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# The Structure of Asymptotic States in a Singular System of Conservation Laws

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We determine the structure of the nonlinear waves to which solutions of a nonstrictly hyperbolic system decay as $t \to +\infty$. The waves in general are not the same as the waves that solve the Riemann problem for the states at $x = \pm\infty$, and solutions do not depend continuously on initial values in the $L^1$-norm. The role of the Lax admissibility condition is explored. 

## INTRODUCTION

We consider the $2 \times 2$ system of conservation laws which model the polymer flood of an oil reservoir. These equations are strictly hyperbolic everywhere except along a curve in state space where the wave speeds in the problem coincide. The Riemann problem and Cauchy problem for this system were solved in [4, 8, 20]. The Lax characteristic condition was used as the admissibility criterion for solutions of the Riemann problem, and the Cauchy problem was solved by demonstrating the convergence of the random choice method. Here we describe the noninteracting waves to which solutions of the Cauchy problem decay asymptotically as $t \to +\infty$. In contrast to the strictly hyperbolic case [2, 11–14] the waves in the asymptotic solution for this nonstrictly hyperbolic problem are, in general, different from the waves in the admissible solution of the Riemann problem with left state $u_L = u_0\ (-\infty)$ and right state $u_R = u_0\ (+\infty)$. (Here $u_0(x)$ denotes the initial data for the Cauchy problem.) Indeed, the asymptotic solution, which is determined by $u_L$, $u_R$, and the initial maximum value of $c$ (the concentration of polymer), can in fact be an inadmissible solution of the Riemann problem $[u_L, u_R]$. An immediate consequence of the analysis



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is that although the admissible solutions of the Riemann problem depend continuously on $u_L$ and $u_R$, and although each solution of the initial value problem is Lipschitz continuous in time in the $L^1$-norm, the initial value problem is not well-posed in $L^1$. In fact, continuous dependence on initial values fails in every $L^p$ for this one-dimensional hyperbolic problem. This lack of continuous dependence parallels the presence of fingering instabilities in the higher dimensional problem. We imagine that continuous dependence is recovered when diffusion is not neglected. However, to our knowledge, this is the first time such a lack of well-posedness has been observed in a pure first order system of hyperbolic conservation laws. In particular, this example violates the stability result proved in [22] for strictly hyperbolic systems.

The analysis also highlights the role of the Lax admissibility criterion in this non-strictly hyperbolic problem. In contrast to the classical rarefaction shocks which violate the Lax condition, the asymptotic solutions of the Riemann problem for this system are not unstable solutions which never appear, but rather are, in general, solutions which are incompatible with the Riemann data in that they spontaneously introduce polymer into the problem. We can improve the convergence of the random choice method by replacing the admissible solution of the Riemann problem in each cell by the asymptotic solution determined by the right and left cell states together with the maximum value of $c$ in each cell. In this case the analysis in [20] can be applied essentially unchanged to obtain convergence of this modified method—and since both the original and modified methods conserve polymer, we expect that both methods generate the same weak solution. From this point of view, the admissible solution of the Riemann problem is unique among all solutions of the Riemann problem which generate the polymer conserving solutions in the random choice method, in that it depends only on left and right cell states, and not on the additional information of the $c$-values in each cell. Another comment concerning the role of the Lax admissibility criterion in this nonstrictly hyperbolic problem seems relevant. If we perturb the system of conservation laws in this example by a small viscosity term $\varepsilon u_{xx}$ on the right-hand side, we argue (see Section 3) that continuous dependence in $L^1$ should be recovered. Moreover, we believe that the weak solutions generated in [20] are limits of the viscous equation as $\varepsilon \to 0$ (at least for some viscosity matrices. We have no proof of this, and the claim must depend on the choice of viscosity matrix. For the sake of discussion, we make the claim for the identity matrix). Assuming this, we can characterize the admissible solutions of the Riemann

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problem as the only asymptotic states that persist under the interchange of limits $t \to +\infty$ and $\varepsilon \to 0$. We wonder whether this perspective can be of help in determining admissibility criteria for problems with more complicated hyperbolic singularities [6, 7, 15, 18].

