\alpha(0,s)ds. \tag{5.10}$$

and complete the proof by showing y^α are locally inertial at p. By the definition of y^α we have

$$\frac{\partial y^\alpha}{\partial x^1} = \omega_1^\alpha \qquad \text{and} \qquad \frac{\partial}{\partial x^j}\frac{\partial y^\alpha}{\partial x^1} = \frac{\partial \omega_1^\alpha}{\partial x^j}, \tag{5.11}$$

which, for $i = 1$, is the identity that leads to the cancellation in (4.6). However, we cannot obtain these identities for the x^2-derivative because the 1-forms ω^α are no longer parallel in the x^1-direction. The following approximate identities are sufficient for the existence of locally inertial frames.

Lemma 5.4. *The coordinates y^α defined in* (5.10) *satisfy for $i,j = 1,2$*

$$\left|\frac{\partial y^\alpha}{\partial x^i} - w_i^\alpha\right|(x^1,x^2) \;\leq\; K_0\big(|x^1| + |x^2|\big), \tag{5.12}$$

$$\left|\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i} - \frac{\partial \omega_i^\alpha}{\partial x^j}\right|(x^1,x^2) \;\leq\; K_0\big(|x^1| + |x^2|\big) \qquad a.e., \tag{5.13}$$

where $K_0 > 0$ is some constant depending only on $\|\Gamma\|_{L^\infty}$ and $\|\mathrm{Riem}(\Gamma)\|_{L^\infty}$.

Proof. The case $i = 1$ follows directly from (5.11). For the case $i = 2$, differentiate (5.10) in the x^2 direction to get

$$\frac{\partial y^\alpha}{\partial x^2}(x^1,x^2) = \int_0^{x^1} \frac{\partial \omega_1^\alpha}{\partial x^2}(s,x^2)ds + \omega_2^\alpha(0,x^2).$$

Using that $\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2}$ converges to $\frac{\partial \omega_1^\alpha}{\partial x^2}$ in $L^1(x(\mathcal{U}))$ as $\epsilon \to 0$, the dominated convergence theorem implies that

$$\frac{\partial y^\alpha}{\partial x^2}(x^1,x^2) = \lim_{\epsilon\to 0}\int_0^{x^1} \frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2}(s,x^2)ds \;+\; \omega_2^\alpha(0,x^2), \tag{5.14}$$

with ϵ convergence in $L^1(x(\mathcal{U}))$. Substituting

$$\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2} = \frac{\partial(\omega_\epsilon)_2^\alpha}{\partial x^1} + \left(\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2} - \frac{\partial(\omega_\epsilon)_2^\alpha}{\partial x^1}\right)$$

into (5.14) gives

$$\left(\frac{\partial y^\alpha}{\partial x^2} - \omega_2^\alpha\right)(x^1,x^2) = \lim_{\epsilon\to 0}\int_0^{x^1}\left(\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2} - \frac{\partial(\omega_\epsilon)_2^\alpha}{\partial x^1}\right)(s,x^2)ds$$

with convergence pointwise almost everywhere. Now, using that $(\omega_\epsilon)^\alpha$ is parallel in the x^2-direction, $\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2} = \Gamma_{12}^\sigma(\omega_\epsilon)_\sigma^\alpha$, we have

$$\left(\frac{\partial(\omega_\epsilon)_1^\alpha}{\partial x^2} - \frac{\partial(\omega_\epsilon)_2^\alpha}{\partial x^1}\right) = -\nabla_1(\omega_\epsilon)_2^\alpha,$$

which leads to

$$\left(\frac{\partial y^\alpha}{\partial x^2} - \omega_2^\alpha\right)(x^1, x^2) = -\lim_{\epsilon\to 0}\int_0^{x^1} \nabla_1(\omega_\epsilon)_2^\alpha(s, x^2)\, ds. \tag{5.15}$$

The Grönwall estimate (5.8) now implies

$$\left|\frac{\partial y^\alpha}{\partial x^2} - \omega_2^\alpha\right|(x^1, x^2) \leq K_0 \int_0^{x^1} |x^2| ds \quad \text{a.e.}$$

which implies the sought after Lipschitz estimate (5.12).

