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ON BLOWUP IN A RESONANT NONSTRICTLY HYPERBOLIC SYSTEM

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Abstract

We discuss the "blowup" in derivatives which occurs in solutions of a 2×2 system of conservation laws in the simplest setting in which wave speeds can coincide, namely, an inhomogeneous equation in which the inhomogeneity is treated as an unknown variable in the problem. For such a system, if we linearize in both variables, the solutions of the linearized equations blow up at a linear rate; while if we fix the inhomogeneity and linearize in the nonlinear variable alone, then solutions will blow up at an exponential rate. We contrast this with what happens in the nonlinear problem. For the nonlinear problem, the authors have shown by the 2×2 Glimm and Godunov methods that solutions and their derivatives remain bounded in terms of a singular transformation of the unknowns, but the conserved quantities can "blow up" in the total variation norm due to the possible formation of oscillations. Such oscillations can appear after finite time due to the formation of interacting time asymptotic states that will later interact: and correspondingly they can also appear as numerical oscillations in any numerical method based on time asymptotic wave pattern (i.e., methods based on the 2×2 Riemann problem). Here we discuss the rate of blowup in this nonlinear problem. We show that there is no bound on the rate of growth in the derivatives (as measured by the total variation of the variables at fixed time) unless the inhomogeneous variable satisfies a certain threshold smoothness condition. We give examples, and motivate a theorem (which represents work in progress) that shows that the growth rate in the nonlinear problem is sublinear under the smoothness assumptions, thus indicating that (in

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terms of growth rate) the nonlinear problem behaves qualitatively more like the less refined linearized system. Our results, which are based on a detailed analysis of the 2×2 Godunov method, can also be interpreted as verifying that the averaging step in the 2×2 Godunov method "wipes out" the numerical oscillations that can occur in the Riemann problem solution step of the method. We interpret this as demonstrating that numerical methods based on the Riemann problem are viable in this non-strictly hyperbolic setting.

1. Introduction

We discuss the issue of "blowup" in the total variation norm for solutions of the 2×2 resonant nonlinear system of conservation laws

$$\begin{aligned} a_t &= 0, \\ u_t + f(a,u)_x &= 0, \end{aligned} \tag{1}$$

$$\begin{aligned} a(x,0) &= a_0(x) \equiv a(x), \\ u(x,0) &= u_0(x), \end{aligned} \tag{2}$$

where $u \in R$, $a \in R$, and we let $U = (a,u)$, $F(U) = (0, f(U))$. System (1) is a system with the two wave speeds $\lambda_0(U) = 0$ and $\lambda_1(U) = \partial f/\partial u$, and is resonant at a state where the two wave speeds coincide, making (1) a nonstrictly hyperbolic system. This is the special case of an $n \times n$ resonant nonlinear system as introduced in [10,11]. (Motivations for this point of view can be found in the work of Marchesin and Paes-Leme, see for example [18].) Note that for fixed a, system (1) is equivalent to the inhomogeneous scalar conservation law $u_t + f(a(x),u)_x = 0$. Examples of resonant systems of this general form have been used to model problems in multi-phase flow, and are related to problems involving transonic flow in a variable area duct [10,11]. In this context, the canonical type of resonant behavior occurs in a neighborhood of a state $U_* = (a_*, u_*)$ under the assumptions that the nonlinear wave family is genuinely nonlinear, the nonlinear wave speed is zero at U_*, and the flux function f is monotone in a at u_* [10,11]. Thus we assume

