Josef Ballmann
Rolf Jeltsch (Eds.)

Reprint

Nonlinear Hyperbolic Equations – Theory, Computation Methods, and Applications

Proceedings of the Second International Conference on Nonlinear Hyperbolic Problems, Aachen, FRG, March 14 to 18, 1988

Notes on Numerical Fluid Mechanics, Volume 24
(Vieweg, Braunschweig 1989)

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Friedr. Vieweg & Sohn   Braunschweig / Wiesbaden

The L^1-Norm Distinguishes the Strictly Hyperbolic from a Non-Strictly Hyperbolic Theory of the Initial Value Problem For Systems of Conservation Laws

Blake Temple*

Department of Mathematics

University of California, Davis

Davis, CA 95616

Abstract

We discuss recent work of the author in which he proves that solutions to systems of two strictly hyperbolic genuinely nonlinear conservation laws are weakly stable in the global L^1-norm. We contrast this with the theory of the initial value problem for a nonstrictly hyperbolic system in which weak stability in L^1 is shown to fail. This is understood from a study of the asymptotic wave patterns to which solutions in this problem decay as $t \to +\infty$. Since solution in both cases have been shown to be stable in the total variation and sup norms, we conclude that the L^1 estimate is the first stability result in a norm that distinguishes the strictly hyperbolic from a nonstrictly hyperbolic theory of the initial value problem.

In this talk we compare the theory of the initial value problem for a 2x2 non-strictly hyperbolic system of conservation laws to the corresponding strictly hyperbolic theory. In terms of the total variation and supnorms the theories look the same. Here we demonstrate that the theories diverge at the L^1-norm. In particular, recent work of the author gives a proof of the weak stability in the global L^1-norm for systems of two strictly hyperbolic equations. In contrast to this, a study of the asymptotic wave structures in a nonstrictly hyperbolic system leads directly to the conclusion that no such stability result holds in a special nonstrictly hyperbolic problem. We first discuss the weak stability result (see "Weak Stability in the global L^1-norm for systems of conservation laws" by Blake Temple, Davis preprint), and then we discuss the asymptotic wave patterns in a simple nonstrictly hyperbolic system with an eye toward seeing how L^1 stability fails (see "The structure of asymptotic states in a singular system of conservation laws" with E. Isaacson, Davis preprint.)

We consider the initial value problem

$$u_t + F(u)_x = 0, \qquad u = (u_1, u_2), \qquad u(x,0) = u_0(x) \qquad F = (F_1, F_2)\,. \tag{C}$$

In the strictly hyperbolic case, Glimm demonstrated in his fundamental paper of 1965 [3] that solutions of (C) generated by the random choice method are stable in the supnorm and in the total variation norm. Indeed, it is stability in the total variation that gives compactness of the approximate solutions, and this resulted in the first existence theory for systems of conservation laws. (We remark that in general we have no proof of uniqueness or continuous dependence for solutions generated by this method.) We state Glimm's result precisely [25].

<u>Theorem (Glimm 1965)</u>: Assume (C) is genuinely nonlinear and strictly hyperbolic in both characteristic fields in a neighborhood of a state $\bar{u} \in \mathbb{R}^2$. Then $\forall\, V > 0$ there exists $\delta << 1$ such that if

$$TV\,\{u_0\,(0)\} < V\,, \qquad \|u_0\,(0) - \bar{u}\|_{sup} < \delta\,,$$

then there exists a solution to (C) satisfying

*This work supported by the NSF under the grant NSF–DMS–86–13450.

$$\mathrm{TV}\,\{u(\cdot,t)\} < C\,V, \qquad \text{(TV)}$$
$$\|u(\cdot,t) - \bar{u}\|_{\sup} < C\,\delta, \qquad \text{(SUP)}$$
$$\|u(\cdot,t) - \bar{u}(\cdot,s)\|_{L^1} \le C|t-s|. \qquad \text{(LIP)}$$

Here C denotes a generic constant, TV denotes the total variation and $\| \; \|_{\sup}$ denotes the supnorm. Note that (LIP) implies that the data is taken on in the L^1 sense.

