ADVANCES IN APPLIED MATHEMATICS 3, 335–375 (1982)

Global Solution of the Cauchy Problem for a Class of 2×2 Nonstrictly Hyperbolic Conservation Laws*

BLAKE TEMPLE

Department of Mathematical Physics, The Rockefeller University, New York, New York 10021

We prove the existence of a global weak solution to the Cauchy problem for a class of 2×2 equations which model one-dimensional multiphase flow, and which represent a natural generalization of the scalar Buckley–Leverett equation. Loss of strict hyperbolicity (coinciding wave speeds with a $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ normal form) occurs on a curve in state space, and waves in a neighborhood of this curve contribute unbounded variation to the approximate Glimm scheme solutions. The unbounded variation is handled by means of a singular transformation; in the transformed variables, the variation is bounded. Glimm's argument must be modified to handle the unbounded variation that appears in the statement of the weak conditions, and this requires that the random choice variable be random in space as well as time.

1. INTRODUCTION

We consider the Cauchy problem for the 2×2 system of nonlinear conservation laws

$$\begin{aligned} s_t + [sG(s,b)]_x &= 0, \qquad s(x,0) = s_0(x), \\ b_t + [bG(s,b)]_x &= 0, \qquad b(x,0) = b_0(x), \end{aligned} \tag{1.1}$$

where $0 \leqslant s \leqslant 1$ and $0 \leqslant b \leqslant s$ are functions of $-\infty \leqslant x \leqslant +\infty$, $t \geqslant 0$. This system arises in one-dimensional oil recovery problems, describing how the addition of a polymer (a water solute that increases the viscosity of water) affects the flow of water and oil in an oil reservoir when a solution of polymer and water is pumped into the reservoir at a constant volumetric

*This research was supported by a National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship, Grant MCS80-17157.

0196-8858/82/030335-41$05.00/0

rate. The equations express Darcy's Law and the conservation of mass under the assumption that the reservoir is uniformly porous. Here s is the saturation of water, c is the concentration of polymer in water, and $b = sc$ is the overall concentration of polymer at any x and t. Thus, system (1.1) represents a natural generalization of the scalar Buckley–Leverett equation, the single hyperbolic equation (two phase oil and water flow) which is obtained in (1.1) when c is constant.

The Cauchy problem in (1.1) is a special case of the general problem

$$u_t + F(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{1.2}$$

with $u = (s, b)$, and $F(u) = (sG(s, b), bG(s, b))$. Since discontinuities can form in the solutions of (1.2) even in the presence of smooth initial data (cf. [4]), we look for weak solutions in the sense of the theory of distributions; i.e., solutions which satisfy

$$\iint_{\substack{-\infty < x < +\infty \\ t \geqslant 0}} u\phi_t + F(u)\phi_x + \int_{-\infty}^{+\infty} u(x,0)\phi(x,0)\,dx = 0 \tag{1.3}$$

for any test function $\phi \in C_0^1(x, t)$. The conservation laws in (1.1), however, differ from the classical conservation laws of gas dynamics (cf. Lax [7]) primarily because $sG(s, b)$ is not convex. This forces strict hyperbolicity and genuine nonlinearity to fail; i.e., the eigenvectors of dF (the first Fréchét derivative of F) become parallel on a one-dimensional curve in state space, so the eigenvalues of dF fail to be everywhere distinct, and fail to be monotone along the integral curves of the corresponding eigenvectors. Equations (1.1) for multiphase flow were derived by Isaacson [2], and in [2] Isaacson also explicitly solved the Riemann problem for arbitrary constant states u_L and u_R.[1] (The Riemann problem is the initial value problem when $u_0(x) = u_L$ for $x \leqslant 0$, $u_0(x) = u_R$ for $x \geqslant 0$.) Here we prove that the Glimm difference scheme [1] converges to a weak solution for arbitrary data of bounded variation in s and c, and for certain data of unbounded variation in s and c; and we assume only the physically motivated conditions on the general shape of the function $g(s, c) = G(s, cs)$ which are required in the solution of the Riemann problem.

The Glimm difference scheme is a scheme by which solutions with arbitrary data are approximated locally by Riemann problem solutions. In the case of system (1.1), new phenomenon arise because strict hyperbolicity

[1] It is interesting to note that system (1.1) also arises in elasticity theory with a different flux function F. In [3], Keyfitz and Kranzer derive these equations, and solve a similar nonstrictly hyperbolic Riemann problem. Many of their techniques, including their analysis of the Riemann invariants and their generalization of the Lax entropy condition, are applicable in the solution of the Riemann problem for system (1.1).

fails. For example, solutions to the Riemann problem for this 2×2 system can have three nontrivial waves (at most two waves appear in 2×2 gas dynamics and other typical convex and strictly hyperbolic 2×2 systems), and in state space these waves can change discontinuously with respect to u_L and u_R. Moreover, the variation of the approximate Glimm scheme solutions can go to infinity as the mesh size vanishes even when the data are of bounded variation. Because the variation of the conserved quantities cannot be bounded in an approximate solution, a modification of Glimm's argument is required to show that the weak conditions hold in the limit, and the argument given here requires the random choice variable to be random in space as well as time. Our technique involves constructing a 1–1 differentiable transformation Ψ: $(s, c) \to (z, c)$, which is regular except where dF is not strictly hyperbolic, and such that the variation of the approximate solutions remains uniformly bounded in the zc-plane. In this way we prove the main theorem, which in particular solves the Cauchy problem for initial data of bounded variation in s and c:

THEOREM 1. *The Cauchy problem for the* 2×2 *system of oil–polymer equations* (1.1) *has a global weak solution for arbitrary initial data of bounded variation in* z *and* c.

Our method involves a detailed study of the Riemann problem solutions to system (1.1), and so in Sections 2 and 3 we sketch the derivation of the equations for oil–polymer flow, and review Isaacson's solution of the Riemann problem [2]. In Section 4 we describe the difference scheme, and discuss the method of obtaining bounds on the variation of the solution at each time step; i.e., we define I-curves, reduce the variation problem to a problem involving "interactions" (these ideas were first formulated in [1]), and suggest the need for a nonregular transformation Ψ. In Section 5 we define the transformation Ψ, estimate the variation of Riemann problem solutions in the coordinates z and c, and apply Helly's theorem to obtain a convergent subsequence of approximate solutions. Finally, in Section 6 we modify Glimm's proof of the convergence of the weak conditions in order to handle the unbounded variation in u and $F(u)$ that can occur in the statement of the weak conditions (1.3). The analysis gives a rate of convergence of order $r^{1/(3+2k)}$ (where r is the mesh length) in case the wave speeds have kth order contact at the curve where strict hyperbolicity fails.

It is interesting to note that in [7], Liu and Wang obtain existence for a system of the form (1.1) under assumptions on G which maintain the linear independence of the eigenvectors of dF, so problems with unbounded variation do not arise there. The only other example of a global existence theorem for 2×2 nonlinear systems of conservation laws is for isothermal gas dynamics [8], where an explicit functional form of the flux function $F(u)$ is required.

2. Derivation of the Equations

We assume that oil, together with solutions of water and polymer in varying concentrations (aqueous phase), are forced to flow through a uniformly porous one dimensional oil reservoir at a constant volumetric rate. The fluids are assumed to be incompressible, and the polymer is assumed to be a solute of water that increases the viscosity of water but is insoluble in oil; and so the particle velocity of the polymer at any point in the reservoir equals the particle velocity of the water at the point. (Water and polymer are viewed as distinct, mutually soluble, incompressible fluids, so that concentrations are determined by volumetric proportions.) Darcy's Law [6] relates the pressure drop across a section of the reservoir to the volumetric flow rate of a viscous fluid through that section; i.e., when only one substance of viscosity μ flows in the reservoir, we have

$$\text{Darcy's Law:} \qquad Q = -\frac{KA}{\mu}\frac{P_2 - P_1}{x_2 - x_1}. \tag{2.1}$$

Here Q is the volumetric flow rate of the fluid through that section of the reservoir between x_1 and x_2, $P_2 - P_1$ is the corresponding pressure drop, and A is the cross sectional area of the reservoir assumed to be unity by choice of units. K is the absolute permeability of the reservoir, which depends on the microscopic structure of the interlocking channels in the porous medium through which the fluid flows; since we assume that the reservoir is uniformly porous, K is constant for flow involving a single viscous fluid. Choosing units so that the volumetric flow rate is unity, Darcy's Law for a single fluid is given locally by

$$1 = Q = -\frac{K}{\mu}P_x. \tag{2.2}$$

But because oil and water are immiscible, a joint flow in the reservoir occurs, on a microscopic level, by forcing the interlocking channels in the medium to partition between those that accept oil flow and those that accept water flow. This effect tends to inhibit the overall flow, since, for example, channels filled with oil can become blocked by surrounding channels of water. Thus, during oil and aqueous flow, the "nonlinear" interlocking of the channels produces relative permeability constants k_0 and k_a which depend at each point on the concentrations of the fluids at that point. In this case, Darcy's Law for joint flow becomes [6]:

$$\begin{aligned} Q_0 &= -\frac{k_0(s)}{\mu_0}P_x, \\ Q_a &= -\frac{k_a(s)}{\mu_a(c)}P_x, \end{aligned} \tag{2.3}$$

for respective oil and water–polymer flow, where $s(x, t)$ is the volumetric proportion of the reservoir at (x, t) occupied by water and polymer, $c(x, t)$ is the volumetric proportion of the polymer in water, $b(x, t) = c(x, t)s(x, t)$ is the volumetric proportion of polymer in the reservoir at (x, t), and $\mu_a(c)$ is the viscosity of the aqueous solution, an increasing function of c. Here Q_0[resp. Q_a] is the volumetric flow rate of oil [resp. water–polymer] and we assume that the overall flow rate $Q = Q_0 + Q_a$ is unity. Using this fact, we can eliminate P_x in (2.3) to conclude that

$$Q_a = \frac{\dfrac{k_a(s)}{\mu(c)}}{\dfrac{k_a(s)}{\mu(c)} + \dfrac{k_0(s)}{\mu_0}} = f(s, c), \tag{2.4}$$

where $f(s, c)$ is the fraction of the total volumetric flow associated with the aqueous phase. We assume that $f(s, c)$ is a smooth function such that $f(s, c)$ increases from zero to one with one inflection point (which represents a maximum value of the first derivative) when c is constant, and such that $f(s, c)$ decreases with increasing c for fixed s, as indicated in Fig. 1. This can be regarded as an experimental fact (cf. [6]). (Mathematically, we assume that $(\partial f/\partial c)(s, c) < 0$ for $0 < s < 1$, and for convenience also assume that the curves $f(s, c)$ all make pth order contact with the s-axis at $s = 0$, some $p > 1$.) The graphs of $k_0(s)$ and $k_a(s)$ have the shapes shown in Fig. 2.