$$\lambda_1(U_*) = 0, \tag{3}$$

$$\frac{\partial}{\partial u}\lambda_1(U_*) \neq 0, \tag{4}$$

$$\frac{\partial}{\partial a} f(U_*) \neq 0. \tag{5}$$

These are generic conditions that generalize to the case when u is a vector as well [10,11], and they imply that solutions of the linearized equations blow up as t tends to infinity, cf. [10,11]. Here we motivate and state a theorem (details to appear in the authors forthcoming paper) that states that solutions of the nonlinear problem can grow at only a sublinear rate under the condition that the function $a(x)$ satisfies a certain threshold smoothness condition which was first identified by Tveito and Winther [23], namely, $Var\{a'(\cdot)\} \leq \infty$. (Here, $Var\{f(\cdot)\} \equiv \int |f'(x)|dx$ denotes the total variation of the function f, a measure of the size of the derivative of f). Our analysis is based on a detailed study of the 2×2 Godunov numerical method, and the result relies on the previous work of the authors [15] where we showed that for the nonlinear problem, $Var\{z(\cdot, t)\}$ remains bounded for all time in solutions generated by the 2×2 Godunov method, where $z = z(a, u)$ is the variable that defines the singular transformation first introduced by Temple in [22]. Since z is related to u by a singular transformation, bounds on $Var\{z(\cdot, t)\}$ do not imply bounds on the total variation of the conserved quantity u, and examples show that in the nonlinear problem, $Var\{u(\cdot, t)\}$ can initially blow up at an arbitrary rate when a is taken to be of bounded variation. Our result is that when $Var\{a'(\cdot)\} < \infty$, the quantity $Var\{u(\cdot, t)\}$ can grow at only a sublinear rate, the rate depending only on $Var\{a'(\cdot)\}$, $Var\{a(\cdot)\}$ and $Var\{z_0(\cdot)\}$. Known counterexamples show that there does not exist a growth rate for $Var\{u(\cdot, t)\}$ that depends only on $Var\{a\}$ even when $u_0(x) \equiv 0$ [22]. Here we show by counterexample that there does not exist a rate depending only on the $Var\{a(\cdot)\}$ and the C^1-norm of $a(x)$. We understand this as follows: the 2×2 Glimm and Godunov method are based on approximating solutions locally by time asymptotic states, and there is no bound on the the total variation of u in the time asymptotic states in terms of the total variation of the initial data. Our result (and also the earlier result in [23]) can be interpreted as saying that when $a(\cdot)$ is sufficiently smooth, i.e., $Var\{a'(\cdot)\} < \infty$, the rate at which the time asymptotic states are taken on in a solution is controlled, this effect being quantified by the statement that $Var\{u(\cdot,t)\}$ grows at a sublinear rate when $Var\{a'(\cdot)\} < \infty$. In the next paragraph we contrast this with the growth rate observed in the linearized equations. Our conclusion is that solutions of the equations obtained by linearizing the $2x2$ nonlinear system about the state U_* blow up at a linear rate due to resonance; and the system obtained by fixing $a(x)$, and linearizing in u alone, blow up at an exponential rate. We conclude, therefore, that in terms of growth rate in the derivatives, the nonlinear problem behaves more like the solutions based on the *less* refined linearization process. As far as we know, the Godunov method is the only numerical method for which both the time independent bound $Var\{z(\cdot,t)\} < CVar\{z_0(\cdot)\}$ as well as the bound $Var\{u(\cdot,t)\} < \infty$ for the conserved quantity have been simultaneously verified. With the latter estimate one can verify that our solutions satisfy the Kruzkov entropy condition (cf.[23]), which implies uniqueness for the scalar inhomogeneous equation, and therefore, as far as we know, our analysis also gives the first proof that the solutions of the scalar equation $u_t + f(a(x),u)_x = 0$ generated by scalar methods must satisfy the same time independent bounds on derivatives that have only been verified via numerical methods based on the construction of time asymptotic states for the 2×2 system (1) [22,15]. The results ini [22] show that the bound $Var\{z(\cdot,t)\} < Const.Var\{z_0(\cdot)\}$, valid for system (1), is a time independent bound on derivatives (as measured under a singular transformation) and thus represents a purely nonlinear phenomenon. As a motivating example, we show that our theorem on linear growth demonstrates that in terms of growth rate, the nonlinear equations behave more like the less refined linearized system. Specifically, consider first the system obtained by linearizing the flux function $F(U) = (0, f(a,u))$ about a state U_*, where $U_* = (a_*, u_*)$ satisfies (3)-(5). In this case, letting $U = U_* + \bar{U}$, we write $F(U) = F(U_*) + dF(U_*)\bar{U} + h.o.t.$, where $h.o.t.$ are higher order in $|\bar{U}|$, and by (3)-(5),

$$dF = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}, \tag{6}$$

where $\beta = \frac{\partial f}{\partial a}(U_*) \neq 0$. Thus the linearized equations for $\bar{U}$ are

$$\bar{U}_t + \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} \bar{U}_x = 0, \tag{7}$$

which have the solution $\bar{u} = \bar{u}_0(x) - \beta\bar{a}'(x)t$. We conclude that for the linearized system, $\bar{u}$ and all x-derivatives of $\bar{u}$ blow up at a linear rate as t tends to infinity. Consider now the more refined linearization procedure based on fixing $a = a(x)$ and linearizing the flux function f about the state $u = u_*$. Setting $u = u_* + \bar{u}$ and omitting higher order terms in $|\bar{u}|$, we obtain $f(a,u) = f(a(x), u_*) + f_u(a(x), u_*)\bar{u}$, so that the equation for $\bar{u}$ based on this procedure is

$$\bar{u}_t + f_a(a(x), u_*)a'(x) + f_{au}(a(x), u_*)a'(x)\bar{u} + f_u(a(x), u_*)\bar{u}_x = 0. \tag{8}$$

Assuming now that $a(0) = a_*$, and $a'(0) = a'_0 \neq 0$, the equation at $x = 0$ is

$$\bar{u}_t = -f_a(a_*, u_*)a'_0 - f_{au}(a_*, u_*)a'_0\bar{u}, \tag{9}$$

which has the solution

$$\bar{u}(0,t) = -\frac{\beta}{\gamma} + (\bar{u}_0 + \frac{\beta}{\gamma})e^{a'_0\gamma t}.$$

Therefore, in this case, $\bar{u}$ and all x-derivatives of $\bar{u}$ blow up to infinity at an exponential rate as t tends to infinity so long as $\gamma \neq 0$ and $\bar{u}_0 + \beta/\gamma \neq 0$, where we set $\gamma = -f_{au}(a_*, u_*)$, $\beta = -f_a(a_*, u_*)$ and $\bar{u}_0 = \bar{u}(0,0)$. Thus we conclude that our result that for solutions of (1), $Var\{u(\cdot,t)\}$ grows at a sublinear rate depending only on $Var\{a'(\cdot)\}$, $Var\{a(\cdot)\}$ and $Var\{z_0(\cdot)\}$ implies that in the nonlinear problem, the grow rate for the derivatives of u corresponds to the growth rate observed in the *less* refined linearization of the nonlinear problem.