We note that not all of the steps in the construction of the asymptotic solutions are obtained rigorously. Our procedure is as follows: first we argue from physical considerations that the maximum concentration of polymer is constant in solutions generated by the random choice method. In fact it is easy to see that this maximum value is nonincreasing even in the approximate solutions of the random choice method [cf., 5, 20], but to show rigorously that it is nondecreasing in the limit solution requires further analysis. Assuming that the maximum concentration of polymer is constant in the weak solutions, we then show that there exists a unique set of noninteracting waves which take $u_L = u(-\infty)$ to $u_R = u(+\infty)$ and which take on the same maximum value $\bar{c}$ as does the initial data. We then claim that these must be the noninteracting waves to which the solution decays asymptotically as $t \to +\infty$. In support of this claim we give a proof that the function $F(t)$, which was shown in [20] to be a positive decreasing function for solutions generated by the random choice method, is in fact minimized on the asymptotic solution. This minimum is taken over all sequences of waves which take $u_L$ to $u_R$ and which also have $\bar{c}$ as the maximum value of the concentration of polymer. Thus assuming that $c$ is conserved in the weak solutions, the only step required to make the argument rigorous is to show that $F(t)$ decreases to its minimum possible value in each solution.

## 1. Preliminaries

We study the $2 \times 2$ system of polymer equations first described by Isaacson [4],

$$\begin{aligned} s_t + (sg)_x &= 0, \\ (cs)_t + (csg)_x &= 0, \end{aligned} \tag{1.1}$$

where $s$ = saturation of water, $(1 - s)$ = saturation of oil, $c$ = concentration of polymer, and $g$ = particle velocity of the aqueous phase. Here $s = s(x, t)$, $c = c(x, t)$, $g = g(s, c)$, and $-\infty < x < +\infty$, $t > 0$, $0 < s < 1$, $0 < c < 1$. The polymer is assumed to move passively with the water (see [4, 20] for details). We let $u = (s, c)$ and consider the initial value problem (1.1) together with

$$u_0(x) = u(x,0), \tag{1.2}$$

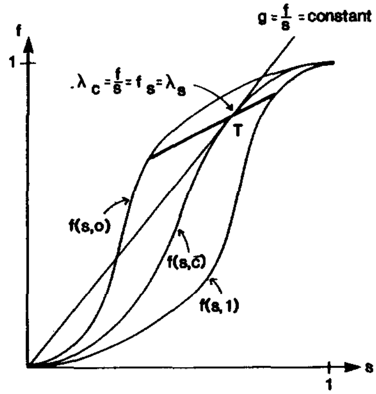


FIGURE 1

which is called the Riemann problem when

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \tag{1.3}$$

We let $f(s, c) = sg(s, c)$, in which case the eigenvalues or wave speeds for system (1.1) are $\lambda_s = f_s$ and $\lambda_c = f/s = g$. Elementary waves corresponding to $\lambda_s$, $\lambda_c$ are called $s$-waves and $c$-waves, respectively. We assume for a constitutive assumption that $f(\cdot, c)$ is $S$-shaped, and that $f(s, c_2) < f(s, c_1)$ for $c_2 > c_1$ (see Fig. 1 and [4, 20] for details). Because $f(\cdot, c)$ is $S$-shaped, there is a one-dimensional curve in $sc$-space on which $\lambda_s = \lambda_c$, and we call this the transition curve. The integral curves of the eigenvectors for the

