Contemporary Mathematics
Volume 108, 1990

NONLINEAR RESONANCE IN INHOMOGENEOUS SYSTEMS OF CONSERVATION LAWS[1,2]

Eli Isaacson[3] and Blake Temple[4]

ABSTRACT: We solve the Riemann problem for a general inhomogeneous system of conservation laws in a region where one of the nonlinear waves in the problem takes on a zero speed. We state generic conditions on the fluxes that guarantee the solvability of the Riemann problem, and these conditions determine a unique underlying structure to the solutions. The inhomogeneity is modeled by a linearly degenerate field. Our analysis thus provides a general framework for studying (what we are calling) resonance between a linear and a nonlinear family of waves in a system of conservation laws. Special cases of this phenomenon are observed in model problems for gas dynamical flow in a variable area duct and in Buckley-Leverett type systems that model multiphase flow in a porous medium.

§1 **INTRODUCTION**: When two different families of waves take on the same wave speed in a nonlinear problem, we say that nonlinear resonance occurs [11,12]. When wave speeds from different families are not distinct, the number of times a pair of waves will interact cannot be bounded apriori. Consequently, since waves are reflected

---

[1] *Mathematics Subject Classification* (1985 Revision): 35L65

[2] The final version of this paper will be submitted for publication elsewhere.

[3] Supported in part by the National Science Foundation and by the Institute of Theoretical Dynamics, UC-Davis, 95616

[4] Supported in part by the National Science Foundation, Grant No. DMS-86-13450 and by the Institute of Theoretical Dynamics, UC-Davis,95616

in other families every time a pair of waves interact, a proliferation of reflected waves can occur by the interaction of a single pair of waves. Here we introduce a general framework in which resonant interaction between a linear and a nonlinear family of waves takes place. Such resonance arises in an inhomogeneous system of conservation laws when a nonlinear family of waves takes on a zero speed. By an inhomogeneous system of conservation laws we mean a system of the general form

$$u_t + f(a,u)_x = 0, \tag{1.1}$$

where we assume that $a=a(x)$ is a variable function of x alone, so that a represents an inhomogeneity in the problem. We express this by the additional conservation law

$$a_t = 0. \tag{1.2}$$

(Systems of this form were previously identified by the authors when they outlined a program for classifying the solutions of nonstrictly hyperbolic systems. [cf. 5,8]) Our general problem thus becomes

$$U_t + F(U)_x = 0, \tag{1.3}$$

where $U=(a,u)$, $F(U)=(0,f(a,u))$, and $u=(u_1,u_2,...,u_n)\varepsilon R^n$, $x\varepsilon R$, $t\geq 0$. System (1.3) is a system of $n+1$ equations in the $n+1$ unknowns a, $u_1,...,u_n$. We assume that for each fixed value of a, system (1.1) is a strictly hyperbolic system of n equations, and that each of the characteristic fields associated with the u variables is either genuinely nonlinear or linearly degenerate [10]. Equation (1.2) produces a linearly degenerate field in system (1.3) with eigenvalue $\lambda_0=0$ and corresponding eigenvector R_0 (i.e., $\nabla\lambda_0\cdot R_0=0$). The remaining eigenvalues,

$$\lambda_1 < \lambda_2 < ... < \lambda_n,$$

of system (1.3) correspond to the eigenvalues of system (1.1) and have corresponding eigenvectors $R_1,...,R_n$ which lie in the hyperplane $a=\text{const}$. We wish to study this system in the neighborhood of a state $U_*=(a_*,u_*)$ at which a nonlinear family of waves in system (1.3) takes on a zero wave speed. Thus we assume that

$$\lambda_k(U_*)=\lambda_0=0, \tag{1.4}$$

and that

$$\nabla\lambda_k\cdot R_k\neq 0. \tag{1.5}$$

In §3 we state a theorem which gives generic conditions on the flux function F that guarantee the unique solvability of the Riemann problem in a neighborhood of a state U_* at which (1.4) and (1.5) hold. The Riemann problem for (1.3) is the initial value problem for piecewise constant initial data

$$U(x,0) = \begin{cases} U_L & \text{for } x < 0, \\ U_R & \text{for } x > 0. \end{cases}$$

The Riemann problem is fundamental to the study of (1.3) because it identifies the elementary waves that propogate, and these are typically shock waves, rarefaction waves and contact discontinuities. The conditions of our theorem determine a unique underlying structure to the solution of the Riemann problem in a neighborhood of U_*.