2. Preliminaries

In this section we state, Theorem 1, our main result on sublinear growth for (1),(2), and then we establish notation and review the results in [15] which are required for the subsequent proof, which is developed in the next section. At the end of the section we show by counterexample that for solutions of (1), $Var\{u(\cdot,t)\}$ does not grow at a rate depending only on the $Var\{a\}$ and the C^1-norm of $a(x)$, the counterexample indicating that condition $Var\{a'(\cdot)\}$ is sharp for sublinear growth. The proof of Theorem 1 is postponed until section 3. We first state the main theorem of this paper. Let U(x,t) be a solution of (1),(2) generated from Godunov's method (to be discussed below) for arbitrary initial data $U_0(x) \subset \mathbf{B}$ of compact support, where **B** is a neighborhood of U_* to be determined below. Assume that U(x,t) is generated from initial data satisfying

$$Var\{a'\} \equiv V_a' < \infty, \tag{10}$$

$$Var\{a\} \equiv V_a < \infty, \tag{11}$$

and

$$Var\{u_0(\cdot)\} \equiv V_u < \infty. \tag{12}$$

In particular, this implies that (see [22]),

$$Var_z\{u(\cdot)\} \equiv V_z < \infty. \tag{13}$$

Let S_f denote the constant

$$C_f \equiv \sup_{\mathbf{B}}\{|f_a|, |f_u|, |f_{aa}|, |f_{au}|, |f_{uu}|\}. \tag{14}$$

Our result is the following theorem, whose proof is sketched in the following sections.

Theorem 1. *Let U(x,t) be a solution of (1),(2) generated from Godunov's method for arbitrary initial data $U_0(x)$, satisfying (10), (11) and (12). Then there exists a constant C depending only on V_a', V_a, V_z and C_f such that*

$$Var_u\{u(\cdot,t)\} \leq Var_u\{u(\cdot)\} + Ct. \tag{15}$$

The proof of Theorem 1 follows from an analysis of the 2×2 Godunov numerical method as applied to (1) which we rewrite as the 2×2 nonstrictly hyperbolic system

$$U_t + F(U)_x = 0, \tag{16}$$

where $U = (a, u)$ and $F(U) = (0, f(u))$. The Godunov scheme is based on the construction of the Riemann problem for (16). We briefly describe the solution of the Riemann problem as outlined in [15]. The Riemann problem, denoted $[U_L, U_R]$, is the initial value problem for initial data given by a jump discontinuity

$$U_0(x) = \begin{cases} U_L & if\ x < 0, \\ U_R & if\ x > 0. \end{cases} \tag{17}$$

Let λ_i and R_i denote the eigenvalues and corresponding unit right eigenvectors of the 2×2 matrix dF, $i = 0, 1$. The eigenvalues are given by $\lambda_0(U) \equiv 0$ and $\lambda_1(U) = \partial f / \partial u$. In [11] it is shown that the assumptions (3) to (5) imply that $\lambda_1(U) = 0$ defines a smooth curve Γ (named the transition curve) in U-space passing through the state $U = U_*$ in a direction transverse to the u-axis, and thus Γ is described by a smooth function $a = T(U)$ in a neighborhood of U_*. Γ is the set of states where system (16) is nonstrictly hyperbolic in a neighborhood of U_*. By substituting $u - T(A)$ for u in system (16), we obtain an equivalent system in which Γ is given by $u = 0$, and so without loss of generality we assume that in a neighborhood of U_*, $\lambda_1(U) = 0$ if and only if $u = 0$. The assumptions (3) to (5) imply that the matrix dF has the Jordan normal form

$$dF = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{18}$$

at each state $U \in \Gamma$. Because (16) is nonstrictly hyperbolic at $U = U_*$, there are in general three waves that solve the Riemann problem [15]. The wave curves for (16) are the integral curves of the eigenvector fields R_0 and R_1 associated with λ_0 and λ_1. The 1-wave curves are given by $a = \bar{a}$, $\bar{a}$ constant, and 1-waves are determined by solutions of the scalar conservation law $u_t + f(\bar{a}, u)_x = 0$. The 0-wave curves are given by $f = const.$ Because of (3)-(5), in a neighborhood of U_*, $f = const.$ defines a smooth curve of nonzero curvature which is tangent to the curves $a = const.$ only at the states $U \in \Gamma$ in the au-plane; and the transition curve Γ intersects the 0-wave and 1-wave curves transversally at U_*. To be consistent with [11,15,22], we assume without loss of generality that $f_u u < 0$ and $f_a < 0$, so that the curves $f = const.$ are convex down in a neighborhood of U_* (see Lemma 3.1 in [11] and Figure 1. of [15]). We restrict attention to solutions of (16) taking values in a neighborhood $\mathbf{B}$ of U_*. Thus let $\mathbf{B}$ denote a neighborhood of U_* bounded above by an integral curve of R_0 and below by an integral curve of R_1, such that the integral curves of R_0 are convex down in $\mathbf{B}$, and such that each integral curve intersects the transition curve Γ transversally at a unique point in $\mathbf{B}$. Assume further that $\partial f/\partial a \neq 0$ in $\mathbf{B}$. Our assumptions (3) through (5) imply that such an open set $\mathbf{B}$ exists in a neighborhood of U_*. Under these assumptions, the set B is an invariant region for Riemann problems for system (1) (cf. [11,15,22]):

Proposition 1. *Let* $\mathbf{B}$ *denote a neighborhood of* U_* *bounded above by an integral curve of* R_0 *and below by and integral curve of* R_1*, such that the integral curves of* R_0 *are convex down in* $\mathbf{B}$*, and such that each integral curve cuts the transition curve* Γ *transversally in* $\mathbf{B}$*. Then* $\mathbf{B}$ *is an invariant region for Riemann problems in the sense that if* $U_L, U_R \in \mathbf{B}$*, then all intermediate states in the solution of the Riemann problem* $[U_L, U_R]$ *are also in* $\mathbf{B}$*.*

We recall also the definition of the singular variable z defined in [22]: for our purpose it suffices to define z by

$$z(a,u) = \begin{cases} +|f(a,u) - f(a,0)| & \text{if } U \text{ lies to the right of } \Gamma, \\ -|f(a,u) - f(a,0)| & \text{if } U \text{ lies to the left of } \Gamma. \end{cases} \tag{19}$$