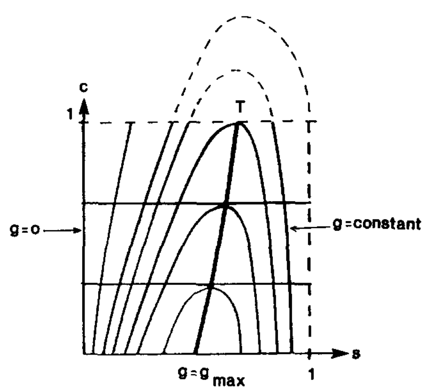


FIGURE 2

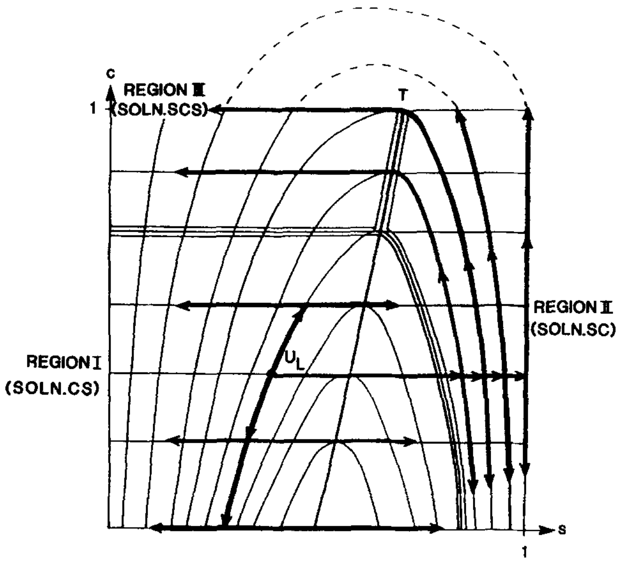


FIG. 3. Riemann problem solution for $u_L$ left of $T$.

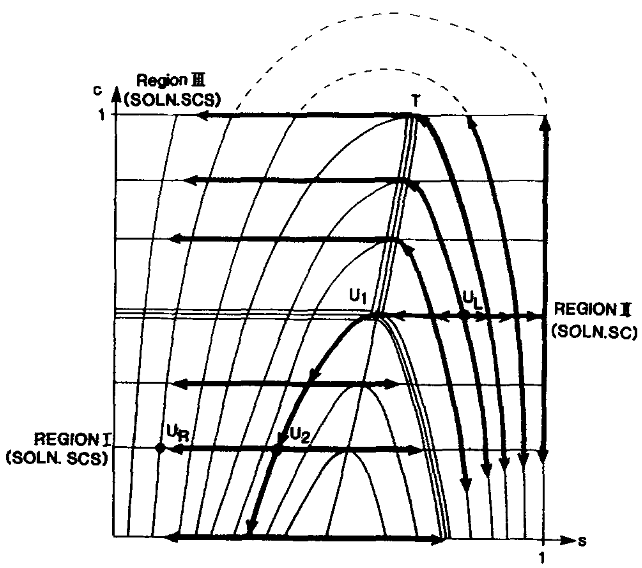


FIG. 4. Riemann problem solution for $u_L$ right of $T$.

fields $\lambda_s$ and $\lambda_c$ are given by $c = \text{constant}$ and $g = \text{constant}$, respectively, so that $\lambda_s$ corresponds to a line field and $\lambda_c$ to a contact field in the sense of [23] (see Fig. 2). The Riemann problem (1.1), (1.3) was solved by E. Isaacson [4] in terms of elementary waves. (The Riemann problem for a system formally equivalent to (1.1) was first solved in [8].) The solution is reproduced in Figs. 3 and 4. Here the directed curve that leads from $u_L$ to $u_R$ in these diagrams traverses the elementary waves that solve the Riemann problem. The Lax characteristic condition was used as an admissibility criterion, and this translated into the condition that $c$-waves cannot cross the transition curve. We note that every solution going from $u_L$ to $u_R$ is of the form an $s$-wave followed by a $c$-wave followed by an $s$-wave. Moreover, the solution of the Riemann problem at time $t > 0$ depends continuously in $L^1_{\text{loc}}$ on the initial states $u_L$ and $u_R$. We refer to [4, 20] for details.