We now prove (5.13). By the dominated convergence theorem, we conclude that (5.15) implies

$$\left(\frac{\partial y^\alpha}{\partial x^2} - \omega_2^\alpha\right)(x^1, x^2) = -\int_0^{x^1} \nabla_1\omega_2^\alpha(s, x^2)\, ds. \tag{5.16}$$

Differentiating (5.16) in x^1-direction gives us

$$\left(\frac{\partial^2 y^\alpha}{\partial x^1 \partial x^2} - \frac{\partial \omega_2^\alpha}{\partial x^1}\right)(x^1, x^2) = -\nabla_1\omega_2^\alpha(x^1, x^2), \tag{5.17}$$

and taking the absolute value, the Grönwall estimate (5.9) gives

$$\left|\frac{\partial^2 y^\alpha}{\partial x^1 \partial x^2} - \frac{\partial \omega_2^\alpha}{\partial x^1}\right|(x^1, x^2) \leq K_0 \left|x^2\right| \;=\; O(|x|),$$

which is the sought after almost everywhere estimate (5.13) for $j = 1$ and $i = 2$.

It remains to prove (5.13) for $i = j = 2$. For this, differentiate (5.15) in the x^2 direction to obtain

$$\left(\frac{\partial^2 y^\alpha}{\partial x^2 \partial x^2} - \frac{\partial \omega_2^\alpha}{\partial x^2}\right)(x^1, x^2) = -\frac{\partial}{\partial x^2}\lim_{\epsilon\to 0}\int_0^{x^1} \nabla_1(\omega_\epsilon)_2^\alpha(s, x^2)\, ds.$$

Note that taking $\frac{\partial}{\partial x^2}$ as a derivative in the weak sense, we can exchange $\lim_{\epsilon\to 0}$ and $\frac{\partial}{\partial x^2}$ by the L^1 convergence of the integrand. Thus by (5.6),

$$\int_0^{x^1} \frac{\partial}{\partial x^2}\nabla_1(\omega_\epsilon)_2^\alpha(s, x^2)\, ds = \int_0^{x^1} \Big((\Gamma_\epsilon)_{2k}^\sigma \nabla_1(\omega_\epsilon)_\sigma^\alpha + (R_\epsilon)_{k12}^\sigma(\omega_\epsilon)_\sigma^\alpha\Big)(s, x^2)\, ds, \tag{5.18}$$

which converges uniformly in x^2 as $\epsilon \to 0$, because the right hand side is continuous in x^2 and bounded in light of the Grönwall estimate (5.8). In light of the Groenwall estimate (5.8), the integrand on the right hand side of (5.18) is in L^∞, we conclude that

$$\left|\frac{\partial^2 y^\alpha}{\partial x^2 \partial x^2} - \frac{\partial \omega_2^\alpha}{\partial x^2}\right|(x^1, x^2) \leq K_0\ |x^1|, \tag{5.19}$$

which implies the sought after bound (5.13) for $i = j = 2$. □

Proof of Theorem 5.2: We show that y^α defined in (5.10) are locally inertial at p. For this, consider the transformation law for connections

$$\Gamma^k_{ij}\frac{\partial y^\alpha}{\partial x^k} = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}. \tag{5.20}$$

Combining (5.5) and (5.13), we obtain

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \Gamma^k_{ij}\omega^\alpha_k + O(|x|).$$

Substituting the previous equation into (5.20) and using that $w^\alpha_k = \frac{\partial y^\alpha}{\partial x^k} + O(|x|)$ by (5.12), the Christoffel symbols Γ^k_{ij} cancel on both sides and we get

$$\Gamma^\alpha_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j} = O(|x|).$$

We then conclude with the sought after estimate (5.2), using that $O(|x|) = O(|y|)$ and that the Jacobians $\frac{\partial y^\beta}{\partial x^i}$ are invertible. This completes the proof of Theorem 5.2. □