(This is slightly different than the definition given in [22,15], but since f is constant along integral curves of R_0 and Γ is given by $u = 0$, the two definitions are essentially equivalent). Since the curves given by $f = Const.$ are convex down in a neighborhood of U_*, we conclude that the mapping

$$u \longrightarrow z$$

is $1-1$ and onto in a neighborhood of U_*, and is regular except at Γ where the

Jacobian vanishes. We let $Var_x U_0$ denote the total variation of U_0 in the variable x, etc. Although for convenience we restrict attention to solutions taking values in a neighborhood $\mathbf{B}$ of U_*, the results extend globally under straightforward assumptions on f. We now indicate by counterexample that for solutions of (1), $Var\{u(\cdot,t)\}$ does not grow at a rate depending only on the $Var\{a\}$ and the C^1-norm of $a(x)$ even when $u_0(x) \equiv 0$, the counterexample indicating that condition $Var\{a'(\cdot)\}$ is sharp for sublinear growth. The counterexample given in [22] shows that for solutions of (16), $Var\{u(\cdot,t)\}$ does not grow at a rate depending only on a through the $Var\{a\}$. For this counterexample it suffices to take $a(x) = a_j$ for $x_j \le x < x_{j+1}$, $u_0(x) \equiv 0$, where

$$a_j = \begin{cases} a_* & if\ j \le 0, \\ a_* + \epsilon j/N & if\ 0 < j \le N, \\ a_* + \epsilon & if\ j \ge N, \end{cases} \tag{20}$$

for some $\epsilon > 0$ and $N \in Z$. In this case, $Var\{u(\cdot,0+)\} = O(\epsilon\sqrt{N})$ in the exact solution because the wave curves a=const. and f=const. have a quadratic tangency at u=0, cf. [22], and thus the growth rate for $Var\{u(\cdot,0+)\}$ is not bounded as $N \to \infty$. Consider now the initial data $U_0^N = (a_N(x), u_0^N(x))$ where $u_0^N(x) \equiv 0$ and $a(\cdot) \in C^1$ is defined by

$$a_N(x) = a_* + \sigma(x)\frac{1}{N}sin(Nx), \tag{21}$$

where $0 \le \sigma(x) \le 1$ is smooth and satisfies $\sigma(x) \equiv 1$ for $0 \le x \le 1$, $\sigma(x) \equiv 0$ for $|x| \ge 5/2$, and $|\sigma'(x)| \le 1$. It is easy to verify that

$$|a_N'(\cdot)| \le 2,$$

and

$$Var\{a_N'(x)\} \le 5.$$

But when $u = 0$, $f_u(a,0) = 0$, so at $t = 0$, (1) reads

$$u_t + f(au)_x = u_t + f_a a_N',$$

and so

$$u_t^N(x,0) \approx a_N'(x),$$

(where we assume without loss of generality that $f_a(a_*, 0) = -1$.) Thus for $0 \le x \le 1$,

$$u_t(x,0) \approx \begin{cases} +1 & if\ x = \frac{(2n+1)\pi}{N} \\ -1 & if\ x = \frac{2n\pi}{N}. \end{cases} \tag{22}$$

and so on the interval $x \in [0,1]$, the solution $u(x,t)$ satisfies

$$Var\{u(\cdot, dt)\} \approx (N/2\pi)dt.$$

We conclude that

$$\frac{d}{dt} Var\{u_N(\cdot, 0)\} \to \infty$$

as $N \to \infty$, thus verifying that the rate of growth of $Var\{u(\cdot, t)\}$ is not bounded by a constant depending on $a(x)$ through $Var\{a(\cdot)\}$ and the C^1-norm of $a(\cdot)$.

3. Godunov Scheme

In this section we sketch the proof of Theorem 1. We state three technical lemmas which are required for the proof. The proofs of the first two lemmas are given here, and then we give the proof of Theorem 1 as a consequence of the three lemmas. The proof of Lemma 3 is technical, and the details will appear in the authors forthcoming paper. We now state and prove the first two lemmas that will be used in the subsequent analysis. To start, let U_1 and U_2 be arbitrary states in $\mathbf{B}$, and define the following second order difference:

$$\Delta^2 f^n(U_2, U_1) \equiv \Delta^2 f^n_{1,2} = f(a_2, u_2) - f(a_2, u_1) - f(a_1, u_2) + f(a_1, u_1). \tag{23}$$

Lemma 1. *The following estimate holds:*

$$|\Delta^2 f^n(U_2, U_1)| \le O(1)\{(\Delta a)^2 + |\Delta a \Delta z|\}, \tag{24}$$

where $\Delta a = a_2 - a_1$ and $\Delta z = |z(a_2, u_2) - z(a_1, u_1)|$, and $O(1)$ denotes a generic constant that depends only on f and $\mathbf{B}$.

Proof: By (20),

$$\Delta^2 f^n_{1,2} = z(a_2, u_2) - z(a_2, u_1) - z(a_1, u_2) + z(a_1, u_1). \tag{25}$$

Now it suffices to verify (24) in the case that both $u_1, u_2 \leq 0$ or both $u_1, u_2 \geq 0$. To see this, note that

$$\begin{aligned}\Delta^2 f^n_{1,2} &= z(a_2,u_2) - z(a_2,0) - z(a_1,u_2) + z(a_1,0) + z(a_2,0) - z(a_2,u_1) - \\ &\quad z(a_1,0) + z(a_1,u_1) \\ &\equiv \Delta^2 f^n_{U_2,U_0^2} + \Delta^2 f^n_{U_0^1,U_1}. \end{aligned} \tag{26}$$