Temple showed in [20] that there exists a singular transformation

$$\psi: (s, c) \mapsto (z, c), \tag{1.4}$$

such that, if $TV\{\psi \circ u_0\} < \infty$, and the sampling is random in space as well as time, then the random choice method converges (modulo a subsequence) to a global weak solution of the initial value problem (we refer to [20] for details). The idea was to construct a functional $F$ on sequences of elementary waves $\gamma_1, \ldots, \gamma_n$ by defining

$$F(\gamma) = \begin{cases} 2|z(u_L) - z(u_R)| & \text{if } \gamma \text{ is a } c\text{-wave with } s(u_R) < s(u_L), \\ 4|z(u_L) - z(u_R)| & \text{if } \gamma \text{ is a } c\text{-wave with } s(u_R) > s(u_L), \\ |z(u_L) - z(u_R)| & \text{if } \gamma \text{ is an } s\text{-wave,} \end{cases} \tag{1.5}$$

$$F(\gamma_1 \ldots \gamma_n) = F(\gamma_1) + F(\gamma_2) + \cdots + F(\gamma_n), \tag{1.6}$$

where $u_L$ and $u_R$ are the left and right states of the elementary wave $\gamma$. Then for an approximate solution of the random choice method, it is shown that $TV\{c(\cdot, t)\}$ and $F(t)$ are positive decreasing functions, where $F(t)$ is the $F$ value of the sequence of elementary waves given in the approximate solution at time $t$. This gives a total variation bound in $zc$-space and leads to the results in [20]. Here we wish to determine the noninteracting waves to which the above weak solutions decay asymptotically as $t \to +\infty$.

## 2. The Asymptotic Solutions

We determine the asymptotic solutions by means of the following claim: Let $u(x, t) = (s(x, t), c(x, t))$ denote a weak solution of (1.1), (1.2) generated by the random choice method in [20]. Let

$$\bar{c}(t) = \sup\{c(x,t): -\infty < x < +\infty\} = C_{\max}.$$

CLAIM 1. The function $\bar{c}(t)$ is constant in time.

We do not prove this claim, but we indicate its truth with the following physical considerations. Let $x(t)$ denote a solution of the ordinary differential equation

$$\frac{dx}{dt} = g(s(x,t), c(x,t)), \tag{2.1}$$
$$x(0) = x_0.$$

Since $g$ is the particle velocity of the aqueous phase, (2.1) defines the particle paths. Since our choice of Riemann problems conserves $c$, we expect these particle paths to be nonintersecting and defined for all time in the weak solutions generated by the random choice method in [20]. Moreover, for smooth solutions

$$\frac{d}{dt}c(x(t),t) = (c_x g) + c_t.$$

But from the second equation in (1.1),

$$s(c_t + gc_x) + (s_t + (sg)_x)c = 0,$$

so that by the first equation in (1.1),

$$c_t + gc_x = 0,$$

and $c$ is constant on particle paths. Furthermore, in discontinuous solutions, the particle paths should never cross a $c$-wave because a $c$-wave also propagates with speed $g$; and, moreover, $c$ is constant across $s$-waves. Thus we expect that $c$ is constant on particle paths of the weak solutions as well. We conclude that $\bar{c}$ is constant in time. The difficulty in proving these statements arises from the fact that sampling errors cause Claim 1 to fail in the approximate solutions of the random choice method. The above remarks actually indicate the following claim which strengthens Claim 1:

CLAIM 2. In the weak solutions generated by the random choice method in [20], the particle paths given in (2.1) are defined and nonintersecting for all time, and $c$ is constant along these particle paths.

Thus Claim 2 implies that the total variation in $c$ is passively transported along particle paths. We determine the asymptotic solutions by the following theorem:

THEOREM 1. *For every pair of states $u_L$ and $u_R$ and value $\bar{c}$, there exists a unique set of noninteracting waves which take $u_L$ to $u_R$, which take on $\bar{c}$ as the maximum value of $c$, and which minimize $F$. These waves are unique in the class of admissible waves that take $u_L$ to $u_R$ and that take on $\bar{c}$ as a maximum value of $c$.*

For every $u_L$, $u_R$, and $\bar{c}$, the solutions of Theorem 1 are diagrammed in Figs. 7, 9. Here $\bar{u}$ denotes the state which lies at the intersection of $c = \bar{c}$ and the transition curve, and $\bar{g} = g(\bar{u})$. The diagrams are classified according to whether $u_L$ lies in one of the following three regions $A$, $B$, or $C$, determined by $\bar{c}$ (see Figs. 5, 6):

$$\text{Region } A = \{ u_L \colon g(u_L) < \bar{g} \text{ and } u_L \text{ lies left of } T \},$$
$$\text{Region } B = \{ u_L \colon g(u_L) > \bar{g} \},$$
$$\text{Region } C = \{ u_L \colon g(u_L) < \bar{g} \text{ and } u_L \text{ lies right of } T \}.$$

