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ON WEAK CONTINUITY AND THE HODGE DECOMPOSITION

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ABSTRACT. We address the problem of determining the weakly continuous polynomials for sequences of functions that satisfy general linear first-order differential constraints. We prove that wedge products are weakly continuous when the differential constraints are given by exterior derivatives. This is sufficient for reproducing the Div-Curl Lemma of Murat and Tartar, the null Lagrangians in the calculus of variations and the weakly continuous polynomials for Maxwell's equations. This result was derived independently by Tartar who stated it in a recent survey article **[7]**. Our proof is explicit and uses the Hodge decomposition.

1. Introduction. The characterization of weakly continuous functionals has been an important tool in some recent developments in partial differential equations. In particular, the Div-Curl Lemma was instrumental in the work of Tartar **[6]** and DiPerna **[3]** on conservation laws, and the characterization of the null Lagrangians in the setting of the calculus of variations Edelen **[4]** was central to the work of Ball **[1]** on polyconvex functions. The most general theorem characterizing weakly continuous functionals is due to Tartar **[5]**: He gives necessary and sufficient conditions for quadratic functionals to be weakly continuous[3] under the general first-order differential constraints[1]

$$A_k^{ij}\frac{\partial u_j^\varepsilon}{\partial x^j} \in \text{ a compact set in } H^{-1}_{\text{loc}}(\Omega), \qquad k = 1,\dots,l. \tag{1.1}$$

Here Ω is an open set in $\mathbf{R}^n$, $x \in \Omega \subseteq \mathbf{R}^n$; $u^\varepsilon \in \mathbf{R}^m$; A_k^{ij}: $i = 1,\dots,n$; $j = 1,\dots,m$; $k = 1,\dots,l$, are constants. We assume summation over repeated indicies. Tartar's result is often refered to as the Quadratic Theorem. While the hypotheses of this theorem are easy to check for any particular function, it is not so clear how one can construct satisfactory functions given only the differential constraints.

In this paper we discuss a method sufficient for constructing weakly continuous polynomials of degree as high as the domain space, the maximum degree possible. With this method one can reproduce all of the weakly continuous functions in the div-curl case, the case of Maxwell's equations of electrodynamics, and the variational case. Our results are contained in the following theorem.

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[3]The theorem also gives necessary and sufficient conditions for quadratic functionals to be *weak-lower-semicontinuous*.

THEOREM 1.1. *Assume that $\alpha_1^\varepsilon, \cdots, \alpha_l^\varepsilon$ are differential forms on $\Omega \subseteq \mathbf{R}^n$ of degree $s_1, \ldots, s_l$ respectively, $s_1 + \cdots + s_l \leq n$. Assume that*

$$\alpha_i^\varepsilon \rightharpoonup \bar{\alpha} \quad \text{in } L^{p_i}(\Omega), \tag{1.2}$$

with $1/p_1 + \cdots + 1/p_l = 1$, and that

$$\mathbf{d}\alpha_i^\varepsilon \in \text{a compact set in } W_{\mathrm{loc}}^{-1,p_i}(\Omega), \qquad i = 1, \ldots, l. \tag{1.3}$$

Then we have

$$\alpha_1^\varepsilon \wedge \cdots \wedge \alpha_l^\varepsilon \rightharpoonup \bar{\alpha}_1 \wedge \cdots \wedge \bar{\alpha}_1 \tag{1.4}$$

in the sense of distributions.

As Tartar indicates [**7**], the Quadratic Theorem can be used to prove Theorem 1.1 when L^2 techniques apply. Our proof is based on the idea of decomposing weakly convergent sequences into weakly convergent and strongly convergent parts using a version of the Hodge decompostion. We obtain a slightly more general result than that obtained directly from the L^2 theory. We feel that this proof gives additional physical insight into the phenomenon of weak continuity and the interaction of oscillations.

The rest of this paper is laid out as follows: In §2 we show how Theorem 1.1 can be used to construct the most common examples of weakly continuous functions. In §3 we prove a specialized version of the Div-Curl Lemma in order to illustrate the ideas of the Hodge decompostion without introducing the language of differential geometry. In §4 we prove Theorem 1.1. Finally, in §5 we make some concluding remarks.

2. Applications of Theorem 1.1. We now present three well-known examples of first order differential constraints in which the weakly continuous functions are obtained directly from Theorem 1.1.