Finally, to obtain the conservation equations, note that since Q_a is the volumetric flow rate of the water–polymer solution, and since particles of water and polymer flow at the same velocity, the particle velocity of the aqueous phase must be $Q_a/s = f(s, c)/s$. This is defined to be $g(s, c)$. By analysing a volume element, we obtain the Eulerian equations of motion of the fluid, which express the conservation of water together with polymer,

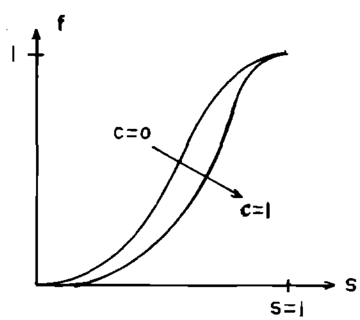


FIGURE 1

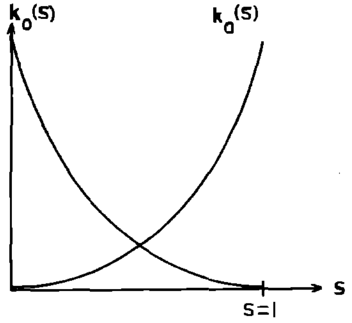


FIGURE 2

and the conservation of polymer, respectively:

$$s_t + [g(s,c)s]_x = 0, \qquad (1.1)$$
$$b_t + [g(s,c)b]_x = 0.$$

These are the desired equations, where properties of

$$g(s,c) = G(s,b) = \frac{f(s,c)}{s}$$

are derived from properties of $f(s, c)$.

3. SOLUTION OF THE RIEMANN PROBLEM

To solve the Riemann problem for system (1.1), we first find the rarefaction curves. Since $F = (sG(s, b), bG(s, b))$ with $g(s, c) = G(s, b)$, we obtain

$$dF = \begin{bmatrix} G + sG_s & sG_b \\ bG_s & G + bG_b \end{bmatrix},$$

where a subscript denotes partial differentiation with respect to that argument. A calculation shows that the eigenvalues of dF are $G = f/s$ and $G + sG_s + bG_b = f_s(s, c)$, and the respective eigenvectors of dF are $(G_b, -G_s)$ and $(1, c)$. Integral curves of the first eigenvector therefore lie along constant G (since ∇G is perpendicular to $(G_b, -G_s)$), and integral curves of the second eigenvector lie along constant c (since $\nabla c = \nabla(b/s) = (-b/s^2, 1/s) = (1/s)(-c, 1)$, which is perpendicular to $(1, c)$). Thus, the two "normal modes" for system (1.1) are solutions that take values on $G =$ const. and propagate with speed G (and so form contact discontinuities), together with solutions that take values on $c =$ const. and propagate with speed f_s.

We now calculate the shock curves for system (1.1). Shock waves are solutions with discontinuities that propagate with speed σ, such that the following jump conditions hold between the states on the left- and right-hand side of the discontinuity [4]:

$$\sigma[u_L - u_R] = [F(u_L) - F(u_R)].$$

This is a system of two equations which, after eliminating σ and simplifying, yields

$$b_R s_L(G_R - G_L) = b_L s_R(G_R - G_L).$$

The solutions to this equation are $G_L = G_R$ and $c_L = c_R$ (obtained by letting $b = sc$), and the respective wave speeds are $\sigma = G_L = G_R$ and $\sigma = (f(u_L) - f(u_R))/(s_L + s_R)$. The first set of conditions confirms the fact that one family of solutions consists of discontinuities which propagate along constant G with speed G, while the second set of conditions implies that solutions in the other family propagate along constant c with wave speed f_s at points of smoothness, and $(f(u_L) - f(u_R))/(s_L - s_R)$ at points of discontinuity. This confirms the fact that solutions propagating at constant c must satisfy the nonconvex scalar conservation law (the Buckley–Leverett equation)

$$s_t + f(s, c)_x = 0, \tag{3.1}$$

obtained from the first equation in (1.1) by identifying $g(s, c) = f(s, c)/s$. Since unphysical weak solutions can occur (cf. [5]), in the next paragraph we apply additional "entropy conditions" to determine the correct solutions that move at constant c or G. We call these physical solutions "s-waves" (saturation waves) and "c-waves" (concentration waves), respectively, and so we label the corresponding wave speeds (eigenvalues) for these families $\lambda_s = f_s$ and $\lambda_c = G = g$. Then the general Riemann problem can be uniquely solved by joining together successive s-waves and c-waves.

Note, first, that since $f(s, c)$ has one inflection point for fixed c, and since $g = f/s$ is the slope of the chord joining the point $(s, f(s, c))$ to the origin (see Fig. 3), the wave speeds $\lambda_s = f_s$ and $\lambda_c = g$ are equal along a curve (labelled "T", the transition curve) in the sc-plane. A study of Fig. 3 now enables us to graph the curves $g =$ const. and $c =$ const. in the sc-plane (Fig. 4), where solutions will be studied. (The conditions on $f(s, c)$ imply that the curves along which g is constant give c as a monotone function of s, except on the transition curve T, where a maximum value of c is taken. These curves continue smoothly to the curve $s = 0$ which defines the curve $g = 0$.) To clarify the relationships between the variables, in Fig. 5 we have drawn this in the sb-plane, the plane of conserved quantities. Now s-waves,

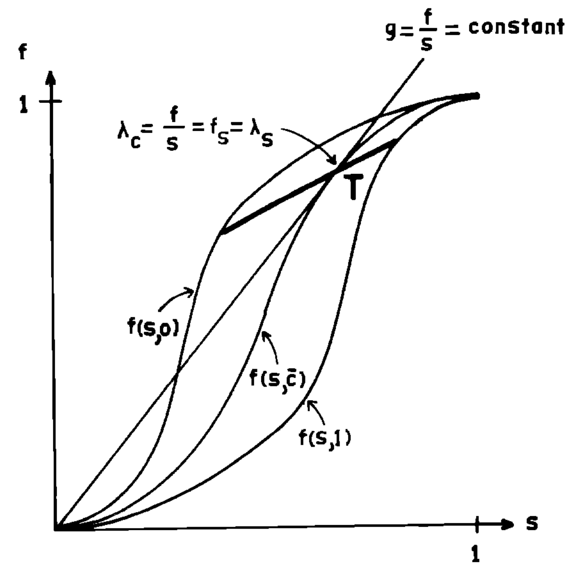


FIGURE 3

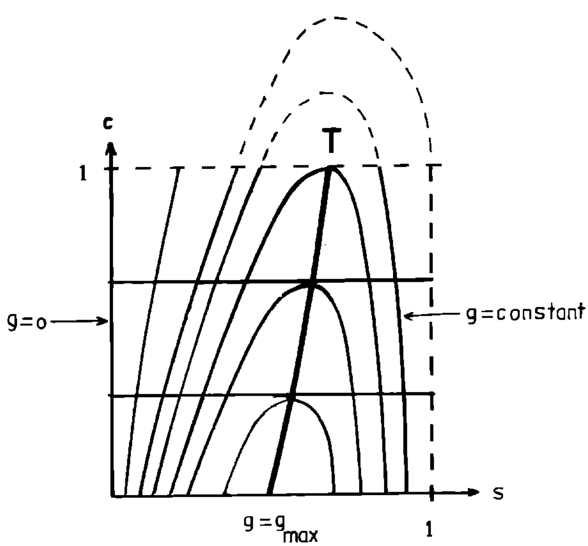


FIGURE 4

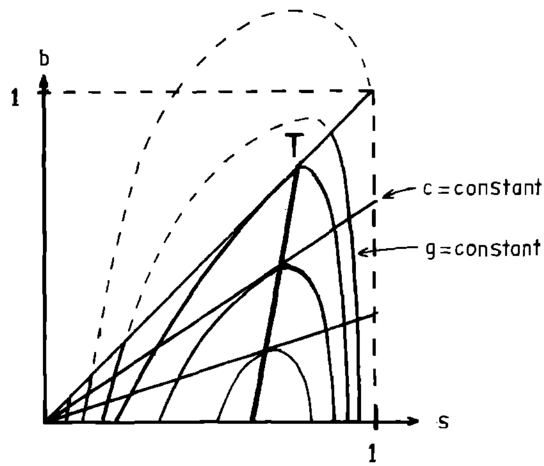


FIGURE 5

solutions that move at constant c, must satisfy the scalar equation (3.1), and so the entropy condition for scalar equations applies; i.e., solutions that move at constant c are obtained by taking "upper convex envelopes" (if $s_L \leqslant s_R$) or "lower convex envelopes" (if $s_L \geqslant s_R$) along the graph of f. This method of constructing s-wave solutions is diagrammed in Fig. 6. To obtain the physically meaningful solutions that move at constant g, we apply the entropy condition of Lax [5] for general $n \times n$ systems of conservation laws.

(E) *A weak solution consisting of adjacent constants u_L and u_R separated by a line of discontinuity in xt-space is an admissible solution (a shock or contact discontinuity) if and only if $n + 1$ of the characteristics generated by u_L and u_R, impinge on the line of discontinuity.*

From Fig. 3 it is easily verified that (E) holds for solutions that move at constant g if and only if the solution does not cross the transition curve T. Thus a c-wave is a solution that connects two states at constant g on the same side of T by a contact discontinuity of speed g. (This application of (E) is due to Keyfitz and Kranzer [3].) The construction of such a solution is in Fig. 7.

Now to solve the general Riemann problem $[u_L, u_R]$ it is only necessary to connect u_L to u_R by s- and c-waves so that the corresponding wave speeds increase from left to right. Such a solution in the xt-plane exists and is unique for every u_L and u_R, and the general solution is given in Figs. 8 and 9. Here Fig. 8 gives the solutions when u_L lies to the left of the transition curve T, while Fig. 9 gives the solutions for u_L to the right of T. These diagrams are read as follows: to solve the Riemann problem $[u_L, u_R]$, follow

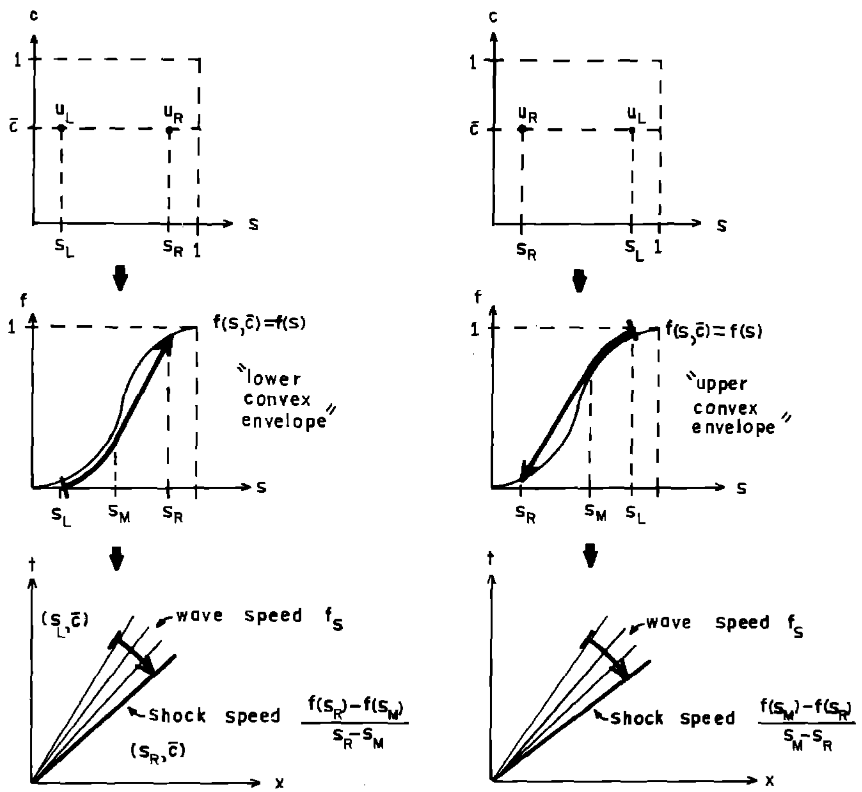


FIG. 6. Riemann problem for nonconvex scalar equation (s-waves).