Finally, it is interesting to point out what goes wrong in the pursuit of the above construction for locally inertial frames in 3-dimensions. Essentially, the analog of (5.13) does not hold in 3-dimensions. That is, defining coordinates in analogy to (5.10) leads to

$$y^\alpha(x^1,x^2,x^3) \equiv \int_0^{x^1} \omega^\alpha_1(s,x^2,x^3)ds + \int_0^{x^2} \omega^\alpha_2(0,s,x^3)ds + \int_0^{x^3} \omega^\alpha_3(0,0,s)ds, \tag{5.21}$$

and the analog of (5.11) again holds. However, since $\nabla_2(\omega_\epsilon)^\alpha_1(x^1,x^2,x^3)$ is not zero when $x_3 \neq 0$, we get in (5.15) an additional error function bounded in L^∞ which is $O(x^3)$, but whose derivative is not $O(x)$. That is, we obtain

$$\left(\frac{\partial y^\alpha}{\partial x^2} - \omega^\alpha_2\right)(x^1,x^2) = -\lim_{\epsilon\to 0}\int_0^{x^1} \nabla_1(\omega_\epsilon)^\alpha_2(s,x^2)\, ds \;+\; \int_0^{x^1} O(x^3)\, ds. \tag{5.22}$$

Thus, differentiating (5.22) in x^2 direction in order to mimic the step leading to equation (5.18) above, the derivative falls on the term $\int_0^{x^1} O(x^3)ds$, the derivative of an L^∞ function, which does not in general produce an error $O(x)$.

5.2. **Locally Inertial Frames in Spherical Symmetry.** We now extend the above constructive method to spherically symmetric spacetimes and thereby complete the proof of Theorem 2.6. That is, we assume a Lipschitz continuous metric tensor in x-coordinates taking the general spherically symmetric form

$$ds^2 = -A(x^1,x^2)\big(dx^1\big)^2 + 2E(x^1,x^2)dx^1dx^2 + B(x^1,x^2)\big(dx^2\big)^2 + C(x^1,x^2)d\Omega^2, \tag{5.23}$$

with components A, B, C and E assumed to be Lipschitz continuous functions, $-(AB+E^2) < 0$ and $C > 0$. Here $d\Omega^2 \equiv d\phi^2 + \sin(\phi)^2 d\theta^2$ is the line element on the unit sphere, $x^3 = \phi \in (0, \pi)$, $x^4 = \theta \in (-\pi, \pi)$, we assume without loss of generality that (x^1, x^2) are centered at $(0, 0)$, with $(x^1, x^2) \in (-1, 1) \times (-1, 1) \equiv \Omega_2$. (General spherically symmetric Lorentz metrics can be transformed to coordinates where the metric takes the form (5.23), [20]. Note coordinates x^i are labeled $i = 1, ..., 4$, to be consistent with earlier notation.) We prove the following theorem:

Theorem 5.5. *Assume $\mathcal{M}$ is a spherically symmetric Lorentz manifold with Lipschitz continuous metric* (5.23) *such that its metric connection Γ^k_{ij} and Riemann curvature tensor R^k_{lij} are bounded in L^∞ in coordinates (x^1, x^2, ϕ, θ). Then the metric* (5.23) *admits locally inertial coordinates at each point $p \in \Omega_2$ within the atlas of $C^{1,1}$ coordinate transformations, in the sense of Definition 5.1.*

We start with the following lemma.