2.1. *The Div-Curl Lemma.*

LEMMA 2.1 (DIV-CURL) [MURAT AND TARTAR]. *Suppose that $u = (u_1^\varepsilon, \ldots, u_n^\varepsilon)$ and $v = (v_1^\varepsilon, \ldots v_n^\varepsilon)$ are sequences of functions on $\Omega \subseteq \mathbf{R}^n$ such that*

$$\left.\begin{matrix} u^\varepsilon \rightharpoonup \bar{u} \\ v^\varepsilon \rightharpoonup \bar{v} \end{matrix}\right\} \quad \text{in } L^2(\Omega),$$

and

$$\left.\begin{matrix} \operatorname{div} u^\varepsilon \\ \operatorname{curl} v^\varepsilon \end{matrix}\right\} \quad \text{in a compact set in } H_{\mathrm{loc}}^{-1}(\Omega).$$

Then

$$u^\varepsilon \cdot v^\varepsilon \rightharpoonup u \cdot v \quad \text{in the sense of distributions.}$$

PROOF. Note that

$$\alpha_I^\varepsilon(x) = u_I^\varepsilon(x)\,\mathbf{d}x^{[1]} + \cdots + u_n^\varepsilon(x)\,\mathbf{d}x^{[n]},$$
$$\alpha_1^\varepsilon(x) = v_1^\varepsilon(x)\,\mathbf{d}x^1 + \cdots + v_n^\varepsilon(x)\,\mathbf{d}x^n$$

satisfy the conditions of Theorem 1.1 in the case $l = 2$, $s_1 = n - 1$, $s_2 = 1$, where

$$\mathbf{d}x^{[i]} = (-1)^{i-1}\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^{i-1} \wedge \mathbf{d}x^{i+1} \wedge \cdots \wedge \mathbf{d}x^n.$$

(The differential form $\mathbf{d}x^{[i]}$ is the Hodge-$*$ dual of $\mathbf{d}x^i$.) Taking the wedge product of α_1^ε and α_2^ε and applying Theorem 1.1 we get

$$(u^\varepsilon \cdot v^\varepsilon)\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n \equiv \alpha_1^\varepsilon \wedge \alpha_2^\varepsilon \rightharpoonup \bar{\alpha}_1 \wedge \bar{\alpha}_2 \equiv (\bar{u}\cdot\bar{v})\,\mathbf{d}x^1 \wedge \cdots \wedge \mathbf{x}^n,$$

which verifies the lemma.

2.2. *Maxwell's equations.* We now consider the weakly continuous functions for sequences under the constraints of Maxwell's equations.

LEMMA 2.2. *Suppose the functions* $(E^\varepsilon, D^\varepsilon, B^\varepsilon, H^\varepsilon)\colon \mathbf{R}^3 \to [\mathbf{R}^3]^4$ *satisfy the equations*

$$\begin{aligned} &\frac{\partial B^\varepsilon}{\partial t} + \operatorname{curl} E^\varepsilon = 0, \quad \operatorname{div} B^\varepsilon = 0, \\ &\frac{\partial D^\varepsilon}{\partial t} - \operatorname{curl} H^\varepsilon = -J^\varepsilon, \quad \operatorname{div} D^\varepsilon = \rho^\varepsilon, \end{aligned} \tag{2.5}$$

for any sequence of data $(\rho^\varepsilon, J_1^\varepsilon, J_2^\varepsilon, J_3^\varepsilon) \in$ *a compact set in* H^{-1}_{loc}. *Then the combinations* $B\cdot H - D\cdot E$, $B\cdot E$, *and* $H\cdot D$ *are weakly continuous; i.e., if* $(E^\varepsilon, D^\varepsilon, B^\varepsilon, H^\varepsilon) \rightharpoonup (\overline{E}, \overline{D}, \overline{B}, \overline{V})$ *in* L^2, *then*

$$\left.\begin{aligned} B^\varepsilon \cdot H^\varepsilon - D^\varepsilon \cdot E^\varepsilon &\rightharpoonup \overline{B}\cdot\overline{H} - \overline{D}\cdot\overline{E}, \\ B^\varepsilon \cdot E^\varepsilon &\rightharpoonup \overline{B}\cdot\overline{E}, \\ H^\varepsilon \cdot D^\varepsilon &\rightharpoonup \overline{H}\cdot\overline{D} \end{aligned}\right\} \quad \textit{in the sense of distributions.}$$

PROOF. We treat space-time as a four-dimensional Euclidean space, letting $x^0 = t$. We define the two-forms of *Faraday*

$$\begin{aligned} \mathbf{F}^\varepsilon = {} & E_1^\varepsilon \mathbf{d}x^1 \wedge \mathbf{d}x^0 + E_2^\varepsilon\,\mathbf{d}x^2 \wedge \mathbf{d}x^0 + E_3^\varepsilon\,\mathbf{d}x^3 \wedge \mathbf{d}x^0 \\ & + B_1^\varepsilon\,\mathbf{d}x^2 \wedge \mathbf{d}x^3 + B_2^\varepsilon\,\mathbf{d}x^3 \wedge \mathbf{d}x^1 + B_3^\varepsilon\,\mathbf{d}x^1 \wedge \mathbf{d}x^2, \end{aligned}$$