The author recently proved the following weak stability result in the global L^1-norm for solutions generated by Glimms method [24]:

$$\|u(\cdot,t) - \bar{u}\|_{L^1} \le G(t, \|u_0(\cdot) - \bar{u}\|_{L^1}) \qquad (L^1)$$

where G is an explicitly constructed smooth function satisfying $G(t,\xi) \to 0$ as $\xi \to 0$ for every fixed $t \ge 0$. Here we assume that $u_0(\pm\infty) = \bar{u}$.

We now contrast this with a corresponding existence theory for a non-strictly hyperbolic system in which (TV), (SUP) and (LIP) have been shown to hold (cf [20]), but (L^1) fails for every smooth function G satisfying $G(t,\xi) \to 0$ as $\xi \to 0$. We conclude that (L^1) gives the first stability result in a norm that distinguishes the two theories. That (L^1) fails in the next example follows directly from an understanding of the asymptotic wave structures to which solutions decay as $t \to +\infty$. This was studied in joint work with E. Isaacson, Dept. of Math., Univ. of Wyoming. Consider the 2x2 system of polymer equations:

$$s_t + f(s,c)_x = 0, \quad u = (s, cs), \quad (sc)_t + \{cf(s,c)\}_x = 0, \quad F = (f, cf). \qquad \text{(P)}$$

In general, system (P) is not strictly hyperbolic when $f(\cdot,c)$ is non-convex. E Isaacson first derived (P) from a simple two component flow problem, and he solved the corresponding nonconvex Riemann problem [4]. In [8], B. Keyfitz and H. Kranzer earlier solved the Riemann problem for a system formally equivalent to (P). In [20] the author proved a global existence theorem by Glimm's method. We state it here in order to compare it with the strictly hyperbolic case:

<u>Theorem (Te)</u>: If $u_0(\cdot)$ is initial data for (P) satisfying

$$\mathrm{TV}\,\{u_0(\cdot)\} < V < \infty, \qquad \|u_0(\cdot) - \bar{u}\|_{\sup} < \delta,$$

then there exists a global weak solution of (P) with initial data u_0 satisfying

$$\mathrm{TV}\,\{u(\cdot,t)\} < C\,V, \qquad \text{(TV)}$$
$$\|u(\cdot,t) - \bar{u}\|_{\sup} < \delta, \qquad \text{(SUP)}$$
$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1} < C|t-s|. \qquad \text{(LIP)}$$

Here total variation is measured in the singular coordinate system of Riemann invariants, and this leads to a modified convergence proof, but formally, the results look the same as in the strictly hyperbolic case of Theorem (Glimm). In joint work with E. Isaacson, we determine the asymptotic waves that these solutions decay to as $t \to +\infty$, and this leads directly to the following result which implies that the two theories diverge on the level of the L^1-norm (cf. [5]).

For the solutions u(x,t) of (P) generated by Theorem (Te) and satisfying $u_0(\pm\infty) = \bar{u}$, the L^1-norm at time t cannot be controlled by the L^1-norm at time $t = 0$, through any nonlinear function; i.e., Theorem ($\neg L^1$): The estimate (L^1) FAILS in general for every smooth G satisfying $G(t,\xi) \to 0$ as $\xi \to 0$ for each fixed t. Specifically, there exists a sequence of solutions $u^\sigma(x,t)$ of (P), $0 \le \sigma \le 1$, such that $u_0^\sigma(\pm\infty) = u_L$,

$$\lim_{\sigma\to 0} \|u_0^\sigma(\cdot) - u_L\|_{L^1} = 0 , \qquad (\neg L^1)$$

but

$$\lim_{\sigma\to 0} \|u^\sigma(\cdot,t) - u_L\|_{L^1} \ne 0 \qquad (\neg L^1)$$

at any $t > 0$.

In the next section we discuss the asymptotic states for solutions of (P) with an eye toward seeing ($\neg L^1$). We comment on the interesting role played by the admissible solutions of the Riemann problem in this nonstrictly hyperbolic problem. In section 3 we return to the strictly hyperbolic case, and discuss the proof of (L^1). The estimate (L^1) is a consequence of the author's decay result [22] which states that

$$\|u(\cdot,t) - \bar{u}\|_{sup} \le F\left(\frac{t}{\|u_0(\cdot) - \bar{u}\|_{L^1}}\right),$$

where $F(\xi)$ is an explicitly constructed function satisfying $F(\xi) \to 0$ as $\xi \to +\infty$, together with the new estimate

$$\|u(\cdot,t) - \bar{u}\|_{L^1} \le \|u_0(\cdot) - \bar{u}\|_{L^1} + C\,\delta\, t \qquad (E)$$

where δ denotes the supnorm of the initial data $u_0(\cdot)$. The details of the proof of this new estimate (E) together with a further discussion can be found in the author's paper [24].