Lemma 5.6. *Assume the metric* (5.23) *is Lipschitz continuous. If the Einstein tensor of the metric* (5.23) *is bounded in L^∞, then $C(x^1, x^2) \in C^{1,1}(\Omega_2)$.*

Proof. An explicit computation of the first three contravariant components of the Einstein tensor G^{11}, G^{12} and G^{22}, yields

$$\begin{aligned} \frac{\partial^2 C}{\partial x^2 \partial x^2} &= \kappa C|g|G^{11} + l.o.t., \\ \frac{\partial^2 C}{\partial x^1 \partial x^2} &= -\kappa C|g|G^{12} + l.o.t., \\ \frac{\partial^2 C}{\partial x^1 \partial x^1} &= \kappa C|g|G^{22} + l.o.t., \end{aligned} \tag{5.24}$$

where $|g| \equiv -AB - E^2$ and $l.o.t.$ denotes terms containing only zero and first order metric derivatives, (c.f. MAPLE). From this we can read off the regularity of C. Namely, when $G^{\mu\nu} \in L^\infty$ and the metric is Lipschitz continuous metric, the right hand side of (5.24) is in L^∞. Thus we conclude that second order weak derivatives of C are in L^∞, which is equivalent to $C \in C^{1,1}$, (c.f. [7]). □

In the proof of the theorem to follow, it is interesting to observe that our assumption that the curvature tensor is bounded in L^∞ comes in at two different points in the argument to imply the existence of locally inertial frames for (5.23) when the connection is only in L^∞. First, we apply Theorem 5.2 to the 2-dimensional metric

$$ds^2 = -A(x^1, x^2)\big(dx^1\big)^2 + 2E(x^1, x^2)dx^1dx^2 + B(x^1, x^2)\big(dx^2\big)^2$$

to obtain coordinates y^α in which the connection is Lipschitz continuous at the center point $p = (0, 0)$, but only for indices running from 1 to 2. An explicit computation then shows that the remaining components involving the angular indices are in fact Lipschitz continuous, one degree smoother than

L^∞, because C is the only differentiated metric component in these connection components, and C is one degree more regular than A, B and E, by Lemma 5.6. This extra degree of regularity in C is crucial because it ensures that the connection coefficients not addressed by our 2-dimensional method, must be Lipschitz continuous, as a second consequence of our assumption that the curvature tensor is bounded in L^∞. The resulting Lipschitz continuity of Γ at p in y-coordinates allows us to introduce a further smooth coordinate transformation, quadratic in y, which breaks the spherical symmetry, and sets the value of the connection to zero at the center point p, while preserving the established Lipschitz continuity at p in y-coordinates.

Proof of Theorem 5.5: For $\alpha = 1, 2$, we introduce the two 1-forms

$$\omega^\alpha = \omega^\alpha_1 dx^1 + \omega^\alpha_2 dx^2$$

as solutions of $\nabla_2 \omega^\alpha = 0$ with variables (x^1, x^2) assuming $(x^3, x^4) = (\phi_0, \theta_0)$ fixed in $(0, \pi) \times (-\pi, \pi)$. That is, ω^1 and ω^2 are solutions of

$$\frac{\partial \omega^\alpha_j}{\partial x^2}(x^1, x^2) - \Gamma^k_{2j}\omega^\alpha_k(x^1, x^2) = 0, \tag{5.25}$$

for initial data $\omega^\alpha_j(x^1, 0, \phi_0, \theta_0)$ at $x^2 = 0$ with $\nabla_1 \omega^\alpha_j(x^1, 0, \phi_0, \theta_0) = 0$ for $j = 1, 2$, c.f. (5.3). Since the angular dependence is kept fixed in (5.25), estimate (5.5) of Lemma 5.3 holds again for both ω^α, that is,

$$\frac{\partial \omega^\alpha_j}{\partial x^i} = \Gamma^k_{ij}\big|_{(\phi_0,\theta_0)} \omega^\alpha_k + O(|x^1| + |x^2|), \tag{5.26}$$

for $\alpha = 1, 2$, where $\Gamma^k_{ij}\big|_{(\phi_0,\theta_0)}$ denotes Γ^k_{ij} evaluated at fixed $(\phi, \theta) = (\phi_0, \theta_0)$.