and *Maxwell*

$$\begin{aligned} \mathbf{M}^\varepsilon = {} & -H_1^\varepsilon\,\mathbf{d}x^1 \wedge \mathbf{d}x^0 - H_2^\varepsilon\,\mathbf{d}x^2 \wedge \mathbf{d}x^0 - H_3^\varepsilon\,\mathbf{d}x^3 \wedge \mathbf{d}x^0 \\ & + D_1^\varepsilon\,\mathbf{d}x^2 \wedge \mathbf{d}x^3 + D_2^\varepsilon\,\mathbf{d}x^3 \wedge \mathbf{d}x^1 + D_3^\varepsilon \mathbf{d}x^1 \wedge \mathbf{d}x^2, \end{aligned}$$

and the dual of the *charge-current* one-form $\mathbf{J}$

$$\begin{aligned} {}^*\mathbf{J}^\varepsilon = {} & \rho^\varepsilon \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 - J_1\,\mathbf{d}x^0 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \\ & - J_2\,\mathbf{d}x^0 \wedge \mathbf{d}x^3 \wedge \mathbf{d}x^1 - J_3\,\mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2. \end{aligned}$$

Maxwell's equations can then be expressed as

$$\mathbf{dF}^\varepsilon = 0, \quad \mathbf{dM}^\varepsilon = {}^*\mathbf{J}^\varepsilon.$$

We now observe that

$$\mathbf{F}^\varepsilon \wedge \mathbf{F}^\varepsilon = B^\varepsilon \cdot E^\varepsilon, \quad \mathbf{M}^\varepsilon \wedge \mathbf{M}^\varepsilon = H^\varepsilon \cdot D^\varepsilon, \quad \text{and} \quad \mathbf{F}^\varepsilon \wedge \mathbf{M}^\varepsilon = B^\varepsilon \cdot H^\varepsilon - E^\varepsilon \cdot D^\varepsilon.$$

The lemma follows directly.

2.3. *The calculus of variations.* Our third example comes from the calculus of variations where we are often concerned with minimizing functionals of the form $\int_\Omega \mathcal{W}(\nabla p(x))\,dx$. Here $p\colon \mathbf{R}^n \to \mathbf{R}^n$ represents the deformation of the region Ω, and $\mathcal{W}$ is the stored energy function. If we write $u = \nabla p$, the problem is then to minimize

$$\int_\Omega \mathcal{W}(u(x))\,dx, \qquad u\colon \mathbf{R}^n \to \mathbf{R}^{n\times n},$$

subject to the differential constraints

$$\frac{\partial u_{ij}}{\partial x_k} - \frac{\partial u_{ik}}{\partial x_j} = 0, \qquad i,j,k = 1,\dots,n. \tag{2.6}$$

Ball constructed the polyconvex functions, a very large class of physically reasonable functions $\mathcal{W}$ for which this problem can be shown to have a solution. In doing so he used the following well-known lemma on weakly continuous functions.

LEMMA 2.3. *Let $f(u)$ be any subdeterminant of the $n \times n$ matrix u. Then f is weakly continuous on sequences satisfying the differential constraints* (2.6); *i.e. if $u^\varepsilon \rightharpoonup \bar{u}$ satisfies* (2.6), *then*

$$f(u^\varepsilon) \rightharpoonup f(\bar{u}) \quad \textit{in the sense of distributions.}$$

PROOF. If we define $\alpha_i^\varepsilon \equiv u_{i_1}^\varepsilon \, \mathbf{d}x^1 \wedge \cdots \wedge u_{in}^\varepsilon \, \mathbf{d}x^n$, then we can represent the differential constraints (2.6) by $\mathbf{d}\alpha_i^\varepsilon = 0$; $i = 1,\dots,n$; and any $l \times l$ subdeterminant f can be written

$$f(u^\varepsilon)\, \mathbf{d}x^1 \wedge \cdots \wedge \mathbf{d}x^n = \alpha_{i_1}^\varepsilon \wedge \cdots \wedge \alpha_{i_l}^\varepsilon \wedge \mathbf{d}x^{[j_1,\dots,j_l]}$$

for some appropriate choice of indices $\{i_1,\dots,i_l\}$ for the rows and $\{j_1,\dots,j_l\}$ for the columns. The lemma follows immediately.

3. The Div-Curl Lemma in $\mathbf{R}^3$. Here we present a proof of a special case of the Div-Curl Lemma that illustrates the idea of the Hodge decomposition without introducing the language of differential geometry.

LEMMA 3.1 (DIV-CURL IN $\mathbf{R}^3$). *Suppose Ω is an open set in $\mathbf{R}^3$ and the sequence of functions $(u^\varepsilon, v^\varepsilon)\colon \Omega \to \mathbf{R}^3 \times \mathbf{R}^3$ satisfies*

$$\left.\begin{matrix} u^\varepsilon \rightharpoonup \bar{u} \\ v^\varepsilon \rightharpoonup \bar{v} \end{matrix}\right\} \quad \textit{in } L^2(\Omega), \tag{3.7}$$

and

$$\left.\begin{matrix} \operatorname{div} u^\varepsilon \\ \operatorname{curl} v^\varepsilon \end{matrix}\right\} \quad \textit{in a compact set in } H^{-1}_{\mathrm{loc}}(\Omega). \tag{3.8}$$