§2 The structure of asymptotic wave patterns for (P).

We view (P) as modeling the polymer flood of an oil reservoir in one space dimension as first developed by Isaacson in [4]. By a polymer flood we mean a two component flow of immiscible fluids, oil and a mixture consisting of water together with polymer. The polymer is a thickener which moves passively with the water and which is assumed to affect the mutual flow of the two components in the porous media. Here, $s \equiv$ saturation of the aqueous phase, $c \equiv$ concentration of polymer in water, $0 \le s \le 1$, $0 \le c \le 1$, and $g(s,c) = \frac{f(s,c)}{s}$ is the particle velocity of the water. In this way (P1) represents conservation of water plus polymer, (P2) represents conservation of polymer, and $f(s,c)$ gives the fraction of the total flow associated with the aqueous component at each position x of the reservoir. The system is determined once the constitutive function $f(s,c)$ is specified. Properties of the flow are determined by quantitative properties of f, and we assume only that $f(\cdot,c)$ is S - shaped for each fixed c, and that $\frac{\partial f}{\partial c} < 0$. (See Fig. 1, cf. [4,20].) These assumptions can be justified by an argument based on Darcy's Law [4].

In this section we describe the structure of the noninteracting waves to which the solutions constructed in [20] decay as $t \to +\infty$. We then discuss the relationship between the admissible solution of a given Riemann problem (P),

$$u_0(x) = \begin{cases} u_L & \text{for } x < 0 \\ u_R & \text{for } x \ge 0 , \end{cases} \qquad (RP)$$

and the asymptotic waves to which a given solution u(x,t) of (P) satisfying

$$u_0(-\infty) = u_L ,\ u_0(+\infty) = u_R \qquad (AS)$$

decays as $t \to +\infty$. In this problem the admissible solution of the Riemann problem is the solution (shown in [4] to be unique) constructed from waves which satisfy the Lax characteristic criterion. Alternatively, these are the solutions which do not spontaneously introduce "extra" polymer into the flow over and above that accounted for in the states u_L and u_R. The noninteracting waves to which a general solution satisfying (AS) decays as $t \to +\infty$ represent an alternate solution of the Riemann problem (P), (RP) which in general is inadmissible by the Lax characteristic criterion. This is because the asymptotic state must account for the "extra" polymer contained in the initial data between $x = -\infty$ and $x = +\infty$. The conclusion then is that in contrast to the classical strictly hyperbolic theory, the asymptotic states do not depend on $u_L = u_0(-\infty)$ and $u_R = u_0(+\infty)$ alone, but on

$$c_{max} = \operatorname*{Sup}_{x} \{c_0(x)\}$$

as well. The analysis leads to the result that the solutions are not well-posed in the L^1-norm (i.e., $(\neg L^1)$ holds) even though the admissible solutions of the Riemann problem depend continuously on u_L and u_R in L^1_{loc}, and despite the fact that the solutions are Lipshitz continuous in time in the L^1-norm. Moreover, the two component flow interpretation indicates that the lack of well-posedness in one dimension may be related to fingering instabilities in higher dimensions. It also appears that well-posedness is retrieved when viscosity is not neglected. In this problem, the admissible solutions of the Riemann problem play an interesting and special role.

We first review the solution of the Riemann problem as first presented by Isaacson [4]. One can easily verify that the eigenvalues of dF (the wave speeds for system (P)) are given by

$$\lambda_s = \frac{\partial f}{\partial s}(s,c)\,, \qquad \lambda_c = \frac{f(s,c)}{s}\,,$$

and the integral curves of the corresponding eigenvectors through a state $\bar{u}$ are given by

$$R\,(\bar{u}) = \{u\colon c(u) = c(\bar{u})\}\,, \qquad R\,(\bar{u}) = \{u\colon g(u) = g(\bar{u})\}\,.$$

Because $f(\cdot,c)$ is S-shaped, it is clear that $\lambda_s = \lambda_c$ on a curve in state space labeled T for the transition curve (see Fig. 1, 2).