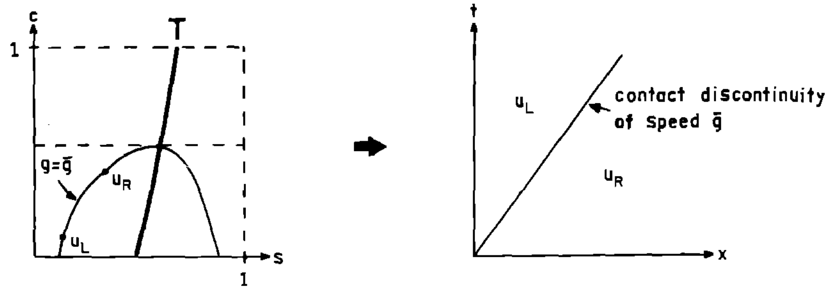


FIG. 7. c-wave solutions.

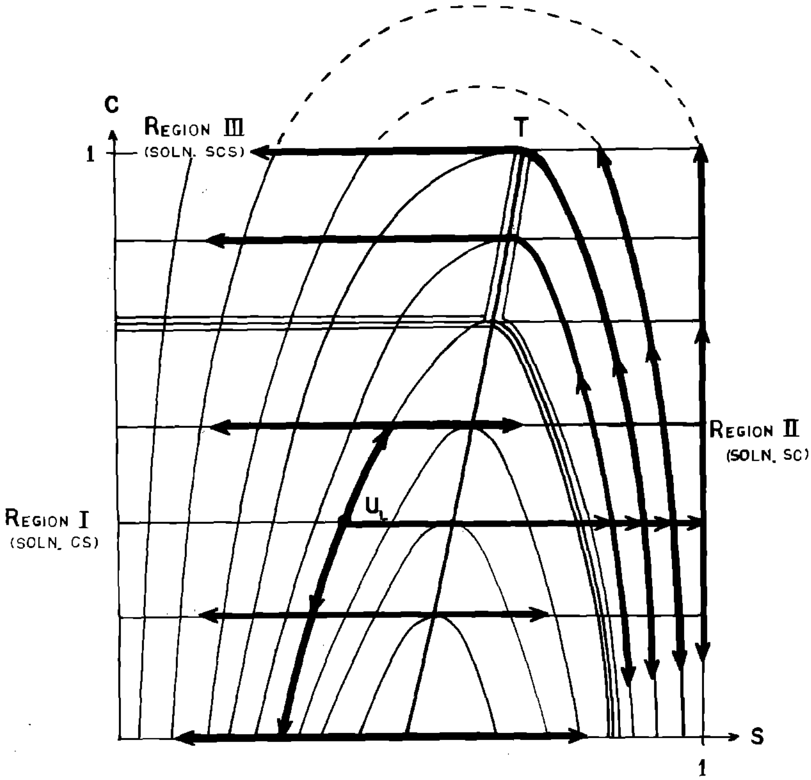


FIG. 8. Riemann problem solution for u_L left of T (i.e., follow the arrows to u_R).

the dark arrows that continuously connect u_L to u_R in either Fig. 8 or Fig. 9. The arrows will cross from one to three s- and c-waves, depending on whether u_R lies in region I, II, or III relative to u_L. Then graph these s- and c-wave solutions in the xt-plane in the direction of the dark arrows, as done in Figs. 6 and 7. The composite of these solutions is the solution to the Riemann problem $[u_L, u_R]$. A sample solution is given in Fig. 10 for the states u_L and u_R plotted in Fig. 9. Note that the waves in an arbitrary Riemann problem solution can be denoted by scS (the s-wave "s" followed by the c-wave "c" followed by the s-wave "S"), where one or more of s, c, or S could be zero. This completes the derivation of the solution to the Riemann problem for system (1.1).

Note that because the coordinate change from (s, b) to (s, c) degenerates at $s = 0$, the Riemann problem solutions are unique in the sc-plane, but fail to be unique in the sb-plane when $s = 0$. For this reason we always assume that data is given first in sc-coordinates. Since $g \geqslant$ const. is an invariant

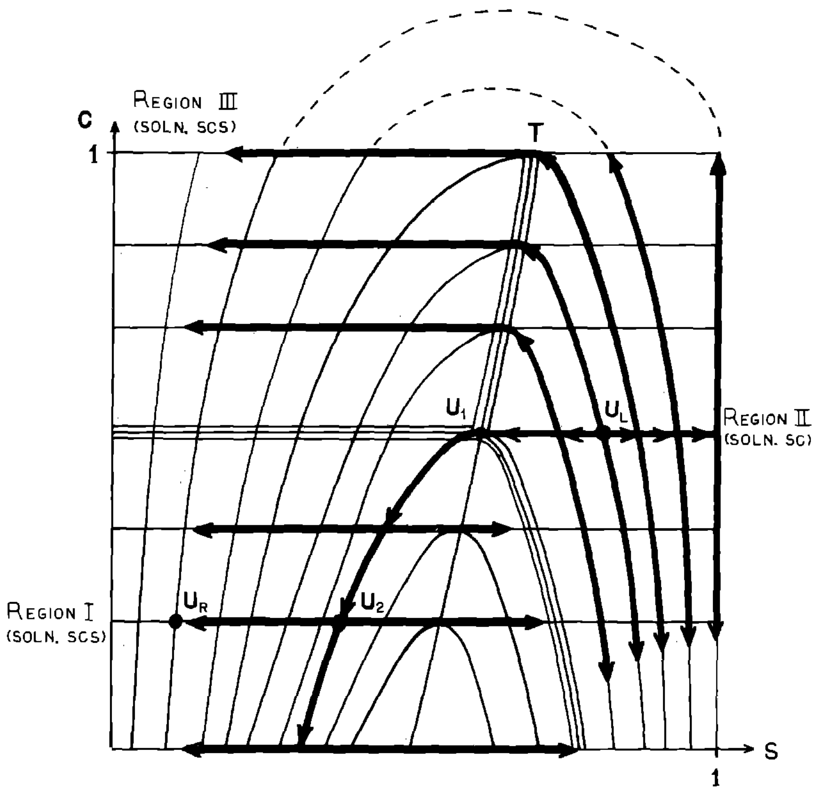


FIG. 9. Riemann problem solution for u_L right of T (i.e., follow the arrows to u_R).

region for Riemann problems, nonuniqueness at $s = 0$ can also be handled by bounding solutions away from $s = 0$.

4. THE GLIMM DIFFERENCE SCHEME

We now apply the Glimm difference scheme to prove the existence of a solution to system (1.1) for general initial data $u_0(x) = \psi(x)$. To establish the scheme, we define a mesh of small rectangles in the xt-plane. Then, in a random way, we approximate the solutions across the bottom of each rectangle by adjacent constant states, so that inside each rectangle, the solution can be approximated by the Riemann problem solutions of Section 3. Helly's theorem can be applied to obtain a convergent subsequence of such approximate solutions, once we obtain a uniform bound on the variation of the approximate solutions as measured under a nonregular transformation of the plane of conserved quantities. This will require only

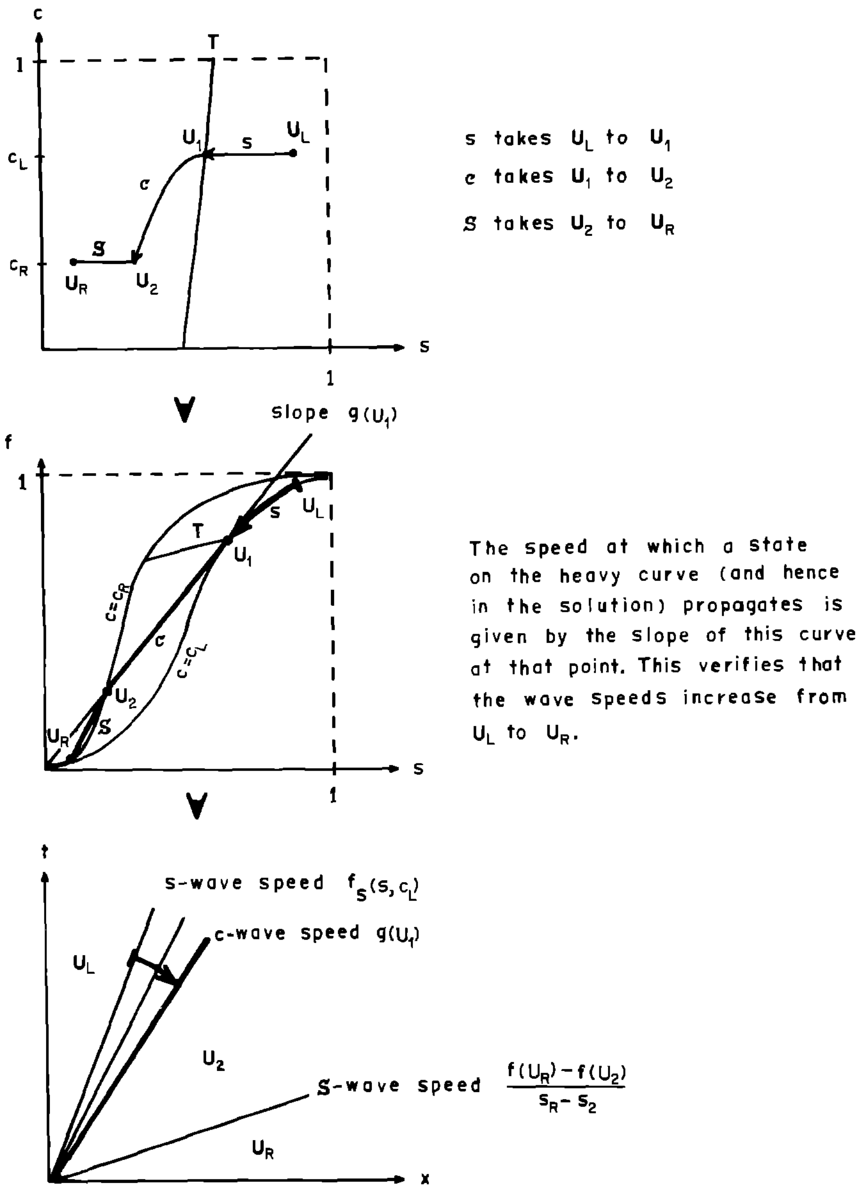


FIG. 10. An explicit solution (for the Riemann problem posed in Fig. 9).

that the initial data $\psi(x)$ have bounded total variation in the transformed plane. A measure space argument is then applied to show that a convergent subsequence satisfies the weak conditions (1.3) in the limit.