The waves in a solution are traversed by the directed curve that takes $u_L$ to $u_R$. The solutions of Figs. 7–9 correspond to either the admissible solution of the Riemann problem $[u_L, u_R]$, or else to a solution of the form $S_1C_1C_2S_2$, where $S_1C_1$ is the admissible solution of the Riemann problem $[u_L, \bar{u}]$ and $C_2S_2$ is the admissible solution of the Riemann problem $[\bar{u}, u_R]$.

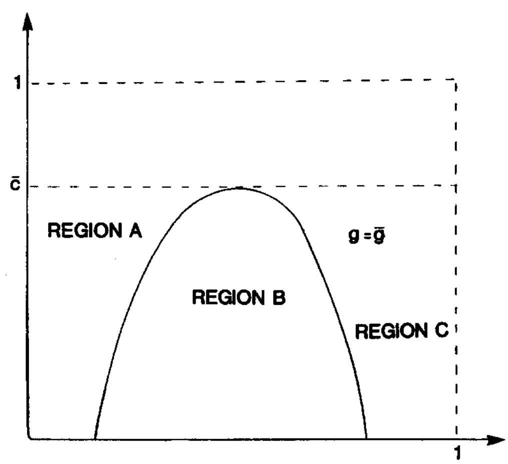


FIG. 5. The three regions for the asymptotic states corresponding to $\bar{c} = \max_x[c_0(x)]$, $sc$-plane.

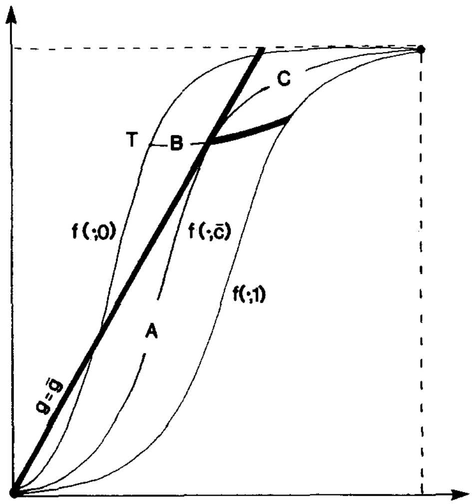


FIG. 6. The three regions for the asymptotic states corresponding to $\bar{c} = \max_x[c_0(x)]$, $sf$-plane.

We call the "asymptotic state" associated to $u_L$, $u_R$ and $\bar{c}$ that solution which is either the solution of the Riemann problem or the solution of type $S_1C_1C_2S_2$ as determined by Theorem 2 and Figs. 7–9. In fact, the asymptotic state takes on the value $c_{\max} = \bar{c}$ only in the sense that we can replace the $c$-wave in the asymptotic state by $c$-waves at the same $g$-value (and hence the same speed) which take on the value $\bar{c}$. When the asymptotic state is viewed as a solution of the Riemann problem, these extra waves travel at the speed of the $c$-wave in the asymptotic state and thus are not observed. We presently discuss the sense in which a general solution decays to the asymptotic state determined by the initial data. We first note that for $u_L$ in Region $A$, the asymptotic state agrees with the admissible solution of the Riemann problem, but when $u_L$ lies in Regions $B$ or $C$, the solution can be strikingly different from the admissible solution. To prove that the waves given in Figs. 7–9 are noninteracting is a matter of checking that the wave speeds are increasing from left to right; verifying that the asymptotic states give the only sequences of noninteracting waves taking $u_L$ to $u_R$ with $c_{\max} = \bar{c}$ is a matter of carrying out the analysis presented in [4] for ensuring uniqueness of admissible Riemann problem solutions. We omit the details.