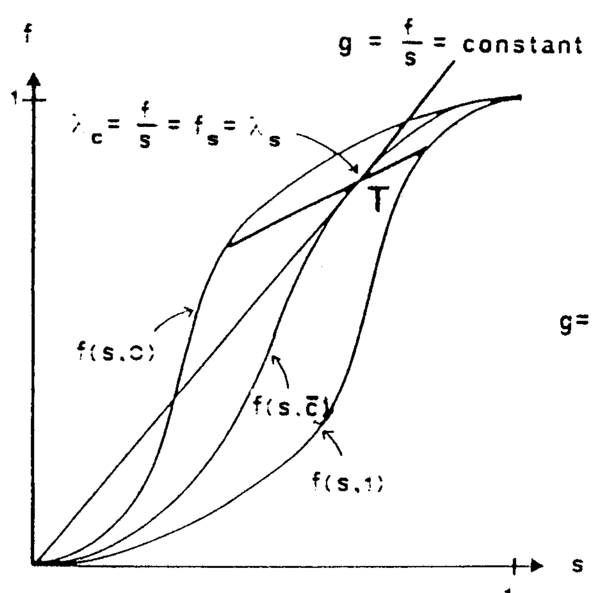

Fig. 1

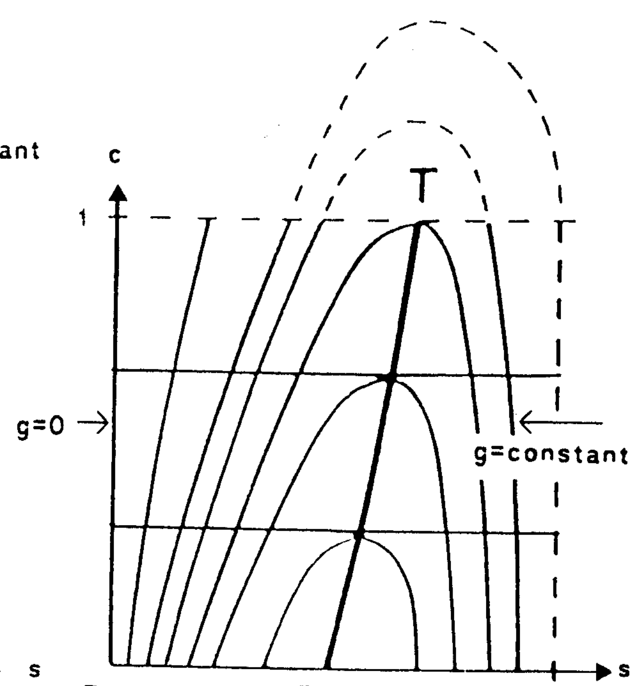

Fig. 2

For this system, the shock and rarefaction curves coincide, and the elementary waves which satisfy the Lax characteristic criterion consist of s-waves and c-waves. Here, s—waves solve the non-convex scalar conservation law which (P) reduces to when c = const., and c-waves are contact discontinuities at g = const. The Lax condition for the c-waves translates into the condition that c—waves cannot cross the Transition curve. The solution of the Riemann problem is summarized in the following theorems (see [4,8]).

Theorem (Is): For each u_L and u_R in the region $0 < s < 1$, $0 < c < 1$, there exists a unique solution to the Riemann problem (P), (RP) in the class of s-waves and c-waves. The solutions are diagramed in Figures 3 and 4. Moreover, these solutions depend continuously on u_L and u_R in L^1_{loc} at each time.