We first define the approximate Glimm scheme solutions $u_{ra}(x, t)$ precisely, and develop notation necessary for the subsequent proof. Let r be a mesh length in x, and let the corresponding mesh length in t be given by $\sigma = Mr$, where

$$\frac{r}{\sigma} = \frac{1}{M} \geqslant \sup_{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant c \leqslant 1}} \{\lambda_s, \lambda_c\}. \tag{4.1}$$

(This is the Courant–Friedrichs–Lewy condition, required to ensure that the Riemann problem solutions do not interact during any one time step). Let (m, n) denote any pair of integers such that $n \geqslant 1$ and $m + n$ is even. (This notation is maintained throughout this paper.) Let R_{mn} be the rectangle of base $2r$ and height σ having the point $(mr, n\sigma)$ at the top center of the rectangle. This forms a staggered grid of rectangles diagrammed in Fig. 10. Let $A = \prod_{m,n}[-1, 1]$ be the measure space product of copies of the interval $[-1, 1]$ equipped with normalized Lebesgue measure, one copy for each rectangle. (That is, the measure of the set $E = \prod_{m,n} E_{mn}$ is $\prod_{m,n} \frac{1}{2}\mu(E_{mn})$, where μ denotes Lebesgue measure.) Let $a \in A$ and write $a = \{a_{mn}\}$. Now to define $u_{ra}(x, t)$ at $t = 0 \cdot \sigma$, approximate the initial data in each interval $[(m-1)r, (m+1)r]$ for m even by the constant value $\psi(mr)$, so that Riemann problems are formed at the bottom center of each rectangle R_{m1} (for $m + 1$ even), lying on the x-axis. Once we have solved a Riemann problem in R_{mn}, let $u_{mn}(x)$ for $(m-1)r \leqslant x \leqslant (m+1)r$ be the corresponding solution that occurs along the top of the rectangle R_{mn} (at time $t = n\sigma$). Then to continue the scheme to time $t = (n+1)\sigma$, approximate the solutions $u_{mn}(x)$ along the top of R_{mn} by the constant function $u_{mn}(mr + a_{mn}r)$. This establishes Riemann problems at the bottom center of the rectangles lying on $t = (n+1)\sigma$, and so defines the function $u_{ra}(x, t)$ by induction. (Note that R_{mn} depends only on r, while $u_{mn}(x)$ depends on both r and a.) We call the points $(mr, 0)$ and $(mr + a_{mn}r, n\sigma)$ mesh points, and for notational convenience we identify a_{mn} with the mesh point $(mr + a_{mn}r, n\sigma)$, so that the value $u_{mn}(mr + a_{mn}r)$ is denoted $u_{mn}(a_{mn})$. We also say that two mesh points are "adjacent" if the corresponding rectangles intersect.

In order to obtain bounds on the variation of the approximate solutions $u_{ra}(x, t)$ at every time level, we need a functional F which measures the variation of a solution along any "I-curve." An I-curve J is defined to be any continuous piecewise linear curve in xt-space that successively connects adjacent mesh points, so that the mesh index n increases monotonically from $x = -\infty$ to $x = +\infty$, and such that J is linear between adjacent mesh

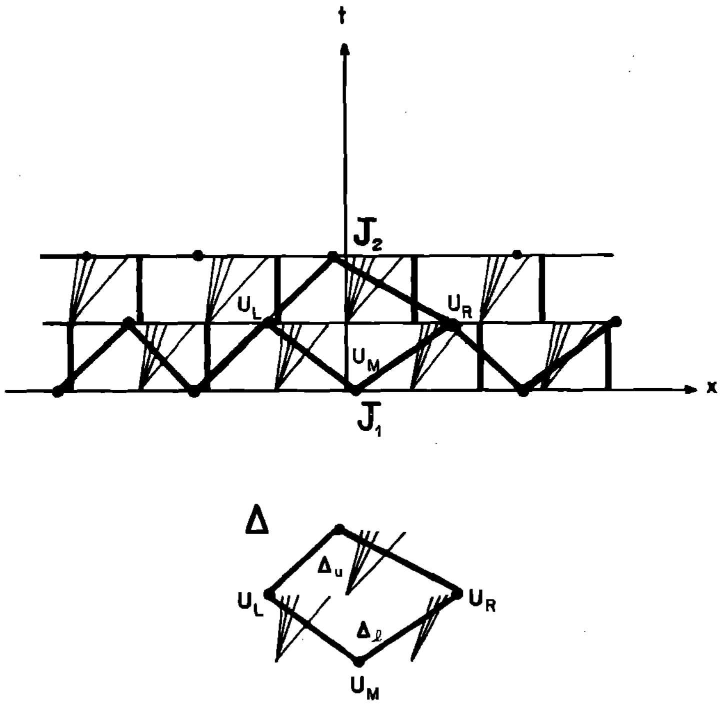


FIGURE 11

points (cf. Fig. 10 and [1]). The main point here is that the unique I-curve that connects the mesh points on $t = n\sigma$ to those on $t = (n + 1)\sigma$ crosses all the waves in the Riemann problem solutions that occur between $t = n\sigma$ and $t = (n + 1)\sigma$. We let Θ denote the initial I-curve that connects mesh points on $t = 0$ to those on $t = \sigma$. Partially order the I-curves by saying that larger curves lie toward larger time and call J_2 an immediate successor of J_1 if both I-curves pass through the same mesh points except at one value of m, where $J_2 \geqslant J_1$. Our method is to define a functional $F(J)$ which dominates the variation on J of $\Psi \cdot u_{ra}(x, t)$, where Ψ is a particular 1–1 function that transforms the plane of conserved quantities in a non-regular way. We then show that $F(J_2) \leqslant F(J_1)$ for every J_2 an immediate successor of J_1, and so by induction we obtain the desired bounds on the variation once we see that $F(\Theta)$is uniformly bounded by the total variation of $\Psi \cdot \psi(x)$. It is important to note that the estimate $F(J_2) \leqslant F(J_1)$ is proved by studying the interaction that occurs in the diamond Δ between J_1 and J_2 as drawn in Fig. 11. Letting Δ_u and Δ_l denote the upper and lower half of Δ as shown in Fig. 11, it is clear that the waves that cross Δ_u solve the Riemann problem $[u_L, u_R]$, while the waves that cross Δ_l solve the consecutive Riemann problems

$[u_L, u_M]$, $[u_M, u_R]$. Thus to show that F decreases between J_1 and J_2, it suffices to study "interactions"; i.e., it suffices to obtain estimates that compare the variation of the waves in the Riemann problem solution $[u_L, u_R]$ to the combined variation in the waves of the solutions $[u_L, u_M]$ and $[u_M, u_R]$, for arbitrary u_M.

Note that we cannot hope to obtain uniform bounds on the total variation of the approximate solutions $u = (s, b)$ because the equations are not strictly hyperbolic at the transition curve T; or more precisely because the curves $g =$ const. and $c =$ const. (curves on which Riemann problem solutions propagate) form a coordinate system that degenerates on T. Indeed, if we choose the initial data $\psi(x)$ to lie on T, as in Fig. 12, the variation in s of the waves that cross the initial I-curve $\mathcal{O}$ will go to infinity as r approaches zero. A second problem that arises because the equations are not strictly hyperbolic is that the waves in the solutions of the Riemann problems, drawn in the sc- or sb-planes, form curves that are discontinuous functions of u_L and u_R. Thus, discontinuous increases in the variation change of the solution between Δ_u and Δ_l can occur as u_L and u_R are smoothly varied. Two key examples are shown in Fig. 13. The first problem above is solved by defining a nonregular transformation that sufficiently flattens the sc-plane near T so as to bound the variation in s that can accumulate around the transition curve; and the discontinuities in the second problem above are resolved by appropriately choosing the functional F.

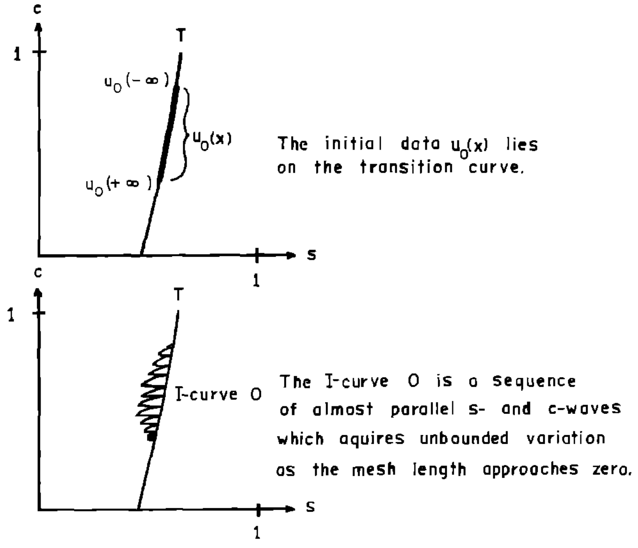


FIG. 12. Example of the appearance of unbounded variation in the approximate solutions.

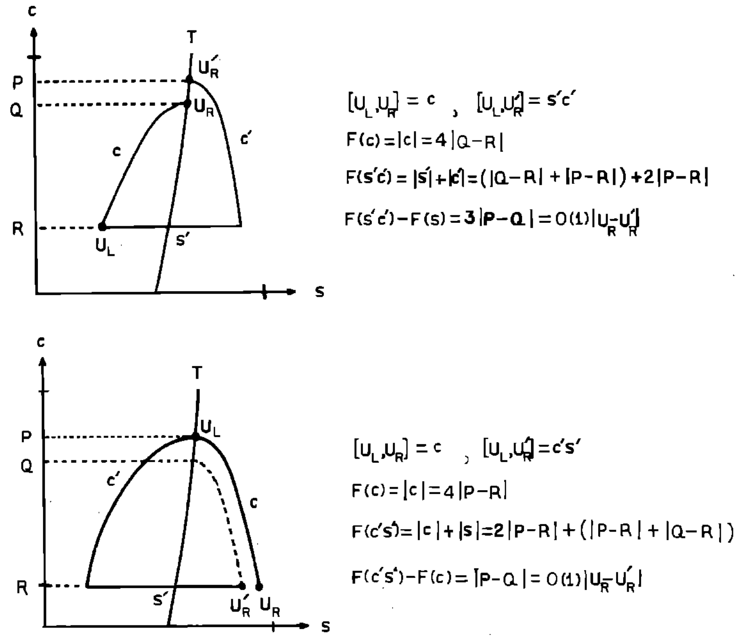


FIG. 13. Two examples demonstrating that the waves in a Riemann problem solution are not continuous with respect to u_L and u_R, but the F-value is.

5. Bounds for Interactions

In this section we show that for any fixed $a \in A$ and any sequence of mesh lengths r that approaches zero, there is a subsequence of approximate solutions $u_{r_k}(x, t) = u_{r_k a}(x, t)$ that converges in L^1 on horizontal lines. This is proven by obtaining a bound on the total variation of a Riemann problem solution $[u_L, u_R]$ in terms of the combined variation in the solutions $[u_L, u_M]$ and $[u_M, u_R]$ for arbitrary states u_L, u_M and u_R; and this requires that variation be measured under a nonregular transformation Ψ.

We let χ denote the map that takes (s, c) to $(s, b) = (s, cs)$, and we let u denote either the point (s, c) or $\chi(s, c)$. Problems with χ^{-1} are easily avoided by always assuming that values of u are prescribed in the sc-plane, the plane where Riemann problems are uniquely solvable. (χ is everywhere differentiable, but fails to be 1–1 and regular at the singular point $s = 0$.) For an arbitrary function $u(x)$, $x \in R$, it is important to keep track of the dependant variables in which total variation is measured. Thus, if Ψ: $(s, c) \to (d, e)$ is a transformation, we let $\mathrm{Var}_{de} u(\cdot)$ or $\mathrm{Var}_{de} u$ denote the total variation of the function obtained from $u(x)$ by transforming values to the de-plane. It is easily verified that if $\mathrm{Var}_{sc} u(\cdot) < \infty$, then also $\mathrm{Var}_{de} u(\cdot)$

$< \infty$ if Ψ is differentiable, but Ψ^{-1} is guaranteed to map functions of bounded variation to functions of bounded variation only if Ψ is also regular (i.e., only if the Jacobian determinant of Ψ is everywhere nonzero). We now define a differentiable transformation Ψ: $(s, c) \to (z, c)$ which is 1–1 and regular except on T. It then follows that the class of functions satisfying $\mathrm{Var}_{zc} u < \infty$ contains all functions of bounded variation in s and c.