Special cases of (1.4) and (1.5) are observed in model problems for gas dynamical flow in a variable area duct and in Buckley-Leverett type systems that model multiphase flow in a porous medium. In the latter case the model equations are not given in the form of an inhomogeneous system of conservation laws, but we show that there exists a Langrangian type transformation that maps the given equations to equivalent systems (in the weak sense) that are of this form. (The transformation was shown to the authors by D. Marchesin and Jorge Patino.) Examples of such systems have been studied by Keyfitz and Kranzer [9] and by the authors [4,16], and under the Lagrangian transformation these turn out to be equivalent to an inhomogeneous scalar conservation law in our theory. Many but not all of the features in the scalar case carry over to the case of an inhomogeneous system of conservation laws. For example, there are in general n+2 waves in a solution even though there are only n+1 equations; Riemann problem solutions depend continuously on the data in xt-space but not in state space; but unlike the scalar case, the wave curves in the case of systems are only Lipschitz continuous curves near the point of resonance. This makes it difficult to apply the implicit function theorem directly, and we show that the existence and uniqueness of solutions of the Riemann problem in a neighborhood of a point of resonance is a consequence of the uniqueness of intersection points of Lipschitz continuous manifolds of complementary dimension. Our goal is to obtain an existence theory for the Cauchy problem using Glimm's method that applies in a neighborhood of a point of resonance in a general inhomogeneous system of conservation laws. Such a theorem in the scalar case can be obtained by methods introduced earlier by the second author [16], but a sharper bound on the total variation of solutions, as well as a quadratic potential interaction functional is required to generalize these methods to systems. Such a quadratic functional has not been found in any other case in which there is no apriori bound on the number of times a pair of waves will interact. In the scalar case the authors recently identified the asymptotic wave structure of solutions as t tends to infinity. Indeed, solutions in general decay to *inadmissible* solutions of the Riemann problem, and as a consequence of this, a lack of continuous dependence on the data in the L^p sense was observed [6]. The authors conjectured that some form of continuous dependence would be retrieved when viscosity effects were included. The analysis gives an explanation of how nonuniqueness of solutions of the Riemann problem is explained in terms of the time dynamics of general solutions, and similar phenomena occur in the case of an inhomogeneous system of conservation laws. The argument for decay in [6] went as follows: In [16] the second author constructed a positive nonincreasing functional defined on solutions at every time t ; and in [6] the authors showed that this functional was minimized on the asymptotic waves patterns among all possible wave patterns that a given solution could take on. The authors believe that a quadratic functional that succeeds for Glimm's method in an inhomogeneous system with resonance would help in completeing the proof of decay and would shed light on the rate of decay in the scalar case. Further directions include generalizing to the case where a(x) is a vector. The authors recently analysed this problem in a case equivalent to the case when u is a scalar, but in the context of the multiphase flow problem [4,16].

§2 **APPLICATIONS:** In this section we describe two settings in which resonance in inhomogeneous systems of conservation laws arise.

Flow in a variable area duct: The equations for gas dynamical flow in a variable area duct with cross-sectional area $a(x)$ are [1]

$$\begin{aligned} &\rho_t + (\rho u)_x = -(a'/a)\rho u, \\ (1.6)\qquad &(\rho u)_t + (\rho u^2+p)_x = -(a'/a)\rho u^2, \\ &E_t + [(E+p)u]_x = -(a'/a)[(E+p)u]. \end{aligned}$$