Thus, assuming that we have verified (24) in the above two cases, if $sign\{u_2\} = sign\{u_1\}$, then

$$\begin{aligned}|\Delta^2 f^n_{1,2}| &\leq |\Delta^2 f^n_{U_2,U_0^2}| + |\Delta^2 f^n_{U_0^1,U_1}| \\ &\leq O(1)\{(\Delta a)^2 + |z_2 \Delta a|\} + \{(\Delta a)^2 + |z_1 \Delta a|\} \\ &\leq O(1)\{(\Delta a)^2 + |\Delta a \Delta z|\},\end{aligned}$$

where we use the fact that $\Delta z = z_1 + z_2$ in this case. Thus we verify (24) in the case that $sign\{u_1\} = sign\{u_2\}$. We do the case $u_1, u_2 \leq 0$. To this end, let $g(a, u_1, u_2)$ be the function defined for $u_1, u_2 \leq 0$ by the formula

$$z(a,u_2) - z(a,u_1) = g(a,u_1,u_2)\{u_2^2 - u_1^2\}. \tag{27}$$

We claim that in some neighborhood of (a_*, u_*, u_*) (we assume this neighborhood consists of the set of all (a, u_1, u_2) such that U, U_1, U_2 all lie in $\mathbf{B}$), g is smooth, nonzero, and $g^{-1} \equiv \frac{1}{g}$ exists and is smooth in this neighborhood intersect the set $u_1, u_2 \leq 0$. We first show that there exist an ϵ such that $|g(a, u_1, u_2)| > \epsilon$ in this set. To see this, note that since $z(a, u_2) - z(a, u_1) = f(a, u_2) - f(a, u_1)$, we know that if $u_1 \neq u_2$, then $f(a, u_1) \neq f(a, u_2)$ if we choose $\mathbf{B}$ small enough so that $\frac{\partial f}{\partial u} \neq 0$ when $u \neq 0$. Thus when $u_1 \neq u_2$,

$$g(a,u_1,u_2) = \frac{f(a,u_2) - f(a,u_1)}{u_2^2 - u_1^2} \neq 0.$$

Moreover, fixing $u_1 \neq 0$, we obtain

$$\frac{\partial f}{\partial u_2} = g(a,u_1,u_2)2u_2 + g_{u_2}(u_2^2 - u_1^2), \tag{28}$$

which implies that when $u_2 = u_1 \neq 0$, we have

$$g(a, u_1, u_2) = \frac{1}{2u_2}\frac{\partial f}{\partial u_1}(a, u_1) \neq 0,$$

where we assume $\mathbf{B}$ is small enough so that $\frac{\partial^2 f}{\partial u^2} \neq 0$ in $\mathbf{B}$. Finally, for $u_1 = 0$, differentiate (28) to obtain

$$\frac{\partial^2 f}{\partial u^2}(a, u) = 2g(a, 0, u) + O(u),$$

which implies that

$$g(a, 0, 0) = \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(a, 0) \neq 0.$$

Thus we have that $|g(a, u_1, u_2)| > \epsilon > 0$ on compact subsets of $\mathbf{B}$, and hence choosing $\mathbf{B}$ small enough, inside $\mathbf{B}$ itself. The smoothness of g and g^{-1} follow at once from the assumed smoothness of g, thus establishing the claim. To complete the proof of Lemma 1, note that

$$z(a_2, u_2) - z(a_2, u_1) = g(a_2, u_1, u_2)|u_2^2 - u_1^2|,$$

$$z(a_1, u_2) - z(a_1, u_1) = g(a_2, u_1, u_2)|u_2^2 - u_1^2|,$$

and thus

$$\Delta^2 f_{1,2} = \{g(a_2, u_1, u_2) - g(a_1, u_1, u_2)\}\{u_2^2 - u_1^2\} = O(1)|\Delta a||\{u_2^2 - u_1^2\}|. \quad (29)$$

Moreover, by (27),

$$\begin{aligned} u_2^2 - u_1^2 &= g^{-1}(a_2, u_1, u_2)\{z(a_2, u_2) - z(a_2, u_1)\} \\ &= g^{-1}(a_2, u_1, u_2)\{z(a_2, u_2) - z(a_1, u_1)\} + g^{-1}(a_2, u_1, u_2)\cdot \\ &\quad \{z(a_1, u_1) - z(a_2, u_1)\} \\ &= O(1)\Delta z + O(1)\Delta a \end{aligned}$$

where we use the fact that z is smooth. Substituting this last line into (29) gives

$$|\Delta^2 f_{1,2}| = O(1)|\Delta a|\{|\Delta a| + |\Delta z|\} = O(1)\{\Delta a^2 + \Delta a\Delta z\},$$

which establishes Lemma 1. Let $U = (a, u) \in \mathbf{B}$ and let a_1 and a_2 correspond to entries of states in $\mathbf{B}$. Define the following second order difference:

$$\Delta^2 f(a_1, a_2, U) = f(a_2, u) - 2f(a, u) + f(a_1, u). \tag{30}$$

Lemma 2. *The following estimate holds:*

$$\Delta^2 f(a_1, a_2, U) = O(1)|\Delta a_1 - \Delta a_2|, \tag{31}$$

where $\Delta a_1 = a - a_1$ *and* $\Delta a_2 = a_2 - a$.