Our definition of Ψ involves locating a unique point in the sc-plane where a given curve $g =$ const. intersects the transition curve T. But if g is sufficiently small, the curves $g = f/s =$ const. are not connected, and do not intersect T (cf. Fig. 3). For this reason we need to continue the curves $g =$ const. into the extended sc-plane in some nonintersecting differentiable way, so that the resulting curves change from strictly increasing to strictly decreasing on some smooth continuation of T. Because the curves $g =$ const. already defined in $[0,1] \times [0,1]$ are smooth and have a derivative dc/ds equal to zero only on T, such a continuation can be done in many ways. So choose any smooth continuation of T to values of c between 1 and 2, and extend the curves $g =$ const. in a smooth way so that they are monotone except at T, where they take a maximum value of c in $[0,2]$. Such an extension is indicated in Fig. 14. For values of $g < 1$, the curves $g =$ const. lie only to the left of T in the sc-plane, and so need only be continued to T on the left. We can now define the z-coordinate of a point $P = (s, c)$ in $[0,1] \times [0,1]$ by appealing to the unique value of $c \in [0,2]$, where the curve $g = g(P)$ intersects the transition curve T. Define $z = z(s, c)$ as follows:

$$\begin{aligned} |z| &= |c(Q) - c(P)| \\ \operatorname{sign} z &= + \text{ if } P \text{ lies to the left of } T \text{ in the } sc\text{-plane} \\ &= - \text{ if } P \text{ lies to the right of } T \text{ in the } sc\text{-plane}, \end{aligned} \tag{5.1}$$

where Q is the point where the curve $g = g(P)$ intersects T (again see Fig. 14), and $c(Q)$ [resp. $c(P)$] is the c-coordinate of Q [resp. P]. Now it is clear that since $g = f/s$, $\partial g/\partial s > 0$ away from T and away from $s = 0$, and so $(\partial z/\partial s)(s, c) > 0$ here also. But at $s = 0$, the curves $f(\cdot, c)$ all have p th order contact with the s-axis at $s = 0$, and so it can be verified that the curves $g =$ const. can be extended to the transition curve near $g = 0$ in such a way that $0 < (\partial z/\partial s)(0, c) < \infty$. Therefore, assuming we have chosen such an extension, $(\partial z/\partial s)(s, c) > 0$ everywhere in $[0,1] \times [0,1]$ except at T, and so

$$\begin{aligned} \Psi : [0,1] \times [0,1] &\to [-2,2] \times [0,1] \\ (s, c) &\mapsto (z, c) \end{aligned} \tag{5.2}$$

is one to one differentiable everywhere, and is regular except at T.

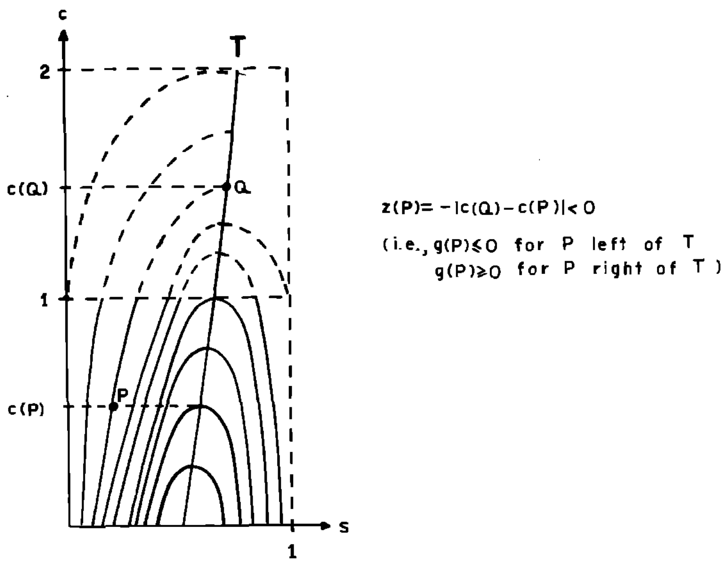


FIG. 14. Defining the coordinate z by continuing the transition curve and the g-const. curves in a differentiable way.

To keep track of the dependent variables, we let u denote either (s, b) or (s, c) but we let w denote (z, c) where it is always assumed that χ and Ψ accomplish the transformations between variables.

We now define the strengths of the waves in a Riemann problem solution. Let c[resp. s] denote a c-wave [resp. an s-wave] that solves a Riemann problem $[u_L, u_R]$. Define

$$\begin{aligned} |s| &= |\Delta z| \\ |c| &= 2\,|\Delta c| \text{ if } s \text{ increases from } u_L \text{ to } u_R \text{ along } c \\ &= 4|\Delta c| \text{ if } s \text{ decreases from } u_L \text{ to } u_R \text{ along } c, \end{aligned} \tag{5.3}$$

where Δz [resp. Δc] denotes the change in z [resp. the change in c] between u_L and u_R. If J is an I-curve, we define

$$F(J) = \sum_J |s_i| + |c_i|, \tag{5.4}$$

where the sum is over all waves that cross J. Refer to Fig. 13 to see that for $J = [u_L, u_R]$, $F(J)$ is continuous with respect to u_L and u_R. By studying interactions in the sc-plane, we now show that among all finite sequences of s- and c-waves that "connect" u_L to u F is minimized by the waves in the Riemann problem solution $[u_L, u_R]$. To make this precise, we generalize the definition of I-curve.

An I-curve has been defined as a piecewise linear curve in xt-space that crosses a sequence of waves from the Riemann problem solutions that appear in the approximate solutions $u_{ra}(x, t)$. Thus the I-curve defines a mapping from a curve in the xt-plane to the curve in the sc-plane which traces out a sequence of s- and c-wave curves given by the waves that J crosses from left to right in the xt-plane. It is this curve in state space that determines $\text{Var}_{sc} J$ and $\text{Var}_{zc} J$. It is convenient for us to generalize the notion of I-curve to include any finite sequence of connected s- and c-wave curves in the sc-plane. By connected we mean that the left state of a wave in the sequence is the right state of its predecessor. Thus an I-curve can be given by listing in order the waves in the sequence, and in this case, any subinterval of the list is also an I-curve. We let $\text{Var}_{sc} J$ and $\text{Var}_{zc} J$ denote the sum of the variations at some fixed time level of the separate s- and c-wave solutions determined by J. Finally, we say that an I-curve joins u_L to u_R if the left state of the first wave is u_L, and the right state of the last wave is u_R. We let $[u_L, u_R]$ denote the unique I-curve that traces the waves in the Riemann problem solution $[u_L, u_R]$. Note that $F(J)$ can still be defined by (5.4) for any I-curve J, and if J_2 is an immediate successor of J_1, then J_1, J_2, Δ_l, and Δ_u are all I-curves. We therefore have that $F(J_2) - F(J_1) = F(\Delta_u) - F(\Delta_l)$, and we use this to prove the following theorem:

THEOREM 5.1. *If J_2 is an immediate successor of J_1, then $F(J_2) \leqslant F(J_1)$.*

Theorem 5.1 is a consequence of the following propositions. Here we let upper and lower case "c" [resp. "s"] denote arbitrary c-waves [resp. s-waves] and we allow waves to have zero strength.

PROPOSITION 5.1. *Let $J' = s'c'S'$ be a generalized I-curve that takes u_L to u_R. Then $F(J) \leqslant F(J')$, where $J = [u_L, u_R] = scS$.*

PROPOSITION 5.2. *Let $J' = c's'C'$ be a generalized I-curve that takes u_L to u_R. Then $F(J) \leqslant F(J')$, where $J = [u_L, u_R] = scS$.*

Since the proofs of Propositions 5.1 and 5.2 involve a study of cases, we postpone the proof until Appendix 1.

LEMMA 5.1. *Let $J_n = b_1 \cdots b_n$ for $b_i = c_i$ or s_i be a generalized I-curve that takes u_L to u_R. Then $F(J) \leqslant F(J_n)$, where $J = [u_L, u_R] = scS$.*

Proof. We first show that there is an I-curve $J' = s'c'S'$ that takes u_L to u_R such that $F(J') \leqslant F(J_n)$. We prove this by induction on the number of c-waves in J_n. Let u_i denote the state that joins b_{i-1} to b_i, and assume that $J_n \neq J' = s'c'S'$ for any such I-curve J'. We show that there exists an I-curve J'' containing at most one c-wave, such that J'' takes u_L to u_R and

$F(J'') \leq F(J_n)$. First, if there exists consecutive waves $b_i b_{i+1}$ in J_n such that $b_i b_{i+1} = s_i s_{i+1}$, then Proposition 5.1 (with $c' = 0$) implies that $F(b_i b_{i+1}) \geq F(s)$, where $[u_i, u_{i+2}] = s$ (since u_i and u_{i+2} lie at constant c). Therefore $F(J_n) \geq F(J_{n-1})$ where $J_{n-1} = b_1 \dots b_{i-1} s b_{i+2} \dots b_n$ and J_{n-1} takes u_L to u_R with the same number of c-waves. Therefore we can assume that no such pair $b_i b_{i+1}$ exists in J_n. But if no consecutive s-waves occur in J_n, then since $J_n \neq s'c'S'$, there must be an i such that $b_i b_{i+1} = c_i c_{i+1}$ or $b_i b_{i+1} b_{i+2} = c_i s_{i+1} c_{i+2}$. Without loss of generality, assume the latter case, where $b_i b_{i+1} b_{i+2}$ takes u_i to u_{i+3}. Then by Proposition 5.2, $F(b_i b_{i+1} b_{i+2}) \geq F(scS)$, where $[u_i, u_{i+3}] = scS$, and so $F(J_n) \geq F(\tilde{J})$ where $\tilde{J} = b_1 \dots b_{i-1} scS b_{i+3} \dots b_n$ takes u_L to u_R. But $\tilde{J}$ has one fewer c-wave than J_n, which proves by induction that there is an I-curve J'' that takes u_L to u_R such that J'' contains at most one c-wave, and such that $F(J_n) \geq F(J'')$. But by the comment above, $F(J'') \geq F(J')$, where J' contains at most one c-wave and J' contains no consecutive s-waves, and hence $J' = s'c'S'$ for some $s'c'S'$ which takes u_L to u_R. Therefore, by Proposition 5.1, we have

$$F(J) \leq F(J') \leq F(J_n).$$

This completes the proof of Lemma 5.1.

Proof of Theorem 5.1. Since $F(J_2) - F(J_1) = F(\Delta_u) - F(\Delta_l)$, we need only show that $F(\Delta_u) - F(\Delta_l) \leq 0$. But Δ_l takes u_L to u_R and contains the waves in the solutions of two consecutive Riemann problems, so that $\Delta_l = s_1 c_1 S_1 s_2 c_2 S_2$. Therefore, since $\Delta_u = [u_L, u_R]$, we have by Lemma 5.1 that $F(\Delta_u) \leq F(\Delta_l)$. This completes the proof of Theorem 5.1.