Similar to (5.10), we define the function y^α for $\alpha = 1, 2$ as

$$y^\alpha(x^1, x^2, \phi, \theta) \equiv \int_0^{x^1} \omega^\alpha_1(s, x^2, \phi_0, \theta_0)ds + \int_0^{x^2} \omega^\alpha_2(0, s, \phi_0, \theta_0)ds, \tag{5.27}$$

(the right hand side evaluated at fixed values of the angular variables), and set $y^3 = \phi$, $y^4 = \theta$. Thus estimates (5.12) and (5.13) hold,

$$\begin{aligned} \frac{\partial y^\alpha}{\partial x^k} &= w^\alpha_k + O(|x^1| + |x^2|), \\ \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} &= \frac{\partial \omega^\alpha_j}{\partial x^i} + O(|x^1| + |x^2|), \end{aligned} \tag{5.28}$$

for $i, j, k = 1, 2$ and $\alpha = 1, 2$, and where the right hand side is evaluated at the fixed angular values (ϕ_0, θ_0). Combining estimates (5.26) and (5.28), we obtain for $\alpha = 1, 2$

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \Gamma^k_{ij}\big|_{(\phi_0,\theta_0)} \frac{\partial y^\alpha}{\partial x^k} + O(|x^1| + |x^2|), \quad i, j = 1, 2. \tag{5.29}$$

To complete the proof, consider again the transformation

$$\frac{\partial y^\alpha}{\partial x^k}\Gamma^k_{ij} = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}. \tag{5.30}$$

Substituting (5.29) we obtain

$$\Gamma^{\alpha}_{\beta\gamma}\frac{\partial y^{\beta}}{\partial x^{i}}\frac{\partial y^{\gamma}}{\partial x^{j}} = \Big(\Gamma^{k}_{ij} - \Gamma^{k}_{ij}\big|_{(\phi_0,\theta_0)}\Big)\frac{\partial y^{\alpha}}{\partial x^{k}} + O(|x^1|+|x^2|). \tag{5.31}$$

Using now that the metric and its inverse are smooth in ϕ and θ, we can Taylor expand Γ^k_{ij} around (ϕ_0,θ_0) to obtain

$$\Gamma^{k}_{ij} - \Gamma^{k}_{ij}\big|_{(\phi_0,\theta_0)} = O(|\phi-\phi_0|+|\theta-\theta_0|)$$

for $i,j=1,2$. Thus, since the Jacobian $\frac{\partial y^\alpha}{\partial x^k}$ is invertible and since $\frac{\partial y^\alpha}{\partial x^k}=0$ for $\alpha=1,2$ and $k=3,4$, we can write (5.31) as

$$\begin{aligned}\Gamma^{\alpha}_{\beta\gamma} &= O(|x^1|+|x^2|+|\phi-\phi_0|+|\theta-\theta_0|)\\ &= O(|y^1|+|y^2|+|\phi-\phi_0|+|\theta-\theta_0|), \qquad \alpha,\beta,\gamma=1,2.\end{aligned} \tag{5.32}$$

Keeping in mind that $y^1(0,0)=0=y^2(0,0)$ and that $y^3=\phi$ and $y^4=\theta$, this is the desired Lipschitz estimate for $\alpha,\beta,\gamma=1,2$.

We now derive a Lipschitz estimate of the form (5.32) for the cases when α,β or $\gamma\neq 1,2$. The transformation to the coordinates y^α defined in (5.27), preserves the spherically symmetric form of the metric representation (5.23). We denote the metric in coordinates y^α by

$$ds^2 = -A(y^1,y^2)(dy^1)^2+2E(y^1,y^2)dy^1dy^2+B(y^1,y^2)(dy^2)^2+C(y^1,y^2)d\Omega^2 \tag{5.33}$$

for Lipschitz continuous metric components A,B,C,E, generally different from the components in (5.23). Computing the Christoffel symbols of (5.33), we find that the non-zero connection coefficient not subject to the Lipschitz estimate (5.32) are given by

