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DEGENERATE SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. We describe systems of conservation laws with the property that the shock and rarefaction curves coincide. We give new examples of such systems. These systems isolate and separate many nonlinear aspects of shock waves.

The author believes that analytical techniques which handle problems of uniqueness, continuous dependence or convergence of finite difference schemes in these systems, would isolate components in a corresponding analysis required for general systems of conservation laws.

We summarize and simplify some of the results in [17,18] which give a theory for the systems of conservation laws which have the property that the shock and rarefaction curves coincide. These systems have milder nonlinearities than systems of conservation laws in general, and represent the simplest setting in which hyperbolic singularities can appear. Moreover, examples of such systems arise in many applications [1,4,5,6,8,12,14,16]. For these systems, as is the case for a scalar conservation law, the nonlinearities affect the wave speeds, but no reflected wave appears after interactions [cf. 15]. The wave speeds are, however, coupled in these systems, a phenomenon not present in scalar conservation laws. Thus systems with coinciding shock and rarefaction curves are a reasonable place for studying some of the problems that appear difficult in the general setting of conservation laws.

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The shock and rarefaction curves coincide in a characteristic family if the rarefaction curves are straight lines (line field) or level curves of the eigenvalue (contact field). Thus for 2×2 systems of conservation laws, there are generically three classes of systems with coinciding shock and rarefaction curves: those with two contact fields (class I), those with one line and one contact field (class II) and those with two line fields (class III). Moreover, under suitable assumptions, these systems cannot be transformed into equations which are either linear or uncoupled.

Below we give conditions on the flux functions which are necessary and sufficient for determining the systems in the above classes (we record here the fact, omitted in [17,18], that there exist nonlinear coupled systems with two contact fields which are in conservative form).

In [19] it is proved that for systems lying outside the class of systems with coinciding shock and rarefaction curves, the solutions are not L^1-contractive in any metric. It would be interesting to locate the systems within the above classes which are L^1-contractive.

Continuous dependence of solutions on the initial data is not known for general systems of conservation laws. It would be interesting to obtain continuous dependence or uniqueness results for systems in class I, II or III.

In [11,15] it is shown that certain finite difference schemes are total variation diminishing for systems in class III; in classes I and II more serious numerical diffusion occurs, and this result is not true. It would be interesting if one could estimate the diffusion in approximate solutions generated by finite difference schemes for equations in classes I or II.

We now summarize and simplify the results in [17,18] regarding systems of conservation laws with coinciding shock and rarefaction curves.

Let

(1) $$U_t + F(U)_x = 0$$

denote an $n \times n$ system of conservation laws. Let (λ, R) denote a real eigenfield for dF, meaning that

$$dF \cdot R = \lambda R,$$

$\lambda \in \mathbf{R}$. Let M denote any 1-dimensional C^3-manifold in U-space. We say that the shock and rarefaction curves coincide on M if M is an integral curve of an eigenvector R, and for every $U_1, U_2 \in M$, U_1 is in the Hugoniot locus of U_2 i.e.,

$$s[U_1 - U_2] = [F(U_1) - F(U_2)]$$

for some scalar s. The following theorem characterizes the coincidence of shock and rarefaction curves:

THEOREM 1. The following are equivalent regarding a 1-dimensional C^3 manifold M:

(i) The shock and rarefaction curves coincide on M.

(ii) M is either a straight line in U-space or else simultaneously an integral of R and a level curve of λ for (λ, R) a real eigenfield for dF.

(iii) System (1) reduces to a scalar conservation law on M.

Here condition (iii) is explained by the following definition:

DEFINITION. System (1) reduces to a scalar conservation law on a 1-manifold M in U-space if in a neighborhood of each point on M there exists a coordinate system φ and a scalar conservation law

(2) $$u_t + f(u)_x = 0$$

such that, if $U = \varphi(u)$, $u(x,t)$ is a weak solution of (2) if and only if $\varphi \cdot u(x,t)$ is a weak solution of (1).

For condition (iii), $f'(u) = \lambda \cdot \varphi(u)$ in the case that the coordinate system u is taken to be a component of U [18].