From (5.1) it is clear that the strength of an s- or c-wave dominates the variation in z and c of that wave, and that $F(J) \geq \mathrm{Var}_{zc} J$ for every I-curve J. Therefore, Theorem 5.1 implies that there is a uniform bound on $\mathrm{Var}_{zc} u_{ra}(\cdot, t)$ at every time step, so long as $F(\Theta) = 0(1)\ \mathrm{Var}_{zc}\psi$. This follows once we show that

$$\mathrm{Var}_{zc} J = 0(1)|w_L - w_R| \tag{5.5}$$

for any Riemann problem $[u_L, u_R] = J$, where $w_L = (z_L, c_L)$ and $w_R = (z_R, c_R)$; for then we can write

$$F(\Theta) \leq 4\sum_i \mathrm{Var}_{zc} J^i \leq 0(1)\sum_i |\Psi \cdot \psi(x_{i+1}) - \Psi \cdot \psi(x_i)| \leq 0(1)\mathrm{Var}_{zc}\psi$$

(where J^i are the Riemann problem solutions traced out by 0, and x_i are the mesh points at $t = 0$). To verify (5.5), define $S(\varepsilon)$ to be the set of points in the sc-plane within a distance of ε from T. Then for u_L and u_R not in $S(\varepsilon)$, (5.5) is clearly true, since Riemann problem solutions have a finite number of waves that globally lie in a bounded set, and which locally involve waves that intersect transversally in the sc-plane; while for u_L and u_R in $S(\varepsilon)$, $\mathrm{Var}_{zc}(J) \leqslant 5|w_L - w_R|$ for ε sufficiently small, by construction of z. This verifies (5.5) and so proves that at each $t \geqslant 0$ (and for any r and $a \in A$),

$$\mathrm{Var}_{zc} u_{ra}(\cdot, t) = 0(1)\mathrm{Var}_{zc}\psi. \tag{5.6}$$

Statement (5.6) implies, by a standard argument (see [1]), that the approximate solutions $w_{ra}(x, t)$ are L^1 continuous in time uniformly in a. This is stated in the following lemma whose proof is given in Appendix 2.

LEMMA 5.2. $\int_{-\infty}^{\infty}|w_{ra}(x, t_2) - w_{ra}(x, t_1)|dx = 0(1)\{|t_2 - t_1| + r\}\mathrm{Var}_{zc}\psi$, *where* $0(1)$ *is independent of* r, a, *and* t_i.

We can now prove the main theorem of this section, which uses the compactness argument in [1] due to Olenik. In what follows, a sequence of functions V_k of one real variable is said to converge to V in L^1_{loc} if, for every $M > 0$, $V_k \to V$ in $L^1[-M, M]$.

THEOREM 5.2. *Let* $a \in A$ *and let* ψ *be any initial data satisfying* $\mathrm{Var}_{zc}\psi < \infty$. *Then for any sequence of mesh lengths* r *which approaches zero, there exists a subsequence* r_k *and a function* u *such that, for any* T, $u_{r_k a}(\cdot, t)$ *converges in* L^1_{loc} *to* $u(\cdot, t)$ *uniformly for* $t \leqslant T$. (*Note that since* $u_{ra}(x, 0) = \psi(mr)$ *for* $(m - 1)r \leqslant x \leqslant (m + 1)r$, $u_{r_k a}(\cdot, 0)$ *also converges to* ψ *in* L^1_{loc}).

Proof. By (5.6), $w_{ra}(\cdot, t)$ has uniformly bounded total variation on every time level, and so by Helly's theorem, a subsequence converges in L^1 on bounded intervals $|x| \leqslant M$ of every horizontal line. By the diagonal process we can achieve this result on the countable dense set of rational times $t = h/k$. Let $w_{r_l} = w_l$ be this subsequence. We apply Lemma 5.2 to show that there is a further subsequence $w_{l(k)}$ that converges uniformly in $L^1[-M, M]$ at every fixed $t \leqslant T < \infty$; i.e., we write

$$\begin{aligned}\int_{-M}^{M}|w_i(x, t) - w_j(x, t)|dx &\leqslant \int_{-M}^{M}\left|w_i(x, t) - w_i\left(x, \frac{h}{k}\right)\right|dx \\ &\quad + \int_{-M}^{M}\left|w_i\left(x, \frac{h}{k}\right) - w_j\left(x, \frac{h}{k}\right)\right|dx \qquad (5.7) \\ &\quad + \int_{-M}^{M}\left|w_j\left(x, \frac{h}{k}\right) - w_j(x, t)\right|dx.\end{aligned}$$

For fixed k, choose $l(k)$ so that $r_k \leqslant 1/k$ and so that if $i, j \geqslant l(k)$, the middle term on the R.H.S. above is uniformly bounded by $1/k$ whenever $h \leqslant kT$. Then by choosing h from these values, and applying Lemma 5.2, the R.H.S. of (5.7) can be bounded by $0(1)(1/k)$ for every $t \leqslant T$. Therefore, $w_{l(k)}$ converges uniformly in $L^1[-M, M]$ at every $t \leqslant T$. Since M and T are arbitrary, we can apply this idea a countable number of times to conclude that there is a further subsequence $w_{r_k}(x, t)$ such that for any $T < \infty$, $w_{r_k}(\cdot, t)$ converges in L^1_{loc} to $w(\cdot, t)$ uniformly in $t \leqslant T$. The uniform continuity of Ψ^{-1} and χ now implies this result for $u_{r_k}(x, t) = \chi \cdot \Psi^{-1} \cdot w_{r_k}(x, t)$ and $u(x, t) = \chi \cdot \Psi^{-1} \cdot w(x, t)$. This concludes the proof of Theorem 5.2.

6. The Weak Solution

We have shown that for any fixed $a \in A$ and any sequence of mesh lengths r approaching zero, if $\text{Var}_{zc}\psi$ is bounded, then there is a subsequence of the $u_{ra}(x, t)$ that converges to some function $u_a(x, t)$. The convergence is in L^1_{loc} of space, and is uniform on bounded intervals of time. It is not true, however, that the solution $u_a(x, t)$ satisfies the weak conditions (1.3) for every $a \in A$. As a simple counterexample, let $a_{mn} = 0$ for every (m, n), and let the initial data be data for a Riemann problem whose solution is a single shock wave. It is easy to see that the limiting solution has the same discontinuity as the Riemann problem solution except that it moves with speed $r/\sigma = 1/M$ and not the shock speed. We prove, on the other hand, that $u_a(x, t)$ is a weak solution to system (1.1) for almost every a in the measure space A, and so conclude the proof of Theorem 6.1. The main point here is that our bound on the variation of the approximate solutions (inequality (5.6)) is given in terms of z and c, so that variation in u_{ra} can accumulate wherever Ψ is not regular (i.e., near the transition curve). Since the weak conditions (1.3) must be stated in terms of the conserved quantities, we must modify Glimm's argument in [1] to handle this unbounded variation in u.

We need to measure how close u_{ra} is to a weak solution of (1.1), so for any C^1 function $\phi(x, t)$ with compact support in $-\infty < x < +\infty$, $t \geqslant 0$ (i.e., $\phi \in C^1_0$), define

$$D(r, a, \phi) = \int_0^\infty \int_{-\infty}^\infty (\phi_t u_{ra} + \phi_x F(u_{ra}))\, dx\, dt + \int_{-\infty}^\infty \phi(x, 0)\psi_r(x)\, dx, \tag{6.1}$$

where $\psi_r(x) = \psi(mr)$ for $(m - 1)r \leqslant x \leqslant (m + 1)r$ and $\text{Var}_{zc}\psi < \infty$. Since u_{ra} is a weak solution of (1.1) in each horizontal strip $n\sigma < t < (n + 1)\sigma$, we can compute

$$D(r, a, \phi) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi(x, n\sigma)(u_{ra}(x, n\sigma) - u_{ra}(x, n\sigma - 0))\, dx. \tag{6.2}$$

We are interested in the error term $D(r, a, \phi)$ when ϕ is only piecewise constant with compact support, and so we take equation (6.2) as the definition of $D(r, a, \phi)$ when ϕ is in this general class of test functions. Note that we cannot then equate (6.2) with (6.1) unless ϕ is also smooth. For every r, we decompose the error term into a sum of integrals defined along the top of each rectangle R_{mn}; i.e., let

$$D_{mn}(r, a, \phi) = \int_{(m-1)r}^{(m+1)r} \phi(x, n\sigma)\big(u_{mn}(a_{mn}) - u_{mn}(x)\big)\, dx \qquad (6.3)$$

so that

$$D(r, a, \phi) = \sum_{m,n} D_{mn}(r, a, \phi). \qquad (6.4)$$

(Here, recall, that $u_{mn}(x) = u_{ra}(x, n\sigma)$ for $(m-1)r \leqslant x \leqslant (m+1)r$, $u_{mn}(a_{mn}) = u_{mn}(mr + a_{mn}r)$, and summations over all (m, n) such that $n \geqslant 1$ and $m + n$ is even). Now for any test function $\phi \in C_0^1$, define

$$\phi_r(x, t) = \phi(mr, n\sigma) \text{ on } R_{mn}. \qquad (6.5)$$

That is, ϕ_r is constant on each of the rectangles in the mesh determined by the mesh length r, and ϕ_r agrees with ϕ at some point in each rectangle. Note that since $\phi \in C_0^2$, ϕ_r has compact support uniformly in r, and moreover

$$\|\phi_r\|_\infty \leqslant \|\phi\|_\infty. \qquad (6.6)$$

Since $\phi \in C_0^1$, ϕ is uniformly Lipschitz in x and t, so that

$$\|\phi - \phi_r\|_\infty \leqslant 0(1)r, \qquad (6.7)$$

where 0(1) depends only on ϕ. Finally, we shall need the following facts which are easily verified:

$$|D_{mn}(r, a, \phi)| \leqslant 0(1)\|\phi\|_\infty r \operatorname{Var}_{sc} u_{mn},$$
$$|D_{mn}(r, a, \phi_r)| \leqslant 0(1)\|\phi\|_\infty r \operatorname{Var}_{sc} u_{mn}. \qquad (6.8)$$

Now consider $D(r, a, \phi_r)$ as a function of $a \in A$. Let $\langle\ ,\ \rangle_A$ denote the inner product for $L^2(A)$, and let $\|\ \|_2$ denote the L^2 norm for A. The main lemma of this section is:

LEMMA 6.1. *For any fixed* $\phi \in C_0^1$,

$$\lim_{r \to 0} \|D(r, \cdot, \phi_r)\|_2^2 = 0.$$

We prove this with the aid of the following propositions.

PROPOSITION 6.1. *For any fixed ϕ and any fixed mesh length r, if $(m, n) \neq (m', n')$, then*

$$\left\langle D_{mn}(r, \cdot, \phi_r), D_{m'n'}(r, \cdot, \phi_r)\right\rangle_A = 0. \tag{6.9}$$

Proof. Without loss of generality, assume $n' \leqslant n$. Note that the value of $D_{m'n'}(r, a, \phi_r)$ depends only on values of a_{ij} lying in the "domain of dependence" of (m', n') (i.e., values of a_{ij} in the triangle centered below $(mr, n\sigma)$ with lateral sides of slope $\pm\sigma/r = \pm M$). Therefore, writing $D_{mn}(a) = D_{mn}(r, a, \phi_r)$, where arguments not appearing are assumed to be held fixed, we have that the value $D_{m'n'}(a)$ is independent of a_{mn}, and so

$$\begin{aligned}\left\langle D_{mn}(\cdot), D_{m'n'}(\cdot)\right\rangle_A &= \int_A D_{mn}(a) D_{m'n'}(a) \\ &= \int_{\hat{A}} D_{m'n'}(a) \int_{-1}^{1} D_{mn}(a)\, da_{mn}\, d\hat{a} = 0\end{aligned}$$

since

$$\begin{aligned}&\int_{-1}^{1} D_{mn}(a)\, da_{mn} \\ &\quad = \phi_r(mr, n\sigma) \int_{-1}^{1} \int_{(m-1)r}^{(m+1)r} \left(u_{mn}(mr + a_{mn}r) - u_{mn}(x)\right) dx\, da_{mn} \\ &\quad = \phi_r(mr, n\sigma) \int_{(m-1)r}^{(m+1)r} \int_{(m-1)r}^{(m+1)r} \left(u_{mn}(\xi) - u_{mn}(x)\right) dx\, d\xi = 0.\end{aligned}$$

Here $\hat{A}$ denotes the measure space obtained from A by deleting the $[-1, 1]_{mn}$ factor, and $\hat{a} \in \hat{A}$. This concludes the proof of Proposition 6.1.