$$\begin{aligned}\Gamma^1_{33} &= \frac{B\dot{C}-EC'}{2(AB+E^2)}, \qquad \Gamma^1_{44}=(\sin\phi)^2\Gamma^1_{33},\\ \Gamma^2_{33} &= \frac{-E\dot{C}-AC'}{2(AB+E^2)}, \qquad \Gamma^2_{44}=(\sin\phi)^2\Gamma^2_{33},\\ \Gamma^3_{13} &= \frac{\dot{C}}{2C}, \qquad \Gamma^3_{23}=\frac{C'}{2C}, \qquad \Gamma^3_{44}=-\sin\phi\cos\phi,\\ \Gamma^4_{14} &= \frac{\dot{C}}{2C}, \qquad \Gamma^4_{24}=\frac{C'}{2C}, \qquad \Gamma^3_{44}=\frac{\cos\phi}{\sin\phi},\end{aligned} \tag{5.34}$$

where $\dot{C}\equiv\frac{\partial C}{\partial y^1}$ and $C'\equiv\frac{\partial C}{\partial y^2}$. Observe that we only differentiate C in the above coefficient components but we never differentiate A, B or E. Since C is $C^{1,1}$ regular by Lemma 5.6, it follows that the components in (5.34) are Lipschitz continuous (as long that $\phi\neq 0$). Combining this with the Lipschitz estimate (5.32), we conclude that $\Gamma^\alpha_{\beta\gamma}$ is Lipschitz continuous at p in coordinate y^α.

The Christoffel symbols in (5.34) are generally non-zero since $\dot{C}$ and C' are non-zero. Since a non-singular coordinate transformation preserving the metric form (5.23) cannot map $\dot{C}$ and C' to zero, we need a transformation that breaks the form (5.23). To complete the proof, we now introduce a coordinate transformation which preserves the Lipschitz continuity at p and maps the Christoffel symbols to zero at the point p. Without loss of generality, we assume that $y(p) = (0, 0, \phi_0, 0)$ for some $\phi_0 \in (0, \pi)$. Since the Christoffel symbols in coordinates y^α are Lipschitz continuous at p (and hence defined at p), we can introduce (for $\mu = 1, ..., 4$) the smooth coordinate transformation

$$z^\mu(y) \equiv \frac{1}{2}\delta^\mu_\alpha \Gamma^\alpha_{\beta\gamma}\big|_p \, y^\beta y^\gamma + \delta^\mu_\alpha \, y^\alpha + c_\beta \, y^\beta + c \tag{5.35}$$

where δ^μ_α denotes the Kronecker symbol and the constants c^μ and the constant coefficients c^μ_β are defined by

$$\begin{aligned} c^\mu &\equiv -\frac{1}{2}\delta^\mu_\alpha \Gamma^\alpha_{33}\big|_p \, {\phi_0}^2 - \delta^\mu_3 \phi_0, \\ c^\mu_\beta &\equiv -\frac{1}{2}\delta^\mu_\alpha \Gamma^\alpha_{\beta 3}\big|_p \phi_0. \end{aligned}$$

By our definition of c^μ and c^μ_β, it follows from (5.35) that

$$z(p) = 0 \qquad \text{and} \qquad \frac{\partial z^\mu}{\partial y^\alpha}\Big|_p = \delta^\mu_\alpha. \tag{5.36}$$

Moreover, (5.35) implies that

$$\frac{\partial^2 z^\mu}{\partial y^\beta \partial y^\gamma}\Big|_p = \delta^\mu_\alpha \Gamma^\alpha_{\beta\gamma}\big|_p. \tag{5.37}$$