We say that resonance occurs in transonic flow because then one of the nonlinear waves in the problem can be zero [cf 11]. Liu was the first to study the initial value problem for these equations from the point of view of Glimm's Random Choice Method, and he proved convergence of the method for solutions taking values in a neighborhood of a state $(\rho,\rho u,E)$ at which none of the wave speeds in the problem is zero (see [11] and references therein). In [12] Liu also gave a fairly complete analysis of a scalar model for (1.6) in which resonance occurs. For systems, however, there is at present no general proof that Glimm's method converges in the transonic regime. To study this case, we rewrite these equations in the form

$$\begin{aligned} &(a\rho)_t + (a\rho u)_x = 0, \\ (1.7)\qquad &(a\rho u)_t + (a\rho u^2+ap)_x = -a'p, \\ &(aE)_t + [a(E+p)u]_x = 0, \end{aligned}$$

with the supplementary equation

$$(1.8)\qquad a_t=0.$$

The equations obtained when the zero order term on the right hand side is dropped yield a mathematical model for the resonant behavior that occurs in the transonic flow. The resulting system falls into our class. Note that the reduced system can be viewed also as the first system to solve in a numerical time splitting method for solving the original problem (1.7) . In the special case that $p=c^2\rho$ (isothermal flow), the energy equation drops out and the zero order term can be incorporated into the fluxes to obtain the system

$$a_t = 0,$$

$$(1.9) \qquad (a\rho)_t + (a\rho u)_x = 0,$$

$$u_t + (u^2/2+c^2 \log \rho)_x = 0.$$

Although this does not supply a physical conservation form for the original problem, it does provide a mathematical model containing a similar nonlinear resonance. For flow in a variable area duct, we believe that these models isolate an important component in the complicated behavior of transonic flow. Marchesin and Paes-Leme [13] studied this system in an analysis of the Riemann problem obtained by taking a to be piecewise constant, and our point of view here was influenced significantly by their analysis.

Buckley-Leverett type systems: We call the following equations the polymer equations because they arise as a model for the polymer flooding of an oil reservoir; i.e., a two phase, three component flow in a porous medium [3,16]:

$$s_t + f(s,c)_x = 0,$$

$$(1.10) \qquad (cs)_t + (cf(s,c))_x = 0.$$

Here s and c correspond to a saturation and a concentration, resp., $0<s,c<1$, $f=f(s,c)$ is a constitutive relation, and the structure of solutions is determined by qualitative properties of f [4,16]. The eigenvalues of system (1.10) coincide when $f_s = f/s$. The Riemann problem for this system was studied by Isaacson in [4], and Keyfitz and Kranzer [9] studied the Riemann problem for the system

$$u_t + [ug(u,v)]_x = 0,$$

(1.11)

$$v_t + [vg(u,v)]_x = 0,$$

which is formally equivalent to the polymer system, and arose in their study of elasticity. The polymer interpretation of these equations suggests a natural Lagrangian transformation of the variables. In this model $g=f/s$ is the particle velocity of the water, and so the particle trajectories are given by solutions of the ordinary differential equation

$$x' = g(s(x,t),c(x,t)).$$

We can thus define a solution dependent mapping of the independent variables (x,t) to (ξ,t) so that ξ=const. defines the particles trajectories in the transformed, or Lagrangian coordinates ξ and t. One can verify that this is implemented through the mapping defined by specifying $x(\xi,t)$ through

$$\frac{\partial x(\xi,t)}{\partial t} = g(x(\xi,t),t),$$

$$x(\xi,0) = \int_0^{\xi} \frac{1}{s_0(x)}\, dx.$$

Rewriting system (1.10) in the ξt-coordinates yields the equivalent system

$$c_t = 0,$$

(1.13)

$$(1/s)_t - g(s,c)_\xi = 0,$$

which is a system of form (1.1), (1.2) when we make the identifications $u=1/s$, $a=c$ and $h=-g$. Systems (1.10) and (1.13) are equivalent in the sense that they determine the same weak solutions under the 1-1 mapping given by Lagrangian change of variables. Furthermore, system (1.13) satisfies the assumptions of our theorem at the points where $\lambda_0=\lambda_1$. Thus writing $a=a(\xi)$, system (1.13) is an example of a scalar inhomogeneous equation in our framework.