By (6.9), $D_{mn}(\cdot)$ is orthogonal to $D_{m'n'}(\cdot)$ with repsect to $L^2(A)$, and so we conclude

$$\|D(r, \cdot, \phi_r)\|_2^2 = \sum_{m,n} \|D_{mn}(r, \cdot, \phi_r)\|_2^2, \tag{6.10}$$

and since ϕ_r has compact support uniformly in r, there are $0(1)r^{-2}$ nonzero terms in this sum ($0(1)$ depending only on ϕ). We cannot now apply (6.8) directly to (6.10) in order to prove Lemma 6.1 (as done in [1]) because here we have no uniform bound on the variation of the approximate solutions in s and c. To remedy this, for each $a \in A$ and mesh length r, we partition the set of all rectangles in the xt-plane according to how far the values $u_{mn}(x)$ lie from the transition curve T (the only place where unbounded variation in the approximate solutions u_{ra} can accumulate.)

Recall that for any small $\varepsilon > 0$, $S(\varepsilon)$ is the strip in sc-space of all points within a distance strictly less than ε of T. Now $u_{mn}(x)$ is a function of $x \in [(m-1)r, (m+1)r]$ whose image consists of the states in the solution of the Riemann problem that occurs at the mesh point $(mr, (n-1)\sigma)$. So for each r and $a \in A$, we can partition the set of all mesh rectangles into two subsets, according to whether the image of $u_{mn}(x)$ lies entirely within $S(\varepsilon)$ or not. Let $R = R(r, a, \varepsilon)$ denote the set of all indices (m, n) which index the u_{mn} whose images lie strictly within $S(\varepsilon)$. Now for every r, ε and ϕ, we estimate (6.10) as follows:

$$\begin{aligned}\|D(r,\cdot,\phi_r)\|_2^2 &= \sum_{m,n} \|D_{mn}(\cdot)\|_2^2 = \sum_{m,n} \int_A |D_{mn}(a)|^2\,da \\ &= \int_A \sum_{m,n} |D_{mn}(a)|^2\,da \\ &\leqslant \sup_{a\in A} \sum_{m,n} |D_{mn}(a)|^2 \\ &\leqslant \sum_{m,n} |D_{mn}(\bar a)|^2 + \varepsilon^2\end{aligned}$$

for some $\bar a \in A$. We can now partition this sum according to whether $(m, n) \in R(r, \bar a, \varepsilon) = R$ or not, and write

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant \sum_{(m,n)\in R} |D_{mn}(\bar a)|^2 + \sum_{(m,n)\notin R} |D_{mn}(\bar a)|^2 + \varepsilon^2. \tag{6.11}$$

In order to show that $\|D(r,\cdot,\phi_r)\|_2^2$ goes to zero with r, we estimate the sums in (6.11) differently. The first sum is estimated as follows: for ε sufficiently small, the sc-variation in any Riemann problem solution which is contained entirely within $S(\varepsilon)$, is dominated by its variation in s, and since there are at most three waves in any solution, this variation must be dominated by 5ε. Thus, if values of u_{mn} lie entirely within $S(\varepsilon)$, then $\mathrm{Var}_{sc} u_{mn} < 5\varepsilon$. We now apply (6.8) and write

$$\begin{aligned}|D_{mn}(r,a,\phi_r)| &= \left|\phi_r(mr, n\sigma)\int_{(m-1)r}^{(m+1)r} (u_{mn}(a_{mn}) - u_{mn}(x))\,dx\right| \\ &\leqslant 0(1)\|\phi\|_\infty r\varepsilon\end{aligned} \tag{6.12}$$

for any u_{mn} whose values lie entirely within $S(\varepsilon)$, and so

$$\sum_{(m,n)\in R} |D_{mn}(\bar a)|^2 \leqslant \sum_{m,n} 0(1) r^2\varepsilon^2,$$

where 0(1) depends only on ϕ. But there are $0(1)r^{-2}$ nonzero terms in this sum, and so

$$\sum_{(m,n)\in R} |D_{mn}(\bar{a})|^2 \leqslant 0(1)\varepsilon^2. \tag{6.13}$$

We now estimate the second sum in (6.11) with the aid of the following proposition:

PROPOSITION 6.2. *If values $u_{mn}(x)$ are not entirely contained within $S(\varepsilon)$, then*

$$\mathrm{Var}_{sc} u_{mn} \leqslant \mathrm{Const}(\varepsilon)\mathrm{Var}_{zc} u_{mn}, \tag{6.14}$$

and for any $\phi \in C_0^1$,

$$\begin{aligned} |D_{mn}(r, a, \phi)| &\leqslant \mathrm{Const}(\varepsilon)\|\phi\|_\infty r\,\mathrm{Var}_{zc} u_{mn}, \\ |D_{mn}(r, a, \phi_r)| &\leqslant \mathrm{Const}(\varepsilon)\|\phi\|_\infty r\,\mathrm{Var}_{zc} u_{mn}, \end{aligned} \tag{6.15}$$

where Const(ε) *denotes a constant that depends only on* ε.

Assuming Proposition 6.2 (whose proof we postpone until after the proof of Lemma 3.1), we obtain

$$\begin{aligned} \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 &\leqslant \sum_{m,n} |r\,\mathrm{Const}(\varepsilon)\mathrm{Var}_{zc} u_{mn}|^2 \\ &\leqslant r^2\,\mathrm{Const}(\varepsilon)^2 \sum_n \sum_m (\mathrm{Var}_{zc} u_{mn})^2 \\ &\leqslant r^2\,\mathrm{Const}(\varepsilon)^2 \sum_n \Big(\sum_m \mathrm{Var}_{zc} u_{mn}\Big)^2, \end{aligned}$$

where, since ϕ is fixed, we have absorbed $\|\phi\|_\infty$ into Const(ε). But for fixed n and any $a \in A$, we have (by (5.6))

$$\sum_m \mathrm{Var}_{zc} u_{mn} \leqslant 0(1)\mathrm{Var}_{zc}\psi$$

and so we can continue

$$\begin{aligned} \sum_{(m,n)\notin R} |D_{mn}(\bar{a})|^2 &\leqslant r^2\,\mathrm{Const}(\varepsilon)^2 \sum_n (\mathrm{Var}_{zc}\psi)^2 \\ &\leqslant r\,\mathrm{Const}(\varepsilon)^2 \end{aligned} \tag{6.16}$$

since there are $0(1)r^{-1}$ nonzero terms in the summation over n. Putting

(6.13) and (6.16) into (6.11) yields

$$\|D(r,\cdot,\phi_r)\|_2^2 \leqslant 0(1)\varepsilon^2 + \text{Const}(\varepsilon)^2 r, \tag{6.17}$$

where the constants are uniform for fixed ϕ. Thus we can choose ε small to make the first term in (6.17) small, and then choose r small at that ε to make the second term small, to conclude

$$\lim_{r\to 0}\|D(r,\cdot,\phi_r)\|_2^2 = 0.$$

This concludes the proof of Lemma 6.1, once we give the

Proof of Proposition 6.2. (We do the case for $D(r, a, \phi)$ in (6.15), the estimate for $D(r, a, \phi_r)$ being obtained similarly.) Since $z(s, c) = 0$ only on the transition curve T, and since $S(\varepsilon)^c$ (the complement of $S(\varepsilon)$ in $[0,1]\times[0,1]$) is compact and bounded away from T, the continuity of $z(s, c)$ implies that there exists a constant $\delta(\varepsilon) > 0$ such that

$$|z(s,c)| \geqslant \delta(\varepsilon) \qquad \text{in } S(\varepsilon)^c. \tag{6.18}$$

Choose $\bar{\varepsilon} > 0$ such that

$$|z(s,c)| \leqslant \frac{\delta(\varepsilon)}{2} \qquad \text{in } S(\bar{\varepsilon}). \tag{6.19}$$

We now consider two cases, depending on whether the image of u_{mn} lies entirely within $S(\bar{\varepsilon})^c$ or not. First, if the values of $u_{mn}(x)$ are so contained, then since $\Psi:(s,c)\mapsto(z,c)$ is one to one and regular on the compact set $S(\bar{\varepsilon})^c$, there exists a Const(ε) such that

$$\text{Var}_{sc}u_{mn} \leqslant \text{Const}(\varepsilon)\text{Var}_{zc}u_{mn}.$$

Thus by (6.8),

$$\begin{aligned}|D_{mn}(r,a,\phi)| &\leqslant 0(1)\|\phi\|_\infty r\,\text{Var}_{sc}u_{mn}\\ &\leqslant \text{Const}(\varepsilon)\|\phi\|_\infty r\,\text{Var}_{zc}u_{mn},\end{aligned} \tag{6.15$_A$}$$

which proves the proposition in the first case.

Now assume that the image of u_{mn} is not entirely contained within $S(\bar{\varepsilon})^c$. Since by assumption this image is not contained within $S(\varepsilon)$, there must exist two points $u_1 = (s_1, c_1)$ and $u_2 = (s_2, c_2)$ in the image of u_{mn} such that $u_1 \in S(\bar{\varepsilon})$ and $u_2 \in S(\varepsilon)^c$. By (6.19), $z(u_2) - z(u_1) \geqslant \delta(\varepsilon)/2$. Therefore

$$\text{Var}_{zc}u_{mn} \geqslant \frac{\delta(\varepsilon)}{2}. \tag{6.20}$$

But the solution to any Riemann problem occurs in the compact set $[0, 1] \times [0, 1]$ in sc-space, and so it is easily seen that there exists a constant $K > 0$ such that the variation in s and c of this solution at any time step is uniformly bounded by K. since $u_{mn}(x)$ is the solution at $t = n\sigma$ of a Riemann problem posed at $(mr, (n-1)\sigma)$, we have

$$\begin{aligned}\operatorname{Var}_{sc} u_{mn} \leqslant K &= \left(K\frac{2}{\delta(\varepsilon)}\right)\frac{\delta(\varepsilon)}{2} \\ &\leqslant \left(K\frac{2}{\delta(\varepsilon)}\right)\operatorname{Var}_{zc} u_{mn} \\ &= \operatorname{Const}(\varepsilon)\operatorname{Var}_{zc} u_{mn},\end{aligned}$$

where we have applied (6.20). Therefore, by (6.8), we can conclude

$$\begin{aligned}|D_{mn}(r, a, \phi)| &\leqslant 0(1)\|\phi\|_\infty r \operatorname{Var}_{sc} u_{mn} \\ &\leqslant \operatorname{Const}(\varepsilon)\|\phi\|_\infty r \operatorname{Var}_{zc} u_{mn}.\end{aligned}$$

This concludes the proof of Proposition 6.2, and so completes the proof of Lemma 6.1. Note, however, that if the curves $g = \text{const.}$ and $c = \text{const.}$ make kth order contact on the transition curve T, we can estimate $\operatorname{Const}(\varepsilon) = 0(1)\varepsilon^{-k-1}$. To see this, define Δs at a point $P = (s, c)$ to be $|s - s'|$ where $Q = (s', c)$ lies on the transition curve. Then the kth order contact of the curves $g = \text{const.}$ and $c = \text{const.}$ means that $z = 0(1)(\Delta s)^{k+1}$ at any P, where $0(1)$ is uniform in P. From this we can conclude that on $S(\varepsilon)^c$, $z \geqslant G\varepsilon^{k+1}$, while at some $\bar{\varepsilon}$ small, $z = 0(1)\bar{\varepsilon}^{k+1} \leqslant \frac{1}{2}G\varepsilon^{k+1}$ in $S(\bar{\varepsilon})$ for some constant G independent of ε. Therefore, $\delta(\varepsilon)$ can be replaced by $G\varepsilon^{k+1}$ in (6.18) and (6.19), so that for $(6.15)_{\mathrm{A}}$ and $(6.15)_{\mathrm{B}}$, $\operatorname{Const}(\varepsilon)$ is $0(1)\varepsilon^{-k-1}$. Thus $\operatorname{Const}(\varepsilon)^2$ is $0(1)\varepsilon^{-2k-2}$ in (6.16) and (6.17). We use this fact at the end of this section to obtain a rate of convergence of the approximate solutions in the presence of this kth order contact along the transition curve.