Then

$$u^\varepsilon \cdot v^\varepsilon \to \bar{u} \cdot \bar{v} \quad \textit{in the sense of distributions.}$$

PROOF. To show that

$$\int_\Omega u^\varepsilon \cdot v^\varepsilon \phi \, dx \to \int \bar{u} \cdot \bar{v} \phi \, dx \tag{3.9}$$

for all $\phi \in C_0^\infty(\Omega)$ it suffices to assume that u^ε and v^ε have compact support. To see this, note that it is sufficient to verify (3.9) with u^ε and v^ε replaced by $\tilde{\phi} u^\varepsilon$ and $\tilde{\phi} v^\varepsilon$ where $\tilde{\phi} \in C_0^\infty(\Omega)$ is a cutoff functions that equals one on the support of ϕ. Our hypotheses (3.7) and (3.8) imply

$$\left.\begin{matrix} \tilde{\phi} u^\varepsilon \rightharpoonup \tilde{\phi} \bar{u}, \\ \tilde{\phi} v^\varepsilon \rightharpoonup \tilde{\phi} \bar{v}, \end{matrix}\right\} \quad \text{in } L^2(\mathbf{R}^3), \tag{3.10}$$

and

$$\left.\begin{aligned}\operatorname{div}(\tilde{\phi}u^\varepsilon) &= \tilde{\phi}\operatorname{div}u^\varepsilon + \operatorname{grad}\tilde{\phi}\cdot u^\varepsilon \rightharpoonup \operatorname{div}(\tilde{\phi}\bar{u}),\\ \operatorname{curl}(\tilde{\phi}v^\varepsilon) &= \tilde{\phi}\operatorname{curl}v^\varepsilon + \operatorname{grad}\tilde{\phi}\times v^\varepsilon \rightharpoonup \operatorname{curl}(\tilde{\phi}\bar{u}),\end{aligned}\right\} \quad \text{in } H^{-1}(\mathbf{R}^3). \tag{3.11}$$

To see that (3.11) holds note that the first term on the right-hand side of each equality converges (at least for a subsequence) directly from the definition of H^{-1}_{loc}. ($f \in H^{-1}_{\mathrm{loc}}(\Omega) \Leftrightarrow f\phi \in H^{-1}(\Omega)\ \forall \phi \in C_0^\infty(\Omega)$.) The convergence of the second term follows from (3.7) and the compactness of the imbedding of $L^2(\Omega)$ into $H^{-1}(\Omega)$.

We now show that there exist scalar fields $\psi^\varepsilon, \bar{\psi}, \eta^\varepsilon, \bar{\eta}$ and vector fields $\gamma^\varepsilon, \bar{\gamma}, \chi^\varepsilon, \overline{\chi}$ such that

$$\begin{aligned} u^\varepsilon &= \operatorname{grad}\psi^\varepsilon + \operatorname{curl}\gamma^\varepsilon, \quad & v^\varepsilon &= \operatorname{grad}\eta^\varepsilon + \operatorname{curl}\chi^\varepsilon,\\ \bar{u} &= \operatorname{grad}\bar{\psi} + \operatorname{curl}\bar{\gamma}, \quad & \bar{v} &= \operatorname{grad}\bar{\eta} + \operatorname{curl}\ \overline{\chi}: \end{aligned} \tag{3.12}$$

and

$$\left.\begin{aligned}\operatorname{grad}\psi^\varepsilon &\to \operatorname{grad}\bar{\psi},\\ \operatorname{curl}\chi^\varepsilon &\to \operatorname{curl}\ \overline{\chi},\end{aligned}\right\} \quad \text{(strongly) in } L^2(\Omega); \tag{3.13}$$

$$\left.\begin{aligned}\operatorname{curl}\gamma^\varepsilon &\rightharpoonup \operatorname{curl}\bar{\gamma},\\ \operatorname{grad}\eta^\varepsilon &\rightharpoonup \operatorname{grad}\bar{\eta},\end{aligned}\right\} \quad \text{(weakly) in } L^2(\Omega); \tag{3.14}$$

$$\left.\begin{aligned}\gamma^\varepsilon &\to \bar{\gamma},\\ \eta^\varepsilon &\to \bar{\eta}\end{aligned}\right\} \quad \text{(strongly) in } L^2(\Omega). \tag{3.15}$$

To see this we extend u^ε by 0 to all of $\mathbf{R}^3$ and let w^ε be the unique solution in $L^2(\mathbf{R}^3)$ of

$$\Delta w^\varepsilon = u^\varepsilon. \tag{3.16}$$

We write

$$w^\varepsilon = \Delta^{-1}u^\varepsilon \tag{3.17}$$

and note that the operator $\Delta^{-1} : H^r_{\mathrm{loc}} \to H^{r+2}_{\mathrm{loc}}$, $r = -1, 0, 1, \ldots$, is continuous. We then set

$$\psi^\varepsilon = \operatorname{div} w^\varepsilon, \quad \gamma^\varepsilon = -\operatorname{curl} w^\varepsilon$$

so that

$$\operatorname{grad}\psi^\varepsilon + \operatorname{curl}\gamma^\varepsilon = \operatorname{grad}\operatorname{div} w^\varepsilon - \operatorname{curl}\operatorname{curl} w^\varepsilon = \Delta w^\varepsilon = u^\varepsilon,$$

and (3.12a) holds.