The main step in the proof of Theorem 1 is the following simple lemma. Let $U(\xi)$, $\xi_1 < \xi < \xi_2$, be a regular C^3-parameterization of a C^3 1-manifold M such that $\dot{U}(\xi) = \alpha(\xi)R$, $\alpha \in \mathbf{R}$, where (λ, R) is a real eigenfield for dF. Let $R(\xi) \equiv R(U(\xi))$, $\lambda(\xi) \equiv \lambda(U(\xi))$.

LEMMA. If $U(\xi)$ lies in the Hugoniot locus of a point U_L for all $\xi \in (\xi_1, \xi_2)$, then

(3) $$\dot{\lambda}(\xi)[U(\xi) - U_L] \cdot N(\xi) = 0$$

for all $\xi \in (\xi_1, \xi_2)$, and all vectors $N(\xi)$ such that $N(\xi) \cdot R(\xi) = 0$.

Thus condition (iii) says that either $\dot{\lambda} = 0$ and λ is constant on M, or $[U(\xi) - U_L] \cdot N(\xi) = 0$ for all ξ and $N(\xi)$, which implies that M is a straight line in U-space [18].

<u>Proof</u>. Let $[U] \equiv U(\xi) - U_L$, $[F] = F(U(\xi)) - F(U_L)$. Then by assumption

(4) $$s[U] = [F] ,$$

where $s \equiv s(\xi)$ is a C^2 function of ξ since $[U]$ and $[F]$ are. Differentiating (4) gives

(5) $$\dot{s}[U] + s\dot{U} = \lambda\dot{U} ,$$

and differentiating again gives

(6) $$\ddot{s}[U] + 2\dot{s}\dot{U} + s\ddot{U} = \dot{\lambda}\dot{U} + \lambda\ddot{U} .$$

Taking the inner product of both sides of (5) with $N(\xi)$ gives

(7) $$\dot{s}[U(\xi) - U_L] \cdot N(\xi) = 0 .$$

Now by (7), if $[U] \cdot N(\xi) \neq 0$ (i.e., $[U]$ is not parallel to $\dot{U}$), then $\dot{s} = 0$, and (5) gives $s = \lambda$. Putting these two conditions into (6) yields

$$\dot{\lambda}\dot{U} = \ddot{s}[U] .$$

Thus whenever $[U] \cdot N(\xi) \neq 0$, we must have $\dot{\lambda} = 0$, and so we can conclude that

$$\dot{\lambda}[U(\xi) - U_L] \cdot N(\xi) = 0$$

for all ξ.

We now restrict our attention to 2×2 systems (1). In this case, let $U = (u,v)$, $F = (f,g)$, and let p, q denote Riemann invariants for (1), by which we mean any C^2 function on U-space whose level curves are integral curves of a real eigenvector field of dF.

<u>DEFINITION</u>. System (1) has a <u>contact field</u> in a neighborhood U of U-space if there exists a Riemann invariant q defined in U, $\nabla q \neq 0$, such that the level curve $q = \lambda$ is an integral curve of R for some eigenfield (λ, R) of dF.

Note that $\nabla q \neq 0$ implies that any system having a contact field is not linear, and is not a nonlinear transformation of a linear system.

<u>DEFINITION</u>. System (1) has a <u>line field</u> if there exists a Riemann invariant q, $\nabla q \neq 0$, such that q = const. is a line of

slope $q = \frac{dv}{du}$, i.e.,

(8) $$q_u + qq_v = 0 .$$

Note that $\nabla q \neq 0$ implies that any system having a line field does not uncouple in this field. (In an uncoupled field, the integral curves are straight lines of constant slope.)

We prove the following simple theorems which give necessary and sufficient conditions for a 2×2 system to have a line or a contact field.

THEOREM 2. A 2×2 system of conservation laws has a contact field if and only if

(9) $$f = uq + F(q)$$

(10) $$g = vq + G(q)$$

for some smooth functions F and G.