§3 THE RIEMANN PROBLEM: We consider the system of equations

$$a_t = 0,$$

$$u_t + f(a,u)_x = 0,$$

where $u=(u_1, u_2, \ldots, u_n)$ and $a \in R$. We can restate this in the form (1.3) by taking $U=(a,u)$ and $F=(0,f)$. Here, $a=a(x)$ is an inhomogeneity in the equations, and $a_t=0$ gives rise to a linearly degenerate field with wave speed $\lambda_0=0$. We consider the Riemann problem for weak solutions in a neighborhood of a state $U_*=(a_*, u_*)$ at which

$$\lambda_1 < \ldots < \lambda_k = \lambda_0 < \ldots < \lambda_n.$$

This represents the simplest example of a coincidence of wave speeds.

THEOREM: *Assume that f satisfies the following conditions in a neighborhood of a state* $U_*=(a_*,u_*)$:

(i) *For each fixed value of* a, *the system*

$$u_t + f(a,u)_x = 0 \qquad \text{(u)}$$

is strictly hyperbolic and genuinely nonlinear with eigenvalues

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n.$$

(ii) $\lambda_k(a_*,u_*) = 0.$

(iii) *The* $n\times(n+1)$ *matrix* $\partial f/\partial U$ *has full rank at* U_*.

(iv) *The directional derivative of* a_0 *in the direction* R_0 *satisfies*

$$\nabla a_0 \cdot R_0 \big|_{U_*} \neq 0,$$

where R_0 *is the unit eigenvector for* $\lambda_0=0$ *and* a_0 *is the a-component of* R_0, *a function of* U.

Under assumptions (i)-(iv), *there exists a unique solution of the Riemann problem in a neighborhood of* U_*, *and in physical space this solution depends continuously on the left and right states. Moreover, for every* f *in this class, the solutions exhibit the same qualitative behavior.*

We indicate the proof here. For details the reader is referred to our forthcoming paper. First of all, the assumption of genuine nonlinearity ($\nabla\lambda_k \cdot r_k \neq 0$) in the k'th field of system (u) guarantees that the equation $\lambda_k=0$ defines a smooth n-dimensional surface locally in R^{n+1}, passing through the state U_*. We call this the transition surface $\mathcal{T}$. By (i), $R_k=(a_k,r_k)$ points along the surface a=const., and the condition $\nabla\lambda_k \cdot r_k \neq 0$ also guarantees that the integral curves of R_k cut the transition surface transversally at $\mathcal{T}$. By (ii) and (iii), $\partial F/\partial U$ has Jordan normal form

$$\text{(J)} \qquad \begin{bmatrix} \lambda_1 \ddots \lambda_{k-1} & & & & O \\ & 0 & 1 & & \\ & & 0 & & \\ & & & \lambda_{k+1} \ddots & \\ O & & & & \lambda_n \end{bmatrix}$$

at U_* because $\partial F/\partial U$ has (full) rank n and $\lambda_0=\lambda_k=0$ at U_*. Moreover, $\partial F/\partial U$ has this normal form for every $U\in\mathcal{T}$ in a neighborhood of U_* since (iii) is an open condition. In particular, this implies that the eigenvectors R_0 and R_k agree on $\mathcal{T}$, and a_0, the 0'th component of R_0, vanishes on $\mathcal{T}$ in this neighborhood. Thus we can conclude that the integral curves for both R_0 and R_k cut the surface $\mathcal{T}$ transversally near the state U_*. Finally, the genericity condition (iv), that $\nabla a_0 \cdot R_0 |_{U_*} \neq 0$, implies that the integral curves of R_0 do not cross the surface $a=a_*$ at states U near U_*, and that the integral curve of R_0 passing through U_* must cross the surface a=const. exactly twice at values of a near $a=a_*$, $a < a_*$. Indeed, a as a function of arclength along the integral curve of R_0 would have an inflection point at U_* if this integral curve stayed within or crossed the surface $a=a_*$ at U_*; and this would imply that $\nabla a_0 \cdot R_0 = 0$ on $\mathcal{T}$, violating (iv). We assume without loss of generality that the integral curve of R_0 lies below the surface $a=a_*$ near the state U_* (see Figure 1). Since all of the conditions (i) through (iv) are open conditions, the above conclusions about the integral curve of R_0 through U_* must also hold for all $U\in\mathcal{T}$ in a neighborhood of U_*.