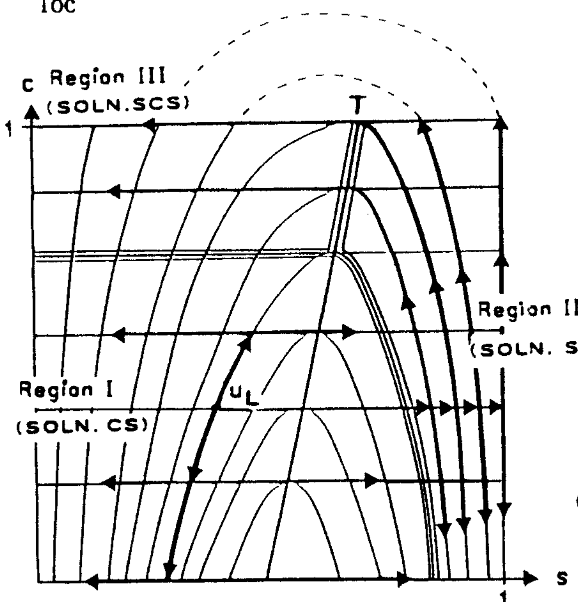

Fig. 3 Riemann problem solution for u_L left of T

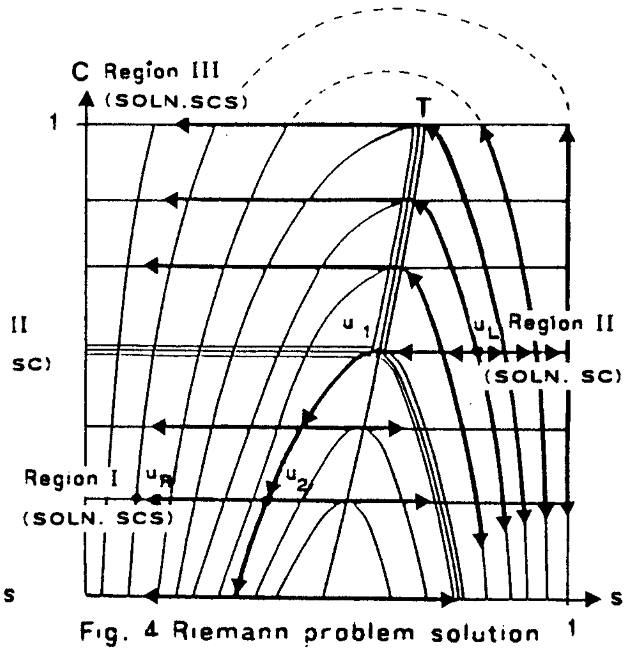

Fig. 4 Riemann problem solution for u_L right of T

The existence Theorem (Te) is obtained by extracting a convergent subsequence from approximate solutions constructed by the random choice method using the solutions of the Riemann problem generated in Theorem (Is). The proof relies on a positive non-increasing function F(t) which is defined on the approximate solutions, and which dominates the total variation of the approximate solutions at time t as measured in the singular coordinate system of Riemann invariants. Because the total variation in the conserved quantities cannot be bounded, a modified convergence proof must be given (see [20] for details). We now ask, what are the noninteracting elementary waves to which these solutions decay as $t \to +\infty$? We answer this by means of the following claim:
Let u(x,t) denote a solution generated by Theorem (Te). For a given u, let x(t) satisfy

$$\frac{dx}{dt} = g(s,c) , \qquad x(0) = x_0 ,$$

so that x(t) describes a particle path of water in the solution.

CLAIM: The particle paths are continuous curves defined and nonintersecting for all $t \geq 0$, and the value of c is constant on each particle path.

We do not give a complete proof of the CLAIM, but we argue for it as follows. Since c-waves move with speed g, we argue first that the particle paths do not cross c-waves in the weak solutions. Since the particle paths are nonintersecting in smooth solutions and Lipshitz continuous across s-waves, we conclude that the particle paths are defined and nonintersecting throughout the weak solutions. Moreover, for smooth solutions,

$$\frac{dc(x(t),t)}{dt} = c_x x_t + c_t = c_t + g\, c_x = 0$$

because equations (P2) gives

$$\begin{aligned} 0 &= c_t\, s + c\, s_t + c\, f_x + f\, c_x \\ &= s\,(c_t + g\, c_x) + c\,(s_t + f_x) \\ &= s\,(c_t + g\, c_x)\,; \end{aligned}$$

and since c is constant across s-waves and we have argued that particle paths don't cross c-waves, we conclude that c is constant on particle paths of the weak solutions. An actual proof of this is made difficult by the fact that the claim is false for approximate solutions of the random choice method. We conclude from the claim that the total variation in c is passively transported along particle paths. Thus in particular, the value

$$\bar{c} = \sup_x c_0(x)$$

satisfies

$$\bar{c} = \sup_x u(x,t)$$

for every $t \geq 0$. We now determine the asymptotic waves through the following theorem:

Theorem (Is, Te): for each $\bar{c}$, u_L and u_R in our domain, there exists a unique set of noninteracting waves taking u_L to u_R, and taking on the value $\bar{c}$ as the maximum value of c at each time. In general, these waves correspond to an inadmissible solution of the Riemann problem. Moreover, the positive nonincreasing function F(t) used in the existence theory is minimized on these waves among all sequences of elementary waves taking u_L to u_R and taking on $\bar{c}$ as the maximum value of c. These waves are diagramed in Figures 5 – 9 according to whether u_L ties in regions A, B or C determined by the value of $\bar{c}$ (see Fig. 5).