We now discuss the sense in which a general solution will decay to the asymptotic state. Assume that $u(x, t)$ is an actual solution generated by the random choice method in [20] satisfying $u_L = u_0\ (-\infty)$, $u_R = u_0\ (+\infty)$, and $\bar{c} = \sup\{c_0(x)\colon -\infty < x < +\infty\}$. In this case, the solution will decay

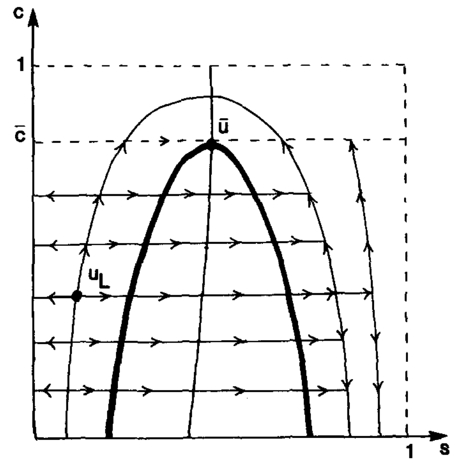


FIG. 7. The asymptotic states for $u_L \in A$.

to the corresponding asymptotic state given in Figs. 7–9. However, an actual solution will decay asymptotically to the waves given by the asymptotic state except that the intermediate $c$-wave appearing in the solution asymptotically will not in general be a sharp discontinuity. Rather, the intermediate wave will consist of states on $g = \bar{g}$, these states including the state $\bar{u}$ whenever the transition curve is crossed. Thus in the actual asymptotic solution, the intermediate wave will be a concatenation of admissible $c$-waves; and the entire asymptotic solution will be the concate-

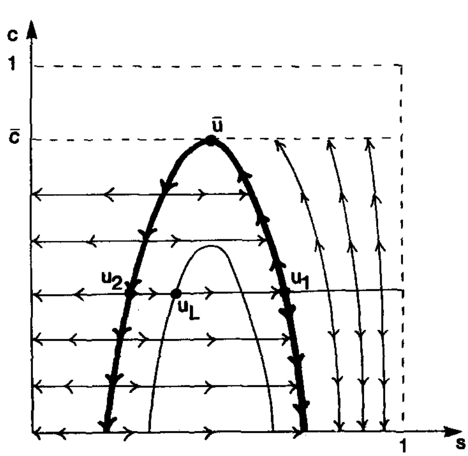


FIG. 8. The asymptotic states for $u_L \in B$.

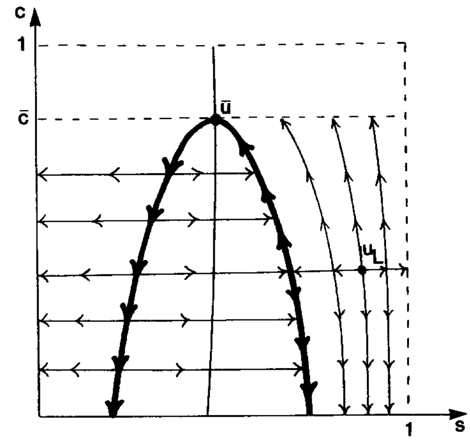


FIG. 9. The asymptotic states for $u_L \in C$.

nation of the two admissible solutions of the Riemann problem $[u_L, \bar{u}]$, $[\bar{u}, u_R]$, with the possible addition of intermediate $c$-waves lying on $g = \bar{g}$. By Claim 2, the $c$-values in a solution are passively transported along particle paths. Thus the variation in $c$ for the actual asymptotic solution must be carried by values of $u$ appearing on $g = \bar{g}$, and so this variation must agree with the variation of $c$ initially. The following theorem gives analytical evidence for the claim that the solutions generated in [20] do indeed decay in the above sense to the asymptotic states given in Theorem 1 as $t \to +\infty$.