The solution of the Riemann problem for arbitrary states U_L and U_R in a neighborhood of U_* is constructed as follows: we let

$T^i{}_t(U_L)$ denote the state t arclength units from U_L along the i-wave curve of U_L, i=1,...,n. (The i-wave curve of U_L consists of all right states that can be connected to U_L by an admissible i-wave, [10].) By (i), all states in the image of $T^i(U)$ lie at level a_L. For a given value of a_R, let $T^R(U_L)$ denote the set of all right states at level a_R that can be connected to U_L by a solution of the Riemann problem consisting of admissible k-waves and 0-waves alone; and let $T^R{}_t(U_L)$ denote the point t arclength units from the transition surface along $T^R(U_L)$. (Choose t to increase in the direction of λ_k.). We say that a 0-wave which connects U_L to U_R on the same integral curve of R_0 by a contact discontinuity of speed zero is *admissible* if the integral curve of R_0 does not cross the transition surface $\mathcal{T}$ between U_L and U_R. (Admissibility here is equivalent to conservation of the total variation of a in Glimm's method [cf 4,9,16]). The curves $T^R(U_L)$ are skethed in figures 2 and 3. Note that $T^R(U_L)$ is a continuous curve at level a_R, but is only Lipschitz continuous due to a possible jump in the derivative at the points labeled Q in figures 2 and 3. The continuity of the curves $T^R(U_L)$ at the special points Q follows from the triple jump condition formulated in [5]. The solution of the Riemann problem for arbitrary U_L and U_R is constructed by finding $t_1,\ldots,t_n$ such that

$$U_R = T^n{}_{t_n} \cdots T^{k+1}{}_{t_{k+1}} \cdot T^R{}_{t_k} \cdot T^{k-1}{}_{t_{k-1}} \cdots T^1{}_{t_1}(U_L).$$

By definition the elementary waves corresponding to the $T^i{}_{t_i}(U_i)$ take U_L to U_R as i ranges from 1 to n, and this determines the unique solution of the Riemann problem near U_*. Since $T^R(U)$ is only Lipschitz continuous, the implicit function theorem is difficult to apply directly to obtain existence and uniqueness of $t_1,\ldots,t_n$ for each pair U_L and U_R in a neighborhood of U_*. Existence and uniqueness is verified by demonstrating the uniqueness of intersection points for Lipschitz continuous manifolds of conplementary dimension. The procedure goes as follows. We first make the definition

DEFN: A function

$$\phi : \{t \varepsilon R^k \ni |t| < \tau\} \to R^n$$

defines a Lipschitz continuous manifold with ε-approximate tangent vectors $W_1,\dots,W_n$ in R^n if

$$\left| \frac{\phi(t+\alpha e_i)-\phi(t)}{\alpha} - W_i \right| < \varepsilon$$

whenever $|t|<\tau$ and $|t+\alpha e_i|<\tau$. It is then not hard to prove the following lemma:

LEMMA 1: Let $\phi = \phi(t_k)$ and $\varphi(t_{n-k})$ define Liplchitz continuous manifolds M^k and N^{n-k} with ε-approximate tangent vectors $W_1, \dots ,W_k$ and $W_{k+1},\dots,W_n$, respectively, which together form a basis for R^n,

$$\phi : R^k \to R^n,$$

$$\varphi : R^{n-k} \to R^n.$$

Here $t_k = (t_1,\dots,t_k)$, $t_{n-k} = (t_{k+1},\dots,t_n)$. Assume further that M^k and N^{n-k} both intersect $B_\delta(u_*)$, the ball of radius δ and center u_*. Then there exists a constant $C > 1$ such that if

$$\varepsilon < 1/(CM_0),$$

then M^k and N^{n-k} intersect each other at a unique point inside of the ball $B_\gamma(u_*)$, where

$$\gamma = CM_0{}^2\delta.$$

Here $M_0>1$ denotes a constant that relates the Euclidean norm on $\mathbf{R}^n$ to the norm determined by the basis $W_1,\ldots,W_n$ in the sense that

$$(1/M_0)|\alpha| < |\alpha_1 W_1 + \cdots \alpha_n W_n| < M_0|\alpha|$$

holds for all $\alpha \,\varepsilon\, \mathbf{R}^n$.