Proof: Lemma 2 follows immediatly from Taylor's theorem. Our analysis is based on solutions of (16) constructed by the Godunov scheme following the work in [15]. Thus let $U_{\Delta x}(x,t)$ denote an approximate solution of the Cauchy problem (1) , (2) generated by the Godunov scheme, for initial data $U_0(x)$ taking values in the neighborhood $\mathbf{B}$ of U_*. Specifically, we discretize $R \times [0, \infty)$ by spacial mesh length Δx and time mesh length Δt such that

$$\frac{\Delta t}{\Delta x} = \delta, \tag{32}$$

where $\delta < 1/(2\lambda)$, and

$$\lambda \equiv \sup_{(a,u)\in\mathbf{B}} \{|\lambda_0|, |\lambda_1|\}. \tag{33}$$

(It suffices to take $\delta < 1/(2\lambda)$, but for convenience in the proof of Lemma 7, we take $\delta = 1/(4\lambda)$). We let $x_n = n\Delta x$, $t_j = j\Delta t$ so that (x_n, t_j) denote the mesh points of the approximate solution. Define

$$S_i = \{(x,t) : t_i \le t < t_{i+1}\}.$$

The approximate solution $U_{\Delta x}$ generated by the Godunov scheme is defined as follows [3,4]: to initiate the scheme at $n = 0$, define

$$U_j^0 \equiv U_{\Delta x}(x, 0) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U_0(x)dx, \quad x_j < x < x_{j+1}.$$

Assuming that $U_{\Delta x}(x,t)$ has been constructed for $(x,t) \in \bigcup_{i=0}^{n-1} S_i$, then we define $U_{\Delta x}$ in S_n as the solution of (1) with the initial values

$$U_j^n \equiv U_{\Delta x}(x, t_n) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U_{\Delta x}(x, t_n-)dx, \quad x_j < x < x_{j+1}. \tag{34}$$

In words, at each time t_n, a piecewise constant approximation $U_{\Delta x}(x, t_n)$ is obtained by taking the arithmetic averages of $U_{\Delta x}(x, t_n-)$ at each interval of the mesh, so that the solution in S_n can be constructed by solving the Riemann problems $[U^n_{j-1}, U^n_j]$ posed at each point of discontinuity $(x_j, t_n), j \in Z$. The Courant-Freidrichs-Levy (CFL) restriction (32) ensures that the Riemann problem solutions in each S_n do not interact before time t_{n+1} [22,15]. Our results rely on the following theorem which was proved in [15]:

Theorem 2. *Assume that the initial data $U_0(x) \in \mathbf{B}$ satisfies the condition $Var_x\{U_0(\cdot)\} < \infty$ and $Var\{a(\cdot)\} < \infty$. Then $U_{\Delta x}(x,t) \in \mathbf{B}$ for all $x, t \geq 0$, $Var_x\{U_{\Delta x}(\cdot, t)\} \leq 4Var_x\{U_0(\cdot)\} < \infty$, and a subsequence of $\{U_{\Delta x}(\cdot, t)\}$ converges boundedly almost everywhere to a weak solution of (1),(2) as Δx tends to zero.*

We let $f^n_j = f(U_{\Delta x}(x_j, t_j))$ and we use the notation $U^n_{j-} \equiv U_{\Delta x}(x_j+, t), U^n_{j+} = U_{\Delta x}(x_{j+1}-, t)$, and for $t_n < t < t_{n+1}$, we let

$$f^n_j = f(U_{\Delta x}(x_j, t)), f^n_{j-} = f(U_{\Delta x}(x_j+, t)) \text{ and } f^n_{j+} = f(U_{\Delta x}(x_{j+1}-, t)).$$

Here, the symbol $+$ refers to the right side and $-$ the left side of the mesh rectangle R^n_j, which we define by $R^n_j \equiv \{(x,t) : x_j \leq x < x_{j+1}, t_n \leq t < t_{n+1}\}$. Using integration by parts it is not difficult to verify that $U_{\Delta x}$ satisfies the difference equation

$$u^{n+1}_j = u^n_j - \delta[f^n_{j+} - f^n_{j-}]. \tag{35}$$

Let $\Delta u^n_j = u^n_j - u^n_{j-1}$, so that by (35) we can write

$$\Delta u^{n+1}_j = \Delta u^n_j - \delta[f^n_{j+} - f^n_{j-} - f^n_{j-1+} + f^n_{j-1-}]. \tag{36}$$

Our procedure is to estimate the right hand side of (36), the idea being to add and subtract terms in such a way as to construct second order differences of the form (23) and (30), together with a remainder term that forms a collapsing sum in the estimate for $Var_u\{U_{\Delta x}\}$ given by $\sum_{j=-\infty}^{\infty} |u^{n+1}_j|$. Since we will be estimating terms at a fixed time level t_n, we will suppress the index n whenever states are assumed to lie at level t_n in an approximate solution $U_{\Delta x}$. Thus we use the notation $f_{i,j}^- = f(a_i, u_{j-}^n)$, $f_{i,j}^+ = f(a_i, u_{j+}^n)$, and we define the first order differences

$$\Delta f_j^- = f_{j,j}^- - f_{j,j-1}^-, \tag{37}$$

$$\Delta f_j^+ = f_{j,j}^+ - f_{j,j-1}^+, \tag{38}$$

together with the following second order differences (cf., (23) and (30))

$$\Delta^2 f_j^- = f_{j+1,j}^- - 2f_{j,j}^- + f_{j-1,j}^-, \tag{39}$$

$$\Delta^2 f_j^+ = f_{j+1,j}^+ - 2f_{j,j}^+ + f_{j-1,j}^+, \tag{40}$$

and

$$\Delta^2 f_{j,j-1}^- = f_{j,j}^- - f_{j,j-1}^- - f_{j-1,j}^- + f_{j-1,j-1}^-, \tag{41}$$

$$\Delta^2 f_{j,j-1}^+ = f_{j,j}^+ - f_{j,j-1}^+ - f_{j-1,j}^+ + f_{j-1,j-1}^+, \tag{42}$$

$$\Delta^2 f_{j,j-1}^{+,-} = f_{j,j}^+ - f_{j,j-1}^+ - f_{j-1,j}^- + f_{j-1,j-1}^-. \tag{43}$$

Our Theorem 1 on sublinear growth for system (1) proceeds as follows.