From the transformation law of connections together with (5.36) and (5.37), we find that the Christoffel symbols in coordinates z^μ vanish at p. Namely, (5.36) and (5.37) imply that the transformation law (5.30) evaluated at p is given by

$$\begin{aligned} \frac{\partial z^\sigma}{\partial y^\alpha}\Gamma^\alpha_{\beta\gamma} &= \frac{\partial^2 z^\sigma}{\partial y^\beta \partial y^\gamma} + \Gamma^k_{\mu\nu}\frac{\partial z^\mu}{\partial y^\beta}\frac{\partial z^\nu}{\partial y^\gamma} \\ &= \delta^\sigma_\alpha \Gamma^\alpha_{\beta\gamma} + \Gamma^\sigma_{\mu\nu}\delta^\mu_\beta \delta^\nu_\gamma, \end{aligned}$$

which implies that the Christoffel symbol in coordinates z^μ satisfies

$$\Gamma^\sigma_{\mu\nu}\Big|_p = 0,$$

for all $\sigma, \mu, \nu \in \{1, ..., 4\}$. Clearly, since the transformation is smooth, it preserves the Lipschitz continuity of Γ at p. Denoting the coordinates z^j by y^α, we proved the sought after Lipschitz estimate (5.1). This completes the proof of Theorem 5.5. □

6. The Curvature of the singular part of Γ

In this section we give the proof of Theorem 2.5. We use the following lemma, which also motivates Definition 2.4.

Lemma 6.1. *Assume Γ is an L^∞ connection with $\mathrm{Riem}(\Gamma) \in L^\infty$ defined on a neighborhood of some $p \in \mathcal{M}$. Then $\alpha\big[\Gamma - \tilde{\Gamma}\big]$, defined in (2.3), is a non-negative L^∞ scalar function on that neighborhood for any Lipschitz tensor $\tilde{\Gamma}$. Moreover, there exists a coordinate system y^α on some neighborhood of p in which*

$$\Gamma^\gamma_{\alpha\beta} \in C^{0,1}$$

if and only if there exists a Lipschitz continuous $(1,2)$*-tensor* $\tilde{\Gamma}^{\sigma}_{\mu\nu}$ *such that*

$$\alpha[\Gamma - \tilde{\Gamma}] = 0$$

in an L^∞ *sense in that neighborhood.*

Proof. Expression $\alpha\big[\Gamma - \tilde{\Gamma}\big]$ defines an invariant scalar because it is the contraction of $\mathrm{Riem}(\Gamma - \tilde{\Gamma})$ with the auxiliary Riemannian metric h, c.f. (2.3). The assumption $\mathrm{Riem}(\Gamma) \in L^\infty$ implies that $Curl(\Gamma)$ is an L^∞ function. Thus, $Curl(\Gamma - \tilde{\Gamma})$ is also an L^∞ function, and since the commutator part of the Riemann curvature (3.1) is undifferentiated, $\mathrm{Riem}(\Gamma - \tilde{\Gamma})$ is also in L^∞ for every Lipschitz tensor $\tilde{\Gamma}$. We conclude that $\alpha\big[\Gamma - \tilde{\Gamma}\big]$ is an L^∞ function for any Lipschitz tensor $\tilde{\Gamma}$.

To show that $\alpha[\cdot]$ is a non-negative L^∞ function, choose a point p and coordinates in a neighborhood of p such that h agrees with the Euclidean metric at p. In these coordinates each of the components of $\mathrm{Riem}(\Gamma)$, enters $\alpha\big[\Gamma\big]$ only squared, so that $\alpha\big[\Gamma\big]$ is a sum of positive numbers at p. Since $\alpha\big[\Gamma\big]$ transforms as a scalar, it follows that it is non-negative in any coordinate system.

Moreover, by the same argument as above it follows that $\alpha[\Gamma - \tilde{\Gamma}]$ vanishes if and only if all components of the Riemann tensor are zero, i.e.,

$$\mathrm{Riem}\big(\Gamma - \tilde{\Gamma}\big) = 0. \tag{6.1}$$

Theorem 2.1 now implies the existence of coordinates y^α in which $\Gamma^{\gamma}_{\alpha\beta}$ is Lipschitz continuous. This proves the backward implication.