Since we are solving (3.16) on all of space and have no boundary conditions to consider, the solution operator Δ^{-1} commutes with Δ and therefore with the differential operators div and curl. Indeed,

$$\begin{aligned}\Delta^{-1}\operatorname{div} f &= \Delta^{-1}\operatorname{div}\Delta\Delta^{-1}f\\ &= \Delta^{-1}\operatorname{div}(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\Delta^{-1}f\\ &= \Delta^{-1}\operatorname{div}\operatorname{grad}\operatorname{div}\Delta^{-1}f\\ &= \Delta^{-1}\Delta\operatorname{div}\Delta^{-1}f\\ &= \operatorname{div}\Delta^{-1}f,\end{aligned}$$

and

$$\begin{aligned}\Delta^{-1}\operatorname{curl} f &= \Delta^{-1}\operatorname{curl}\Delta\Delta^{-1} f\\ &= \Delta^{-1}\operatorname{curl}(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\Delta^{-1} f\\ &= \Delta^{-1}(-\operatorname{curl}\operatorname{curl}\operatorname{curl})\Delta^{-1} f\\ &= \Delta^{-1}(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\operatorname{curl}\Delta^{-1} f\\ &= \Delta^{-1}\Delta\operatorname{curl}\Delta^{-1} f\\ &= \operatorname{curl}\Delta^{-1} f.\end{aligned}$$

Thus,

$$\psi^\varepsilon \operatorname{div}\Delta^{-1} u^\varepsilon = \Delta^{-1}\operatorname{div} u^\varepsilon.$$

Now by the convergence of $\operatorname{div} u^\varepsilon$ in $H^{-1}_{\mathrm{loc}}(\Omega)$ (at least for a subsequence) and the continuity of Δ^{-1} from $H^{-1}(\Omega)$ to $H^1(\Omega)$ we get

$$\operatorname{grad}\psi^\varepsilon = \operatorname{grad}(\Delta^{-1}\operatorname{div} u^\varepsilon) \to \operatorname{grad}(\Delta^{-1}\operatorname{div} u^\varepsilon) = \operatorname{grad}\bar{\psi} \quad \text{(strongly) in } L^2(\Omega),$$

so (3.13a) holds. Since $\operatorname{curl}\gamma^\varepsilon = u^\varepsilon - \operatorname{grad}\psi^\varepsilon$, (3.14a) follows directly from (3.7) and (3.13a). To see that (3.15a) holds note that (3.7) and the continuity of Δ^{-1} from $L^2(\Omega)$ to $H^2(\Omega)$ imply that $\gamma^\varepsilon = -\operatorname{curl}\Delta_{-1} u^\varepsilon$ is bounded in $H^1(\Omega)$. Strong convergence in $L^2(\Omega)$ follows by compact imbedding.

The corresponding results for v^ε follow in similar fashion except that the roles of div and curl have been reversed.

With (3.13)–(3.15) in hand we complete the proof. We consider

$$\int_\Omega u^\varepsilon \cdot v^\varepsilon \phi\, dx = \int (\operatorname{grad}\psi^\varepsilon + \operatorname{curl}\gamma^\varepsilon)\cdot(\operatorname{grad}\eta^\varepsilon + \operatorname{curl}\chi^\varepsilon)\phi\, dx$$

for any $\phi \in C_0^\infty(\Omega)$. The product has four terms, and $\int_\Omega \operatorname{grad}\eta^\varepsilon \cdot \operatorname{curl}\gamma^\varepsilon \phi\, dx$ is the only one which is not the product of a weakly convergent sequence and a strongly convergent sequence. But here we can integrate by parts and get

$$\begin{aligned}\int_\Omega \operatorname{grad}\eta^\varepsilon \cdot \operatorname{curl}\gamma^\varepsilon \phi\, dx &= -\int_\Omega [\eta^\varepsilon \operatorname{div}\operatorname{curl}\gamma^\varepsilon \phi + \eta^\varepsilon \operatorname{curl}\gamma^\varepsilon \cdot \operatorname{grad}\phi]\, dx\\ &= -\int_\Omega \eta^\varepsilon \operatorname{curl}\gamma^\varepsilon \cdot \operatorname{grad}\phi\, dx,\end{aligned}$$

and once again this is the product of a weakly convergent sequence with a strongly convergent sequence. Thus, the entire integral converges, and our proof is complete.

Note that the key to the proof is in decomposing each of the weakly convergent sequences u^ε and v^ε into the sum of weakly and strongly convergent parts and showing that the two weakly convergent parts are in some way orthogonal. If we think of the weakly convergent parts as oscillations superimposed on the strongly convergent parts the key is to show that the oscillations do not interact.