Definition 1. We say that *the mesh point* (x_i, t_j) *falls into case one* (we write $T_j^n \in C_1$) if

$$|\Delta f_j^-| \leq (2\delta)^{-1} |\Delta u_j^n|,$$

and

$$\Delta u_j^n \Delta f_j^- \leq 0.$$

If either of these fails, then we say $T_j^n \in \bar{C}_1$.

Definition 2. We say that *the mesh point* (x_i, t_j) *falls into case two* (we write $T_j^n \in C_2$) if

$$|\Delta f_j^+| \leq (2\delta)^{-1} |\Delta u_j^n|,$$

$$\Delta u_j^n \Delta f_j^+ \geq 0.$$

If either of these fails, then we say $T_j^n \in \bar{C}_2$.

Note that the type T_j^n depends on the wave structures in the Riemann problems posed at (x_{j-1}, t_n) and (x_{j+1}, t_n) as well as (x_j, t_n). Theorem 1 is a consequence of the following Lemmas:

Lemma 3. *If $T_j^n \in C_1$, then the following estimate holds:*

$$|\Delta u_j^{n+1}| \leq |\Delta u_j| - \delta|\Delta f_j^-| + \delta|\Delta f_{j+1}^-| + E_1(j), \tag{44}$$

where

$$E_1(j) = \delta|\Delta^2 f_j^-| + \delta|\Delta^2 f_{j,j-1}^-|. \tag{45}$$

Proof: By adding and subtracting the appropriate terms, it is straightforward to verify that (35) implies the identity

$$\Delta u_j^{n+1} = \Delta u_j - \delta\Delta f_{j+1}^- + \delta\Delta f_j^- - \delta\Delta^2 f_j^- - \delta\Delta^2 f_{j,j-1}^-. \tag{46}$$

Statement (44) follows from (46) by using $\Delta u_j^n \Delta f_j^- \leq 0$, together with the estimate $|\Delta f_j^-| \leq \delta^{-1}|\Delta u_j|$ of Definition 1.

Lemma 4. *If $T_j^n \in C_2$, then the following estimate holds:*

$$|\Delta u_j^{n+1}| \leq |\Delta u_j| + \delta|\Delta f_{j-1}^+| - \delta|\Delta f_j^+| + E_2(j), \tag{47}$$

where

$$E_2(j) = \delta|\Delta^2 f_{j-1}^+| + \delta|\Delta^2 f_{j-1,j-2}^+|. \tag{48}$$

Proof: Statement (35) implies the identity

$$\Delta u_j^{n+1} = \Delta u_j - \delta\Delta f_j^+ + \delta\Delta f_{j-1}^+ - \delta\Delta^2 f_{j-1}^+ - \delta\Delta^2 f_{j-1,j-2}^+. \tag{49}$$

Thus (47) follows from (49) by using $\Delta u_j^n \Delta f_j^+ \geq 0$, together with the estimate $|\Delta f_j^-| \leq \delta^{-1}|\Delta u_j|$ of Definition 2.

Corollary 1. *The following estimate holds for l=1,2:*

$$\begin{aligned} |E_l(j)| \leq\ & O(1)\{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\} \\ & + O(1)\{(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\}, \end{aligned} \tag{50}$$

where $Var_x U_j^n$ denotes the total variation in x of the approximate solution $U_{\Delta x}(x, t_j+)$ in the interval $x_j- \le x \le x_j+$, and $O(1)$ denotes a generic constant depending only on f and the supnorm bounds on the solution. Note that by Theorem 2,

$$\sum_{k=-\infty}^{+\infty} Var_x U_k^n \le 8V_x. \tag{51}$$

Proof: By lemmas 1 and 2 of the previous section, it follows that

$$|\Delta^2 f_j^n| \le O(1)|\Delta a_{j+1} - \Delta a_j|, \tag{52}$$

$$|\Delta^2 f_{j,j-1}^-| \le O(1)|\Delta a_j|\{Var_x U_j^n + Var_x U_{j-1}^n\}. \tag{53}$$

By the same argument, the latter estimate also holds for $\Delta^2 f_{j,j-1}^+$, and so (50) is evident. The following lemma is the main technical lemma used in the proof of Theorem 1:

Lemma 5. *If $T_j^n \in C_1 \cap C_2$, then*

$$|\Delta u_j^{n+1}| \le |\Delta u_j| - \delta|\Delta f_{j+1}^-| - \delta|\Delta f_j^+| + E_3(j), \tag{54}$$

where

$$\begin{aligned} |E_3(j)| \ \le\ & O(1)\{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\} \\ & + O(1)\{(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\}; \end{aligned} \tag{55}$$

If $T_j^n \in \bar{C}_1 \cap \bar{C}_2$, then

$$|\Delta u_j^{n+1}| \le |\Delta u_j| + \delta|\Delta f_{j-1}^+| + \delta|\Delta f_{j+1}^-| + E_4(j), \tag{56}$$

where

$$\begin{aligned} |E_4(j)| \ \le\ & O(1)\{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\} \\ & + O(1)\{(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\}; \end{aligned} \tag{57}$$

And if $T_{j-1}^n \in \bar{C}_2$ and $T_j^n \in \bar{C}_1$, then

$$|\Delta f_{j-1}^+| + |\Delta f_j^-| \le E_5(j-1), \tag{58}$$

where

$$\begin{aligned} |E_5(j)| \leq\ & O(1)\{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\} \\ & +O(1)\{(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\}. \end{aligned} \tag{59}$$

Proof: The inequalities (54)-(58) follow from a detailed study of the possible admissible Riemann problem types T_{j-1}^n, T_j^n that satisfy the specified conditions. Details are omitted.