Now given U_L , and U_R , define

$$\phi(t_k) = T^R{}_{t_k} \circ T^{k-1}{}_{t_{k-1}} \circ \cdots \circ T^1{}_{t_1} U_L \ ,$$

$$\varphi(t_{n-k}) = T^{-(k+1)}{}_{t_{k-1}} \circ \cdots \circ T^{-n}{}_{t_n} U_R \ .$$

Here T^{-j} denotes the inverse of the function T^j . The existence and uniqueness of solutions of the Riemann problem then follows directly from the next lemma (details will appear in the authors' forthcoming paper.):

LEMMA 2: For U_L and U_R in a δ-neighborhood of U_* , ϕ and φ define Lipshitz continuous manifolds at level $a=a_R$, with ε-approximate tangent vectors $W_j = PR_j(U_*)$ for $j<k$, and $W_j = R_j(U_*)$ for $j \geq k$, where P denotes the matrix that projects onto the tangent space of $\mathcal{T}$ at $U=U_*$, and

$$\varepsilon = o(\delta) \ .$$

We note that it is the continuity of the curves $T^R(U_L)$ that leads to the existence and uniqueness of solutions of the Riemann problem, and to the fact that solutions of the Riemann problem depend continuously on the left and right states U_L and U_R. The Lipschitz continuity of the wave curves follows directly from the fact that the equations are posed in conservation form.

In conclusion, the general structure of the solutions in a neighborhood of U_* can be described as follows: to leading order the waves in the 0,k-characteristic families correspond to the waves in the Riemann problem solution for the scalar inhomogeneous equation; the general solution is obtained by adjoining to these waves the faster and slower waves from families $i \neq k$. Thus, the Riemann problem solutions of the scalar inhomogeneous equation give the canonical structure of solutions under our generic assumptions, just as the scalar homogeneous equation determines the local structure to leading order in the strictly hyperbolic case.

REFERENCES

[1] Courant and Freidrichs, *Supersonic flow and shock waves, John Wiley, New York, 1948.*

[2] H. Freisthuler, Rotational Degeneracy of Hyperbolic Systems of Conservation Laws, *Arch. Rat. Mech. Anal.,* to appear

[3] J.Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715

[4] E. Isaacson, Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery, Rockefeller University preprint

[5] E. Isaacson, D. Marchesin, B. Plohr, and B. Temple, The Reimann

problem near a hyperbolic singularity: the classification of solutions of quadratic Riemann problems I, *SIAM J. Appl. Math.*, **48** No. 5 (1988)

[6] E. Isaacson and B. Temple, The structure of asymptotic states in a singular system of conservation laws, *Adv. Appl. Math.*, to appear

[7] E. Isaacson and B. Temple, Analysis of a singular system of conservation laws, *Jour. Diff. Equn.*, 65 No. 2 (1986)

[8] E. Isaacson and B. Temple, Examples and classification of non-strictly hyperbolic systems of conservation laws, *Abstracts of AMS*, January 1985.

[9] B. Keyfitz and H. Kranzer, A system of non-strictly hyperbolic conservation laws arising in elasticity theory, *Arch. Rat. Mech. Anal.* **72** (1980), 219-241

[10] P. D. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537-566

[11] T. P. Liu, Quasilinear Hyperbolec Systems, *Comm. Math. Phys.*, **68**, 141-172 (1979)

[12] T.P. Liu, Scalar example

[13] D. Marchesin and P. J. Paes-Leme, PUC report.

[14] D. Schaeffer and M. Shearer, The classification of 2x2 systems of conservation laws, with application to oil recovery, with Appendix by D. Marchesin, P. J. Paes-Leme, D. Schaeffer, and M. Shearer, *Comm. Pure Appl. Math.*, **15** (1987), pp. 141-178.

[15] J.Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer-Verlag, Berlin, Niw York, 1980

[16] B. Temple, Global solution of the Cauchy problem for a class of 2x2 nonstrictly hyperbolic conservation laws, *Adv. in Appl. Math.* **3** (1982), 335-375