THEOREM 2. *Among all connected sequences of admissible s-waves and c-waves which take* $u_L$ *to* $u_R$ *and which take on* $\bar{c}$ *as a maximum value of c, the functional F defined in* (1.5), (1.6) *is minimized on the asymptotic state. Moreover, among all sequences of waves having* $c_{\max} = \bar{c}$ *and having a given total variation in c, F is minimized on an asymptotic solution constructed from the asymptotic state by replacing the c-wave by a sequence of c-waves which lie at the same g-value as the c-wave in the asymptotic state and which account for the initial total variation in c which is conserved.*

Since $F$ is positive decreasing in approximate solutions of the random choice method, Theorems 1 and 2 argue strongly that solutions decay to the asymptotic state given in Figs. 7–9 in the sense discussed above. We now construct a proof of Theorem 2 using the results in [20]. First, one can verify from Figs. 7–9 that the asymptotic state is either the admissible solution of the Riemann problem, or else it is the solution $S_1C_1C_2S_2$, where $S_1C_1$ solves the Riemann problem $[u_L, \bar{u}]$ and $C_2S_2$ solves the Riemann problem $[\bar{u}, u_R]$. (We use the notation of [20].) Again, we call these the asymptotic states even though an actual solution asymptotically looks like the solution constructed from the asymptotic state by replacing the $c$-wave in the asymptotic state by a sequence of $c$-waves at the same $g$-value which account for the total variation in $c$ which is conserved in the solutions. If, however, $F$ is minimized on the asymptotic state among all connected elementary waves taking $u_L$ to $u_R$ with $c_{\max} = \bar{c}$, then it is clear that among all sequences with a given total variation in $c$, $F$ is minimized when $c$-waves with this total variation are included in the $c$-wave of the asymptotic solution. Thus it suffices to show that among all sequences of connected elementary waves $\gamma_1 \ldots \gamma_n$ having $c_{\max} = \bar{c}$, $F$ is minimized on the asymptotic state. In the case when the asymptotic state is the solution of the Riemann problem, this is just Lemma 5.1 of [20]. Thus we need only consider the case where the asymptotic waves are given by $S_1C_1C_2S_2$, as above. Since $\gamma_1, \ldots, \gamma_n$ have $c_{\max} = \bar{c}$, we can partition $\gamma_1 \ldots \gamma_n$ into

$$\gamma_1 \ldots \gamma_n = \alpha_1 \ldots \alpha_p \beta_{p+1} \ldots \beta_n,$$

where $\alpha_1 \ldots \alpha_p$ take $u_L$ to $\bar{u}$, $\beta_{p+1} \ldots \beta_n$ take $\bar{u}$ to $u_R$, and $c(\bar{u}) = \bar{c}$. By Lemma 5.1 of [20], the $F$ value of the sequence decreases when we replace $\alpha_1 \ldots \alpha_p$ by the solution $S_1C_1S_1$ of the Riemann problem $[u_L, u_R]$, and we replace $\beta_{p+1} \ldots \beta_n$ by the solution $\tilde{S}_2\tilde{C}_2\tilde{S}_2$ of the Riemann problem $[\bar{u}, u_R]$. More precisely,

$$F(\gamma_1 \ldots \gamma_n) \geq F\big(S_1C_1S_1\tilde{S}_2\tilde{C}_2\tilde{S}_2\big).$$

Now one can verify that our assumption that the asymptotic state is not the admissible solution of the Riemann problem implies that either $S_1C_1$ takes $u_L$ to $\bar{u}$ or else $\tilde{C}_2\tilde{S}$ take $\bar{u}$ to $u_R$. This follows from the fact that one of the two Riemann problems must contain waves that cross the transition curve. Without loss of generality, assume $S_1C_1$ takes $u_L$ to $\bar{u}$. In this case

$$F\big(S_1\tilde{S}_2\tilde{C}_2\tilde{S}_2\big) \geq F(C_2S_2)$$

because $C_2S_2$ solves the Riemann problem for $[\bar{u}, u_R]$. Thus

$$F(\gamma_1 \ldots \gamma_n) \geq F(S_1C_1C_2S_2),$$

where $S_1C_1C_2S_2$ is the asymptotic state. This establishes Theorem 2.