We conclude from Theorem (Is, Te) that the solutions generated in Theorem (Te) decay to the noninteracting waves determined by $u_0(-\infty) \equiv u_L$, $u_1(+\infty) \equiv u_R$ and $\bar{c} = \operatorname{Max}_x c_0(x)$. A proof here would be complete were one to show rigorously that F(t) decreases to its minimum possible value in each solution.

In order to contrast the situation here with the classical strictly hyperbolic case, consider the example of the asymptotic state corresponding to the values $u_R \equiv u_L$ and $\bar{c} = c(\bar{u})$ diagramed in Figure 8, and corresponding to u_L in Region B. This is the region for which the structure of asymptotic states differ strikingly from the structure of asymptotic states in a strictly hyperbolic problem. For example, assume that the initial data is given by

$$u_0(x) = \begin{cases} u_L & x \leq 0 \ , \\ \bar{u} & 0 < x < \sigma \ , \\ u_L & x \geq \sigma \ . \end{cases}$$

The exact solution, which corresponds to the asymptotic state $u_0(-\infty) = u_L = u_0(+\infty)$, $\bar{c} = c(\bar{u})$, is drawn in Figure 10. In a strictly hyperbolic problem such a solution would decay to zero, because the admissible solution of the Riemann problem for $u_0(-\infty) = u_L = u_0(+\infty)$ is the constant solution $u \equiv u_L$ (cf [2, 11–14]). For (P), however, the solution decays to a solution containing two strong nonlinear s–waves separated by a contact discontinuity. We can now observe Theorem ($\neg L^1$) by taking the limit $\sigma \to 0$. Indeed, when $\sigma = 0$, the solution is the constant state $u \equiv u_L$, but for $\sigma > 0$ the solution at times $t > 0$ is far from the solution $u \equiv u_L$ in the L^1–norm. This occurs despite the Lipshitz continuity of the solutions in L^1. We conclude that a small amount of polymer at $x = 0$, $t = 0$ drastically alters the flow in this model.

The admissible solutions of the Riemann problem play a different role in the theory of this non–strictly hyperbolic problem than they play in the classical strictly hyperbolic theory of Lax. We explore this difference in the following comments.

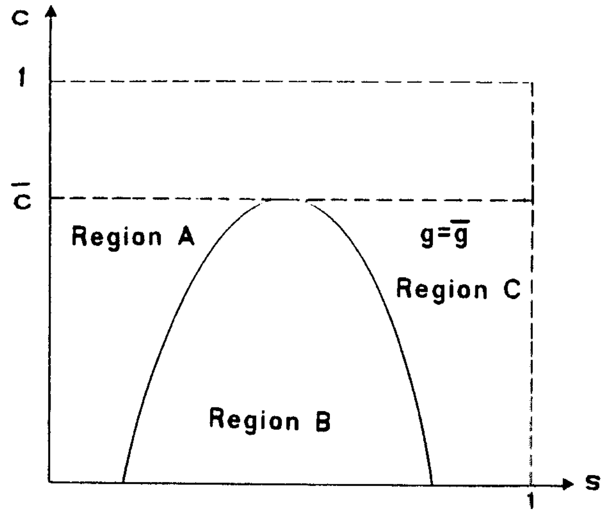

Fig. 5 The three regions for the asymptotic states corresponding to $\bar{c} = \max_x [c_0(x)]$, sc-plane