4. The weak continuity of wedge products. In this section we prove Theorem 1.1. We begin by introducing some standard notation from differential geometry. Let $\Lambda^k(\Omega)$ denote the space of differential forms of degree k defined over $\Omega \subseteq \mathbf{R}^n$ with standard coordinates system x. For each k, $0 \le k \le n$, $\mathbf{d}\colon \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ denotes the exterior derivative. Recall that $\mathbf{d} \equiv 0$ for

$k \geq n$, and $\mathbf{dd} \equiv 0$ $\forall k$ Let $\langle\cdot,\cdot\rangle$ be the inner product on $\Lambda^k(\Omega)$ defined by taking $\{\mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ as an orthonormal basis. This defines a natural L^2 inner product between elements $\alpha, \beta \in \Lambda^k(\Omega)$ by

$$\langle \alpha, \beta \rangle_{L_2(\Omega)} = \int_\Omega \langle \alpha, \beta \rangle \, dx,$$

for $\alpha, \beta \in \Lambda^k(\Omega)$. Let $\delta \colon \Lambda^{k+1}(\Omega) \to \Lambda^k(\Omega)$ denote the formal adjoint of $\mathbf{d}$ with respect to the L^2 inner product on $\Lambda^k(\Omega)$; *i.e.*,

$$\langle \mathbf{d}\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2},$$

for all $\alpha \in \Lambda^k(\Omega), \beta \in \Lambda^{k+1}(\Omega)$.

For each k, let $\Delta \colon \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ denote the Laplace-Beltrami operator defined by

$$\Delta\alpha = \mathbf{d}\delta\alpha + \delta\,\mathbf{d}\alpha.$$

As in the precious section we define Δ^{-1} to be the solution operator for the Laplace-Beltrami operator on all of space; we note that

$$\Delta^{-1} \colon W^{r,p}_{\text{loc}} \to W^{r+2,p}_{\text{loc}} \tag{4.18}$$

is well defined and continuous for $1 < p < \infty$ and $r = -1, 0, 1, \ldots$ and that since Δ^{-1} is the solution operator on all of space and we need not worry about boundary conditions we have

$$\Delta\Delta^{-1}\alpha = \Delta^{-1}\Delta\alpha = \alpha. \tag{4.19}$$

Here $W^{r,p}(\Omega)$, $r \geq 0$, is the Sobolev space of functions whose distributional derivatives up to order r can be represented by L^p functions, and $W^{-1,p}(\Omega)$ is the dual of $W_0^{r,p^*}(\Omega), 1/p + 1/p^* = 1$, where W_0^{r,p^*} is the completion of $C_0^\infty(\Omega)$ in the W^{r,p^*} topology.

We now prove a lemma on the Hodge decomposition of a weakly convergent sequence.

LEMMA 4.1 (HODGE DECOMPOSITION). *Assume that $\alpha^\varepsilon \colon \Omega \to \Lambda^k(\Omega)$ has compact support in Ω, that*

$$\alpha^\varepsilon \rightharpoonup \bar{\alpha} \quad \textit{in } L^p(\Omega), \tag{4.20}$$

and that

$$\mathbf{d}\alpha^\varepsilon \in \textit{ a compact set in } W^{-1,p}_{\text{loc}}(\Omega), \tag{4.21}$$

for $1 < p < \infty$. Then there exist functions

$$\left.\begin{aligned} \chi^\varepsilon &\to \overline{\chi}, \\ \psi^\varepsilon &\to \psi, \\ \mathbf{d}\psi^\varepsilon &\rightharpoonup \mathbf{d}\bar{\psi}, \end{aligned}\right\} \quad \textit{in } L^p(\Omega) \tag{4.22}$$

such that

$$\alpha^\varepsilon = \mathbf{d}\psi^\varepsilon + \chi^\varepsilon \rightharpoonup \mathbf{d}\bar{\psi} + \overline{\chi} = \bar{\alpha}.$$

PROOF. We begin by writing

$$\alpha^\varepsilon = \Delta\Delta^{-1}\alpha^\varepsilon = \delta\,\mathbf{d}\Delta^{-1}\alpha^\varepsilon + \mathbf{d}\delta\Delta^{-1}\alpha^\varepsilon. \tag{4.23}$$

We note that $\mathbf{d}$ and Δ^{-1} commute:

$$\mathbf{d}\Delta^{-1}\alpha^{\varepsilon} = \Delta^{-1}\mathbf{d}\alpha^{\varepsilon}. \tag{4.24}$$