## 3. Failure of Well-Posedness

We consider the states $u_L$, $u_1$, $\bar{u}$, and $u_2$ diagrammed in Figs. 8 and 10. The asymptotic solution for $u_L = u_R$ is diagrammed in Fig. 10. Here, the

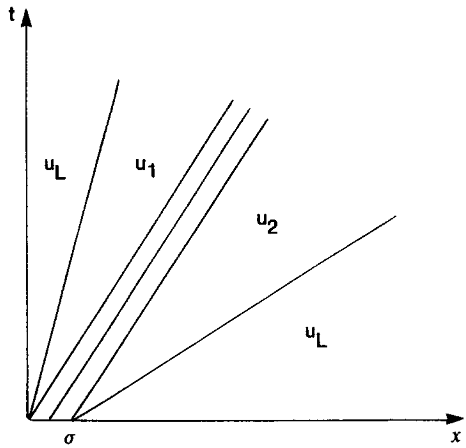


FIG. 10. The asymptotic state for $u_L$ and $\bar{c}$ diagrammed in Fig. 8.

states that lie between $u_1$ and $u_2$ in the solution are values of $u$ on $g(u) = \bar{g}$. Let $u_\sigma(x, t)$ denote this solution when the $c$-wave has width $\sigma > 0$. Lack of continuous dependence is observed by letting $\sigma \to 0$, since this asymptotic solution does not tend to the solution of the Riemann problem for $u_L = u_R$, which is the constant solution $u = u_L$. Nevertheless, $u_\sigma(x, t)$ is an admissible solution because it is the concatenation of the two admissible Riemann problem solutions $[u_L, \bar{u}]$ and $[\bar{u}, u_R]$. To understand the lack of well-posedness, consider what the solution $u_\sigma(x, t)$ represents physically. Since

$$s(u_2) < s(u_L) < s(u_1),$$

the solution $u_\sigma(x, t)$ represents a solution which is displacing oil; i.e., oil is being displaced from the $u_1$ region to the $u_2$ region in Fig. 10. This is due to the presence of a small strip of polymer between $u_1$ and $u_2$ which is enhancing the displacement of the oil. In two dimensions, this process is unstable because the low viscosity fluids are displacing the higher viscosity fluids; i.e., this represents a fingering instability. Thus we expect that in two dimensions fingering would occur, the interface of polymer would collapse, and the solution would quickly evolve into a solution near the constant solution $u_L$ when $\sigma \ll 1$. In this way the higher dimensional instability helps restore well-posedness. Also, we expect that if diffusion were present, then the spike of polymer between $u_1$ and $u_2$ would diffuse and again the solution would quickly decay to approximately the constant $u = u_L$ when $\sigma \ll 1$. Thus we also expect that well-posedness is restored when diffusion is not neglected. At this time we do not have proofs for either of these statements. It is interesting to note that the lack of continuous dependence described above occurs despite the fact that each weak solution generated in [20] is Lipschitz continuous in time in the $L^1$ norm.

We believe that the weak solutions generated in [20] are limits of the viscously perturbed equation as $\varepsilon \to 0$. If this is indeed the case (we have no proof), then we can also characterize the admissible solution of the Riemann problem as follows: Let $u^\varepsilon(x, t)$ denote a solution of the initial value problem for the viscous equation

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{3.1}$$

where $u = (s, cs)$ and $f$ are given in (1.1). Let $\mathscr{A}_1$ and $\mathscr{A}_2$ denote the asymptotic states defined by

$$\mathscr{A}_1 \equiv \lim_{t \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon,$$

$$\mathscr{A}_2 \equiv \lim_{\varepsilon \to 0} \lim_{t \to \infty} u^\varepsilon.$$

If solutions of (1.1) are limits of solutions of (3.1) as $\varepsilon \to 0$, then $\mathscr{A}_1$ is the actual asymptotic solution determined by $u_0(-\infty) = u_L$, $u_0(+\infty) = u_R$ and $\bar{c} = \max c_0(x)$. However, our example in Fig. 10 indicates that the limit $\mathscr{A}_2$ should be the admissible solution of the Riemann problem $[u_L, u_R]$. In this case the admissible solutions of the Riemann problem are special because $\mathscr{A}_1 \equiv \mathscr{A}_2$ only when the asymptotic state is the admissible solution of the Riemann problem. Thus the admissible solutions are the ones for which it is valid to interchange the limits in (3.1).

## Acknowledgments

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