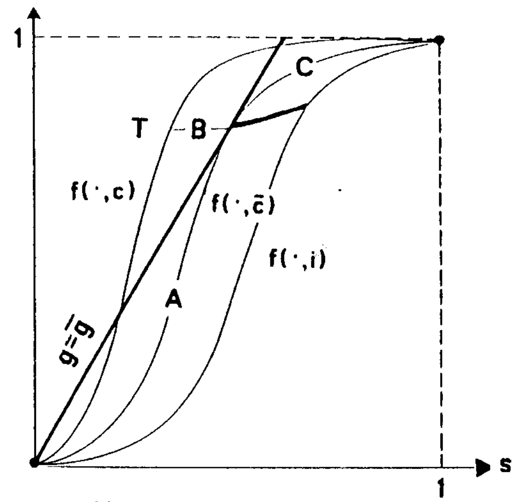

Fig. 6 The three regions for the asymptotic states corresponding to $\bar{c} = \max_x [c_0(x)]$, sf-plane

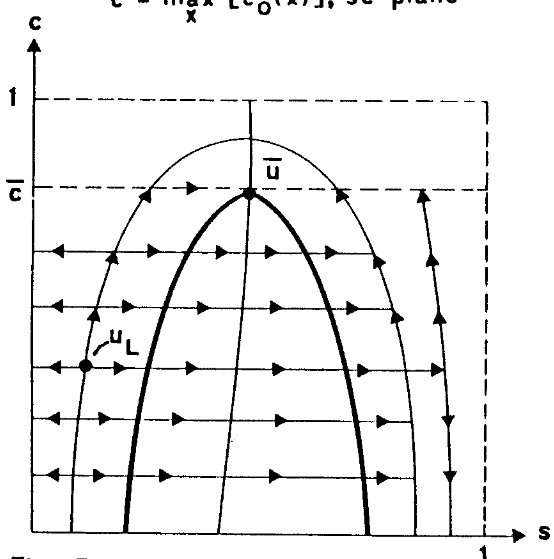

Fig. 7 The asymptotic states for $u_L \epsilon$ A

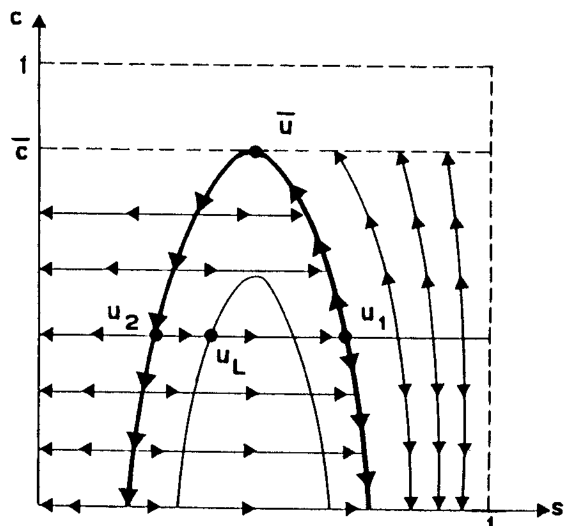

Fig. 8 The asymptotic state for $u_L \epsilon$ B

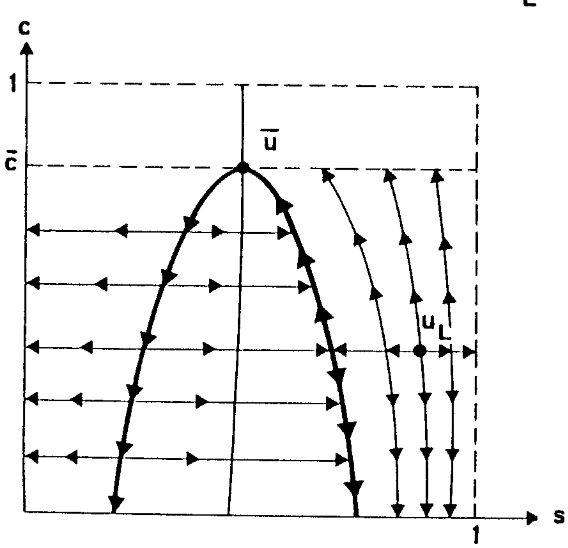

Fig. 9 The asymptotic state for $u_L \epsilon$ C

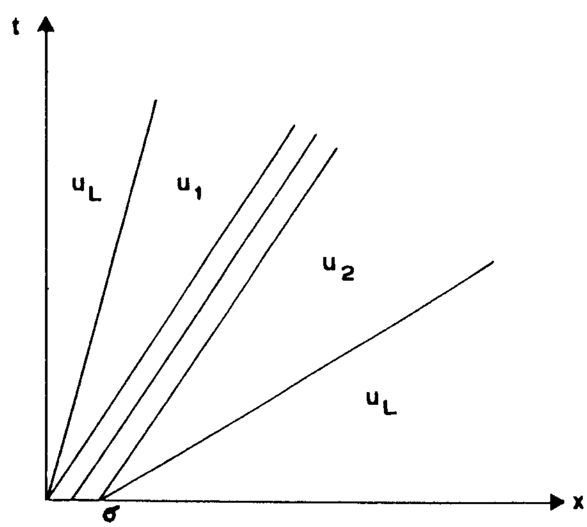

Fig. 10 The asymptotic state for u_L and $\bar{c}$ diagrammed in Fig. 8

Comments

(1) The classical strictly hyperbolic theory of conservation laws is a generalization of the theory of Euler's equations in gas dynamics. One can take the point of view that the Riemann problem is relevant because it represents the local asymptotic state in a general flow. By the scale invariance of the equations, the flow should locally look like an asymptotic state, and Glimm's theorem can be viewed as a justification of this picture; the random choice method replaces the solution locally by an asymptotic state. For system (P), the asymptotic solutions are not the admissible solution of the Riemann problem, but in fact one can speed up the convergence of the random choice method by replacing the solution of the Riemann problem by the asymptotic solution in each cell. Since the limit solution in this case conserves c–values, we expect this to generate the same solution as that generated in Theorem (P). The admissible solutions of the Riemann problem are special in that all asymptotic wave structures are concatenations of these. Thus the admissible solutions can be characterized as the only solutions of the Riemann problem which give convergence to the polymer conserving solution by the random choice method, but which require only the values of u_L and u_R in each cell, and not the further information of $\bar{c}$.

(2) From the example above, it appears that continuous dependence in L^1 is recovered when diffusion is not neglected. For example, if ϵu_{xx} is added to the right hand side of (P), then we expect the spike in Fig. 10 to diffuse away as $t \to +\infty$, and the solution to decay to the constant state $u \equiv u_L$. Moreover the rate of decay would increase as $\sigma \to 0$, so we expect continuous dependence in L^1 as $\sigma \to 0$.