Indeed, applying $\Delta^{-1}\mathbf{d}$ to (4.23) and using $\mathbf{dd} = 0$ we have

$$\begin{aligned}\Delta^{-1}\mathbf{d}\alpha^{\varepsilon} &= \Delta^{-1}\mathbf{d}\Delta\Delta^{-1}\alpha^{\varepsilon} = \Delta^{-1}\mathbf{d}(\mathbf{d}\delta - \delta\,\mathbf{d})\Delta^{-1}\alpha^{\varepsilon} \\ &= \Delta^{-1}\mathbf{d}(\delta - \delta\,\mathbf{d})\,\mathbf{d}\Delta^{-1}\alpha^{\varepsilon} = \Delta^{-1}\Delta\,\mathbf{d}\Delta^{-1}\alpha^{\varepsilon} = \mathbf{d}\Delta^{-1}\alpha^{\varepsilon}.\end{aligned}$$

Thus we can write (4.23) as

$$\alpha^{\varepsilon} = \delta\Delta^{-1}\,\mathbf{d}\alpha^{\varepsilon} + \mathbf{d}\delta\Delta^{-1}\alpha^{\varepsilon} \equiv \chi^{\varepsilon} + \mathbf{d}\psi^{\varepsilon}. \tag{4.25}$$

Similarly,

$$\bar{\alpha} = \delta\Delta^{-1}\mathbf{d}\bar{\alpha} + \mathbf{d}\delta\Delta^{-1}\bar{\alpha} \equiv \overline{\chi} + \mathbf{d}\bar{\psi}. \tag{4.26}$$

From the continuity of Δ^{-1} from $W^{-1,p}(\Omega)$ to $W^{-1,p}(\Omega)$ we conclude that

$$\Delta^{-1}\mathbf{d}\alpha^{\varepsilon} \in \text{ a compact set in } W^{1,p}(\Omega),$$

and hence that

$$\chi^{\varepsilon} \equiv \delta\Delta^{-1}\mathbf{d}\alpha^{\varepsilon} \in \text{ a compact set in } L^{p}(\Omega).$$

Similarly, $\psi^{\varepsilon} \equiv \delta\Delta^{-1}\alpha^{\varepsilon}$ is bounded in $W^{1,p}(\Omega)$, and by compact imbedding

$$\psi^{\varepsilon} \in \text{ a compact set in } L^{p}(\Omega).$$

These compactness results and the uniqueness weak limits give us (4.22a, b). The weak convergence of $\mathbf{d}\psi^{\varepsilon} = \alpha^{\varepsilon} - \chi^{\varepsilon}$ follows immediately, and the lemma is proved.

We now prove a lemma that shows that the product of certain weakly convergent sequences converges weakly as well.

LEMMA 4.2. *For $\psi_i^{\varepsilon}\colon \Omega \to \Lambda^{s_i}(\Omega)$, suppose that*

$$\begin{aligned}&\psi_i^{\varepsilon} \to \bar{\psi}_i \quad \text{in } L^{p_i}(\Omega),\\ &\mathbf{d}\psi_i^{\varepsilon} \rightharpoonup \mathbf{d}\bar{\psi}_i \quad \text{in } L^{p_i}(\Omega), \qquad i = 1, \dots, k,\end{aligned} \tag{4.27}$$

with $1 \le 1/q_k = 1/p_1 + \cdots + 1/p_k$. Then

$$\mathbf{d}\psi_1^{\varepsilon} \wedge \cdots \wedge \mathbf{d}\psi_k^{\varepsilon} \rightharpoonup \mathbf{d}\bar{\psi}_1 \wedge \cdots \wedge \mathbf{d}\bar{\psi}_k \quad \text{in the sense of distributions,} \tag{4.28}$$

and if $q_k > 1$ then

$$\mathbf{d}\psi_1^{\varepsilon} \wedge \cdots \wedge \mathbf{d}\psi_k^{\varepsilon} \rightharpoonup \mathbf{d}\bar{\psi}_1 \wedge \cdots \wedge \mathbf{d}\bar{\psi}_k \quad \text{in } L^{q_k}(\Omega) \tag{4.29}$$

PROOF. We prove this by induction on k. The case $k = 1$ is immediate. Now assume the lemma holds for a fixed k. Let $1 \le 1/q_{k+1} = 1/q_k + 1/p_{k+1}, p_{k+1} > 1$ and $s = n - s^1 - \cdots - s_k$. First note that the convergence in the sense of distributions holds: For any $\phi\colon \Omega \to \Lambda^{s}(\Omega)$ in $C_0^{\infty}(\Omega)$, integration by parts and the fact that $\mathbf{dd}\gamma \equiv 0$ imply

$$\int_{\Omega} \mathbf{d}\psi_1^{\varepsilon} \wedge \cdots \wedge \mathbf{d}\psi_k^{\varepsilon} \wedge \mathbf{d}\psi_{k+1}^{\varepsilon} \wedge \phi = -\int_{\Omega} \mathbf{d}\psi_1^{\varepsilon} \wedge \cdots \wedge \mathbf{d}\psi_k^{\varepsilon} \wedge \psi_{k+1}^{\varepsilon} \wedge \mathbf{d}\phi.$$

This converges to the desired limit since $\mathbf{d}\psi_1^{\varepsilon} \wedge \cdots \wedge \mathbf{d}\psi_k^{\varepsilon}$ converges weakly in $L^{q_k}(\Omega)$ ($q_k > 1$) by the induction hypothesis, and ψ_{k+1}^{ε} converges strongly in $L^{p_{k+1}}(\Omega)$ (and hence in $L^{q_k^*}(\Omega)$) by (4.27a). Thus, the product converges in $L^1(\Omega)$ and hence in the sense of distributions.