To prove the forward implication, assume there exists coordinates y^α in which $\Gamma^{\gamma}_{\alpha\beta}$ is Lipschitz continuous in some neighborhood. Theorem 2.1 implies that there exists a $C^{0,1}$ tensor $\tilde{\Gamma}$ such that $\mathrm{Riem}(\Gamma - \tilde{\Gamma}) = 0$. It now follows from Definition 2.4 that $\alpha[\Gamma - \tilde{\Gamma}] = 0$. This completes the proof. □

Proof of Theorem 2.5: For the forward implication, assume there exists coordinates y^α in which $\Gamma^{\gamma}_{\alpha\beta}$ is Lipschitz continuous in some neighborhood of p. Lemma 6.1 implies that there exists a $C^{0,1}$ tensor $\tilde{\Gamma}$ such that $\alpha[\Gamma - \tilde{\Gamma}] = 0$. Moreover, by Lemma 6.1, $\alpha[\Gamma - \tilde{\Gamma}]$ is a non-negative L^∞ function. Thus $\tilde{\Gamma}$ minimizes (2.4) and $\alpha_M(\Gamma) = 0$ for any $M > \|\tilde{\Gamma}\|_{C^{0,1}}$, and hence the curvature of the singular part of Γ equals zero.

For the backward implication, assume the curvature of the singular part of Γ equals zero. Then $\alpha_M(\Gamma) = 0$ for some constant $M > 0$. Then there exists a sequence of $\tilde{\Gamma}_k \in C^{0,1}$ with $\|\tilde{\Gamma}_k\|_{C^{0,1}} \leq M$ such that $\alpha[\Gamma - \tilde{\Gamma}_k] \to 0$ as $k \to \infty$. By Arzela-Ascoli, the space

$$\mathcal{S} \equiv \Big\{ \tilde{\Gamma} \in C^{0,1} \;\Big|\; \|\tilde{\Gamma}\|_{C^{0,1}} \leq M \Big\}$$

is compact, implying there exists a convergent subsequence $\tilde{\Gamma}_{k_j} \to \tilde{\Gamma}_0 \in \mathcal{S}$ with $\alpha[\Gamma - \tilde{\Gamma}_0] = 0$. Lemma 6.1 now implies the existence of coordinates y^α in which $\Gamma^{\gamma}_{\alpha\beta}$ is Lipschitz continuous. This completes the proof. □

7. Conclusion

We prove that the question whether there exists a $C^{1,1}$ coordinate transformation which smooths an L^∞ symmetric connection Γ to $C^{0,1}$ in some neighborhood is equivalent to the existence of a Lipschitz continuous $(1,2)$-tensor $\tilde{\Gamma}$ such that $\Gamma - \tilde{\Gamma}$ is Riemann-flat in that neighborhood. Since the addition of a Lipschitz tensor does not change the jumps in the singular part of the connection, this says, essentially, that a regularity singularity is present at a point p if and only if the singular part of the spacetime connection (the shock set) *cannot* be extended to a Riemann-flat connection in a neighborhood of p. In light of this result, we defined the "curvature of the singular part" of an L^∞ connection with L^∞ Riemann curvature tensor, and proved that it is zero if and only if the connection can be smoothed to $C^{0,1}$ within the atlas of $C^{1,1}$ coordinate transformations.

Somewhat surprisingly, one can modify the method of proof of Theorem 2.3 in the special case of a spherically symmetric $C^{0,1}$ metric with an L^∞ curvature tensor, can always be mapped in a neighborhood of any point p, to a locally inertial frame at p, within the $C^{1,1}$ atlas, even though the regularity issue of whether the metric can be smoothed to $C^{1,1}$, remains open. This applies to solutions of the Einstein-Euler equations generated by the Glimm scheme, [8]. The existence of locally inertial frames implies that the Coriolis terms discussed in [16] are removable and, using our terminology, *no* strong regularity singularities exist in spherically symmetric spacetimes. In summary, the space of L^∞ connections with L^∞ curvature tensors provides a consistent general framework for shock wave theory in General Relativity, and the problem whether *weak* regularity singularities exist in spherically symmetric spacetimes, or whether *weak* or *strong* regularity singularities exist at more complicated (asymmetric) shock wave interactions, remains an open problem for which the results here provide a new geometric perspective.

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