(3) We believe that the weak solutions generated by Theorem (Te) are limits of the viscously perturbed equations as $\epsilon \to 0$. If this is indeed the case (we have no proof), then we can also characterize the admissible solutions of the Riemann problem as follows: Let $u^\epsilon(x,t)$ denote a solution of the initial value problem for the viscous equation

$$u_t + f(u)_x = \epsilon u_{xx}, \qquad (P^\epsilon)$$

where u and f are given in (P). Let Q_1 and Q_2 denote the asymptotic states defined by

$$Q_1 \equiv \lim_{t\to\infty} \lim_{\epsilon\to 0} u^\epsilon,$$

$$Q_2 \equiv \lim_{\epsilon\to 0} \lim_{t\to\infty} u^\epsilon. \qquad (*)$$

If solutions of (P) are limits of solutions of (P^ϵ) as $\epsilon \to 0$, then Q_1 is the actual asymptotic solution determined by $u_0(-\infty) = u_L$, $u_0(+\infty) = u_R$ and $\bar{c} = \text{Max } c_0(x)$. However, our example indicates that the limit Q_2 should be the admissible solution of the Riemann problem $[u_L, u_R]$. In this case, the admissible solutions of the Riemann problem are special because $Q_1 \equiv Q_2$ only when the asymptotic state is the admissible solution of the Riemann problem. Thus the admissible solutions are the ones for which it is valid to interchange the limits in (*). (This comment was suggested to the author by Philip Collela of Lawrence Livermore Laboratories).

(4) In the polymer flood interpretation of (P) it is clear that the narrow "spike" in the example of Figure 10 is unstable to figuring in higher dimension. We wonder whether a lack of continuous dependence corresponds to the presence of higher dimensional instabilities in some general setting.

In conclusion, we comment that system (P) probably represents the simplest setting in which one finds a singular hyperbolic problem. It is surprising that one can give an almost complete analysis of the initial value problem in this case. We hope that this study of the Riemann problem and the structure of asymptotic states can help to shed light on the role of admissibly criteria and the non–uniqueness of Riemann problem solutions in more complicated problems in which strict hyperbolicity is lost.

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