PROOF. The function q is a Riemann invariant of a contact field if and only if

(11) $$q[u] = [f] ,$$

(12) $$q[v] = [g] ,$$

for all jumps at q = const. But (11) and (12) hold if and only if

(13) $$[f - qu] = 0 ,$$

(14) $$[g - qv] = 0 ,$$

for all jumps at q = const.; and this is equivalent to

$$f = uq + F(q) ,$$

$$g = vq + G(q) .$$

THEOREM 3. A 2×2 system of conservation laws has a line field if and only if

$$g = fq + H(q)$$

for some smooth function H, and some smooth solution q of Burgers equation

(15) $$w_u + ww_v = 0 .$$

PROOF. A smooth solution q of (15) is a Riemann invariant if and only if the value of q at (u,v) is the slope of the integral curve of a line field through (u,v). Thus by Theorem 1, a smooth solution q of (15) is a Riemann invariant for system (1) if and only if

$$s[u] = [f] ,$$
$$s[v] = [g] ,$$

for some scalar s, whenever

$$\frac{[v]}{[u]} = q .$$

But this condition is equivalent to

$$q = \frac{[g]}{[f]}$$

for all jumps at q = const., which holds if and only if

$$[qf - g] = 0$$

for all jumps at q = const. This final condition is equivalent to

$$g = qf + H(q)$$

for some smooth function H and some smooth solution q of (15).

The following corollaries characterize the systems in classes I, II and III:

COROLLARY 1. A 2×2 system of conservation laws has two independent contact fields with Riemann invariants p and q if and only if

$$f = uq + F_1(q) = up + F_2(p) , \tag{16}$$

$$g = vq + G_1(q) = vp + G_2(p) , \tag{17}$$

for some smooth functions F_i, G_i, $\nabla p \neq \nabla q$. Conditions (16) and (17) hold if and only if p and q as functions of u and v are given by the inverse of the map

$$\binom{u}{v} = \begin{pmatrix} F_1(q) - F_2(p) \\ G_1(q) - G_2(p) \end{pmatrix} \frac{1}{p - q} . \tag{18}$$

PROOF. Statement (16) and (17) follow directly from Theorem 2, and (18) is obtained by solving for u and v in (16) and (17).

We can use (18) to construct a 2×2 nonlinear coupled system of conservation laws having two contact fields as follows: choose $F_i(w) = w^2$, $G_i(w) = w^3$ for $i = 1,2$. In this case

$$\binom{u}{v} = \begin{pmatrix} -(q + p) \\ -(q^2 + pq + p^2) \end{pmatrix} . \tag{19}$$

We can invert (19) by obtaining $p = -(q + u)$ from the first equation in (19) and substituting this into the second equation. By solving the resulting quadratic equation in q, we obtain

$$(20) \qquad \binom{p}{q} = \frac{1}{2}\begin{pmatrix} -u \pm \sqrt{-4v - 3u^2} \\ -u \mp \sqrt{-4v - 3u^2} \end{pmatrix}.$$

Choosing $-$ for p and $+$ for q in (20), we obtain $f = -u^2 - v$ and $g = u^3 + uv$ by substituting (20) into (16), (17). Thus the 2×2 system

$$(21) \qquad \begin{aligned} u_t - (u^2 + v)_x &= 0, \\ v_t + (u^3 + uv)_x &= 0, \end{aligned}$$

has two contact fields with wave speeds p and q given by (20).

COROLLARY 2. A 2×2 system of conservation laws has one contact field and one line field if and only if

$$(22) \qquad f = (u + a)q,$$

$$(23) \qquad g = (v + b)q,$$

for some constants a, b, and some smooth function q of u and v. In this case q is the wave speed and Riemann invariant for the contact field, and

$$(24) \qquad p = \frac{v + b}{u + a}$$

is the Riemann invariant of the line field which satisfies (15).

PROOF. (See (2.10)-(2.18) of [17].)

Systems of type (22), (23) with $a = b = 0$ have arisen in applications, and were studied in [5,8,16].

COROLLARY 3. A 2×2 system of conservation laws has two independent line fields if and only if

$$(25) \qquad f = \frac{H_1(p) - H_2(q)}{q - p},$$

$$(26) \qquad g = \frac{qH_1(p) - pH_2(q)}{q - p}$$

for some smooth functions H_1 and H_2, and two smooth independent solutions p and q of (15).

PROOF. By Theorem 3, system (1) has two independent line fields if and only if

(27) $g = fp + H_1(p)$

and

(28) $g = fq + H_2(q)$

for smooth independent solutions p and q of (15). Thus solving for f and g in (27), (28) gives (25), (26).

The system

$$u_t + \left\{ \frac{u}{1 + u + v} \right\}_x = 0 ,$$

$$v_t + \left\{ \frac{kv}{1 + u + v} \right\}_x = 0 ,$$

for $0 < k < 1$, $u > 0$, $v > 0$, is an example of a system in class III arising in the study of multicomponent chromatography [1,14].

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