If $q_{k+1} > 1$ we note that $\mathbf{d}\psi_1^\varepsilon \wedge \cdots \wedge \mathbf{d}\psi_{k+1}^\varepsilon$ is bounded in $L^{q_{k+1}}(\Omega)$ by Hölder's inequality. By Banach-Alaoglu there is an element $\Psi \in L^{q_{k+1}}(\Omega)$ and a weakly convergent subsequence such that $\mathbf{d}\psi_1^\varepsilon \wedge \cdots \wedge \mathbf{d}\psi_{k+1}^\varepsilon \rightharpoonup \Psi$. It follows that

$$\Psi = \mathbf{d}\bar{\psi}_1 \wedge \cdots \wedge \mathbf{d}\bar{\psi}_{k+1} \tag{4.30}$$

by the uniqueness of limits.

We now prove Theorem 1.1:

PROOF. Let $s = n - s_1 - \cdots - s_l$. Then for any test function $\phi \colon \Omega \to \Lambda^s(\Omega)$ in $C_0^\infty(\Omega)$ we show that

$$\int_\Omega \alpha_1^\varepsilon \wedge \cdots \wedge \alpha_l^\varepsilon \wedge \gamma\phi \to \int_\Omega \bar{\alpha}_1 \wedge \cdots \wedge \bar{\alpha}_l \wedge \phi. \tag{4.31}$$

Using the same localization arguments as in the previous section, we note that we can assume without loss of generality that the α_i^ε have compact support. With this assumption made, Lemma 4.1 gives us

$$\alpha_i^\varepsilon = \mathbf{d}\psi_i^\varepsilon + \chi_i^\varepsilon \rightharpoonup \mathbf{d}\bar{\psi} + \overline{\chi}_i = \bar{\alpha}_i,$$

where

$$\left.\begin{array}{l} \psi_i^\varepsilon \to \bar{\psi}_i \\ \chi_i^\varepsilon \to \overline{\chi}_i \end{array}\right\} \quad \text{in } L^{p_i}(\Omega), \qquad \mathbf{d}\psi_i^\varepsilon \rightharpoonup \bar{\psi}_i \quad \text{in } L^{p_i}(\Omega).$$

Thus,

$$\int_\Omega \alpha_1^\varepsilon \wedge \cdots \wedge \alpha_l^\varepsilon \wedge \phi = \int \Omega(\mathbf{d}\chi_1^\varepsilon + \psi_1^\varepsilon) \wedge \cdots \wedge (\mathbf{d}\chi_l^\varepsilon + \psi_l^\varepsilon) \wedge \phi.$$

By Lemma 4.2,

$$\mathbf{d}\psi_{i_1}^\varepsilon \wedge \cdots \wedge \mathbf{d}\psi_{i_s}^\varepsilon \rightharpoonup \mathbf{d}\bar{\psi}_{i_1}^\varepsilon \wedge \cdots \wedge \mathbf{d}\bar{\psi}_{i_s}^\varepsilon \tag{4.32}$$

for all sequences $1 \le i_1 \le \cdots \le i_s \le l$. Therefore, every term in (4.32) is a product of a strongly convergent sequence in $L^p(\Omega)$ wedged with a weakly convergent sequence in $L^{p^*}(\Omega)$, $1/p + 1/p^* = 1$, except for $\mathbf{d}\psi_1^\varepsilon \wedge \cdots \wedge \mathbf{d}\psi_l^\varepsilon$ which converges in the sense of distributions, again by Lemma 4.2. This completes the proof of the theorem.

5. Comments. We conclude with a few unrelated comments and open questions:

1. In the examples of §2 weakly convergent sequences of differential forms are constructed out of linear combinations of weakly convergent sequences of functions u^ε. Thus, we have used Theorem 1.1 as a method of generating polynomials in u of degree less than or equal to n, the dimension of the domain space. (If $l > n$, then $\alpha_1^\varepsilon \wedge \cdots \wedge \alpha_l^\varepsilon \equiv 0$.) Note that this is the maximal degree modulo coefficients depending on components of u over which we have complete control (cf. Dacorogna [**2**, p. 30]).

2. Can all weakly continuous functions under first-order differential constraints be constructed in this manner? We are unaware of any examples that cannot, but we are so far unable to prove even that all weakly continuous *quadratic* functions are of this form.

3. In the search for a more general characterization for weakly continuous functions we consider the question: Can one classify the differential operators and quadratic maps analogous to the exterior derivative and wedge product for which a generalized version of Theorem 1.1 holds?

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