We use (44) through (58) in the proof of the following proposition:

Proposition 1. *The following estimate holds:*

$$\sum_{k=i}^{j} |\Delta u_k^{n+1}| \leq \sum_{k=i}^{j} |\Delta u_k^n| + F_i^1 + F_j^2 + E(k), \tag{60}$$

where F_i^1 and F_j^2 are defined by

$$F_i^1 = \begin{cases} -\delta|\Delta f_i^-| & if\ (j,n) \in \mathbf{C_1} \\ +\delta|\Delta f_{i-1}^+| & if\ (j,n) \in \bar{\mathbf{C}}_1, \end{cases} \tag{61}$$

$$F_j^2 = \begin{cases} -\delta|\Delta f_j^+| & if\ (j,n) \in \mathbf{C_2} \\ +\delta|\Delta f_{j+1}^-| & if\ (j,n) \in \bar{\mathbf{C}}_2, \end{cases} \tag{62}$$

and $E(k)$, defined to be $E(k) \equiv \sum_{l=1}^{5} E_l(k)$, satisfies

$$\begin{aligned} |E(k)| \leq\ & O(1)\{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\} \\ & +O(1)(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\}. \end{aligned} \tag{63}$$

Proof: The estimate (63) follows directly from (50), (55), (57), and (59). We prove (60) by induction. To start the induction, note that when $i - j = 0$, (60) reduces to

$$|\Delta u_k^{n+1}| \leq |\Delta u_k^n| + F_i^1 + F_i^2 + \sum_{k=1}^{4} E_k(j). \tag{64}$$

Estimate (64) follows directly from (44),(47),(54) and (56) depending on the possible values for $F_i^1+F_i^2$. Thus, (60) holds for i-j=0. Suppose now that (60) **holds** for i and j. We show that it holds for $i-1$ and j as well, thus verifying (60) by induction. In fact, by (64),

$$|\Delta u_{i-1}^{n+1}| \leq |\Delta u_{i-1}^n| + F_{i-1}^2 + F_{i-1}^1 + \sum_{k=1}^{4} E_k(i-1). \tag{65}$$

Putting this in (60), we obtain

$$\sum_{k=i-1}^{j} |\Delta u_k^{n+1}| \leq \sum_{k=i-1}^{j} |\Delta u_k^n| + F_{i-1}^1 + F_j^2 + (F_i^1 + F_{i-1}^2) + \sum_{k=i}^{j} E(k) + \sum_{l=1}^{4} E_l(i-1). \tag{66}$$

Thus in order to prove the proposition, it is sufficient to show that

$$F_i^1 + F_{i-1}^2 \leq E_5(i-1). \tag{67}$$

By definition, $F_i^1 + F_{i-1}^2 = 0$ if either $T_i \in \bar{C}_1, T_{i-1} \in C_2$ or $T_i \in C_1, T_{i-1} \in \bar{C}_2$; and $F_i^1 + F_{i-1}^2 \leq 0$ if $T_i \in C_1, T_{i-1} \in C_2$. But by (58) of Lemma 5, if $T_i \in \bar{C}_1, T_{i-1} \in \bar{C}_2$, then $|\Delta f_{j-1}^+| + |\Delta f_j^-| \leq E_5(i-1)$. Thus (60) is established.

Proof of Theorem 1: Let $U_{\Delta x}$ be an approximate solution generated by the Godunov scheme from compactly supported inital data $U_0(\cdot)$ satisfying (10), (11) and (12) and taking values in **B**. Since $Var\{z_{\Delta x}\}(\cdot,t)$ is uniformly bounded, $f(u_{\Delta x})(x,t)$ tends to a constant state as x tends to plus or minus infinity; and thus F_j^1 and F_j^2 tend to zero as $|j|$ tends to infinity. Thus taking $i=-\infty$ and $j=+\infty$ in (60) and using (63), we obtain

$$\sum_{k=-\infty}^{+\infty} |\Delta u_k^{n+1}| \leq \sum_{k=-\infty}^{+\infty} |\Delta u_k^n| + O(1) \sum_{k=-\infty}^{+\infty} \{|\Delta a_{j+1} - \Delta a_j| + |\Delta a_j - \Delta a_{j-1}|\}$$

$$+O(1) \sum_{k=-\infty}^{+\infty} (|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n) \cdot \tag{68}$$

$$\cdot \sum_{k=-\infty}^{+\infty} |\Delta a_k - \Delta a_{k-1}| \leq V_a' \Delta x, \tag{69}$$

$$\sum_{k=-\infty}^{+\infty} \{(|\Delta a_j| + |\Delta a_{j-1}|)(Var_x U_j^n + Var_x U_{j-1}^n + Var_x U_{j-2}^n)\} \leq 6 V_x V_a' \Delta x, \tag{70}$$

where we use the inequality

$$|a'(x)| \leq Var\{a'\},$$

which applies because $Var\{a(\cdot)\} < \infty$ and $Var\{a'(\cdot)\} < \infty$ imply that $a'(x) \to 0$ as $|x| \to \infty$. Putting these in (68) and using CFL yields

$$\sum_{k=-\infty}^{+\infty} |\Delta u_k^{n+1}| \leq \sum_{k=-\infty}^{+\infty} |\Delta u_k^n| + C\Delta t, \tag{71}$$

where C depends only on V_a, V_a', V_z, δ and C_f. Moreover, (71) applies to any weak solution obtained as a limit of approximate solutions $U_{\Delta x}$, and thus we complete the proof of Theorem 1.

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