The fact that $\|D(r, \cdot, \phi_r)\|_2^2$ approaches zero as r approaches zero for fixed ϕ implies that, given any sequence of mesh lengths r approaching zero, there exists a subsequence r_k such that $D(r_k, a, \phi_{r_k}) \to 0$ as $r_k \to 0$ for almost every $a \in A$. But $D(r, a, \phi)$ as defined in (6.1) is linear in ϕ, so that for $\phi \in C_0^1$ we can write

$$D(r, a, \phi) = D(r, a, \phi_r) + D(r, a, \phi - \phi_r). \tag{6.21}$$

Therefore, Lemma 6.1 implies that (for fixed ϕ) $D(r_k, a, \phi) \to 0$ as $r_k \to 0$ for almost every $a \in A$ if we can show that $D(r, a, \phi - \phi_r) \to 0$ as $r \to 0$

for all $a \in A$. This is

LEMMA 6.2. *For any fixed $a \in A$ and $\phi \in C_0^1$,*

$$\lim_{r \to 0} |D(r, a, \phi - \phi_r)| = 0. \tag{6.22}$$

Proof. Let ε be any small positive number and let $a \in A$ and $\phi \in C_0^1$ be fixed. By the triangle inequality

$$\begin{aligned} |D(r, a, \phi - \phi_r)| &\leqslant \sum_{m,n} |D_{mn}(r, a, \phi - \phi_r)| \\ &= \sum_{(m,n) \in R} |D_{mn}(r, a, \phi - \phi_r)| \\ &\quad + \sum_{(m,n) \notin R} |D_{mn}(r, a, \phi - \phi_r)|, \end{aligned} \tag{6.23}$$

where $R = R(r, a, \varepsilon)$. We use (6.12) to estimate the first sum and (6.15) to estimate the second sum.

$$\begin{aligned} \sum_{(m,n) \in R} |D_{mn}(r, a, \phi - \phi_r)| &\leqslant \sum_{m,n} 0(1) \|\phi - \phi_r\|_\infty r\varepsilon \\ &\leqslant \sum_{m,n} 0(1) r^2 \varepsilon \\ &\leqslant 0(1)\varepsilon, \end{aligned} \tag{6.24}$$

where we have applied (6.7) and the fact that there are $0(1)r^{-2}$ terms in this sum (0(1) depending only on ϕ).

$$\begin{aligned} \sum_{(m,n) \notin R} |D_{mn}(r, a, \phi - \phi_r)| &\leqslant \sum_{m,n} \mathrm{Const}(\varepsilon) \|\phi - \phi_r\|_\infty r \,\mathrm{Var}_{zc} u_{mn} \\ &\leqslant \mathrm{Const}(\varepsilon) r^2 \sum_n \Big(\sum_m \mathrm{Var}_{zc} u_{mn} \Big) \\ &\leqslant \mathrm{Const}(\varepsilon) r^2 \sum_n \mathrm{Var}_{zc} \psi \\ &\leqslant \mathrm{Const}(\varepsilon) r, \end{aligned} \tag{6.25}$$

where we have applied (6.7), (5.6), and the fact that there are $0(1)r^{-1}$ terms in the summation over n. Replacing (6.24) and (6.25) in (6.23) yields (at any fixed a and ϕ)

$$|D(r, a, \phi - \phi_r)| \leqslant 0(1)\varepsilon + \mathrm{Const}(\varepsilon) r. \tag{6.26}$$

Since 0(1) and Const(ε) are independent of r, we can choose ε small to make the first term in (6.26) small, and then choose r small at that ε to conclude

$$\lim_{r\to 0}|D(r, a, \phi - \phi_r)| = 0$$

for every $\phi \in C_0^1$ and $a \in A$. This completes the proof of Lemma 6.2, and so we can conclude from Lemma 6.1 together with (6.21) that, given any fixed $\phi \in C_0^1$ and any sequence of mesh lengths r approaching zero, there exists a subsequence r_k such that

$$\lim_{r_k\to 0} D(r_k, a, \phi) = 0 \qquad \text{for a.e. } a \in A. \tag{6.27}$$

We now extend this result to hold uniformly over $\phi \in C_0^1$.

LEMMA 6.3. *Given any sequence of mesh lengths r approaching zero there exists a subsequence r_k and a set N of measure zero in A, such that, if $a \in A \setminus N$, then*

$$\lim_{r_k\to 0} D(r_k, a, \phi) = 0$$

for any $\phi \in C_0^1$.

Proof. Choose a countable set of functions $\{\phi^i\} \subset C_0^1$ such that $\{\phi^i\}$ is dense in C^1, uniformly on compact sets. That is, assume that every function $\phi \in C_0^1$ is the limit of a sequence of functions from $\{\phi^i\}$ such that every function in this sequence has its support within a single compact set (such a set of functions can easily be constructed via the diagonal process). By the diagonal process we can obtain from (6.27) that, given any sequence of mesh lengths r approaching zero, there exists a subsequence r_k and a set N of measure zero in A, such that if $a \in A \setminus N$, then

$$\lim_{r_k\to 0} D(r_k, a, \phi^i) = 0$$

for every ϕ^i. But for ϕ in C_0^1, the definition (6.1) of $D(r, a, \phi)$ is equivalent to the integral equation (6.2); and (6.2) is linear in ϕ and involves only first derivatives of ϕ. Therefore, since u_{ra} and $F(u_{ra})$ are uniformly bounded, (6.2) implies that

$$|D(r, a, \phi^i - \phi)| \leqslant 0(1)\|\phi^i - \phi\|_{C^1}$$

where 0(1) depends only on the set in which ϕ^i and ϕ have compact support. Now let ϕ be an arbitrary element of C_0^1, and choose $\{\phi^j\}$ in $\{\phi^i\}$ so that ϕ^j converges to ϕ in C^1, and assume that the supports of ϕ^j and ϕ all lie within a single compact set. Let $\delta > 0$ be arbitrary, and choose j_0 large so that

$$0(1)\|\phi^{j_0} - \phi\|_{C^1} \leqslant \frac{\delta}{2}.$$

For any fixed $a \in A \setminus N$, choose r_0 small so that for $r_k < r_0$,

$$\left|D\left(r_k, a, \phi^{j_0}\right| < \frac{\delta}{2}.$$

Then for $r_k < r_0$ we have

$$\begin{aligned} |D(r_k, a, \phi)| &\leqslant \left|D\left(r_k, a, \phi^{j_0}\right)\right| + \left|D\left(r_k, a, \phi^j - \phi^{j_0}\right)\right| \\ &\leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Since δ is arbitrary, we have proven Lemma 6.3.

Proof of Theorem 1. Let $\psi(x)$ be any initial data satisfying $\mathrm{Var}_{zc}\psi < \infty$, and let r be any sequence of mesh lengths approaching zero. By Lemma 6.3, there exists a subsequence r_k and a set N of measure zero in A such that, if $a \in A \setminus N$, then

$$\lim_{r_k \to 0} D(r_k, a, \phi) = 0$$

for every $\phi \in C_0^1$. Now by Theorem 5.2, there exists a further subsequence, call it r_k, and a function u_a, such that $u_{r_k a}(x, t)$ converges to $u_a(x, t)$ and $F(u_{r_k a}(x, t))$ converges to $F(u_a(x, t))$ in $L^1_{\mathrm{loc}}(x, t)$. Therefore, applying the Lebesgue Dominated Convergence Theorem, we conclude that

$$\begin{aligned} 0 &= \lim_{r_k \to 0} D(r_k, a, \phi) \\ &= \iint_{\substack{-\infty < x < +\infty \\ t \geqslant 0}} u_a \phi_t + F(u_a)\phi_x \, dx \, dt + \int_{-\infty}^{\infty} \psi(x,0)\phi(x,0)\, dx \end{aligned}$$

for any $\phi \in C_0^1$ and $a \in A \setminus N$; and so $u_a(x, t)$ is a weak solution to system (1.1) with initial data $\psi(x)$. This completes the proof of Theorem 6.1.

When the curves $g =$ const. and $c =$ const. make kth order contact on the transition curve (or equivalently, when the wave speeds λ_s and λ_c make kth order contact on the transition curve), we can use (6.17) and (6.26) together with the fact that $\mathrm{Const}(\varepsilon) = 0(1)\varepsilon^{-k-1}$ (obtained in Proposition 6.2) to obtain a rate of convergence for $\|D(r, \cdot, \phi)\|_2^2$. That is, we have

$$\begin{aligned} \|D(r, \cdot, \phi)\|_2^2 &= \int_A |D(r, \cdot, \phi)|^2 = \int_A |D(r, \cdot, \phi_r) + D(r, \cdot, \phi - \phi_r)|^2 \\ &\leqslant \int_A |D(r, \cdot, \phi)|^2 + \int_A |D(r, \cdot, \phi - \phi_r)|^2 \\ &\quad + \int_A |D(r, \cdot, \phi_r)||D(r, \cdot, \phi - \phi_r)|. \end{aligned}$$

Applying (6.17) to estimate the first integral, and (6.26) to estimate the second two integrals we obtain

$$\|D(r,\cdot,\phi)\|_2^2 \leqslant \mathrm{Const}(\varepsilon)^2 r + 0(1)\varepsilon^2 + (\mathrm{Const}(\varepsilon)r + 0(1)\varepsilon)^2$$
$$+ (\mathrm{Const}(\varepsilon)r + 0(1)\varepsilon)\int_A |D(r,\cdot,\phi_r)|. \qquad (6.28)$$

But $\int_A |D(r,\cdot,\phi_r)| = 0(1)$, where 0(1) depends only on ϕ, because

$$\int_A |D(r,\cdot,\phi_r)| = \int_E |D(r,\cdot,\phi_r)| + \int_{A\setminus E} |D(r,\cdot,\phi_r)|$$
$$\leqslant \int_A |D(r,\cdot,\phi_r)|^2 + 1$$
$$\leqslant \mathrm{Const}(\varepsilon)^2 r + 0(1)\varepsilon^2 + 1 = 0(1),$$

where $E = \{a \in A : |D(r,\cdot,\phi_r)| \geqslant 1\}$. Therefore, (6.28) becomes

$$\|D(r,\cdot,\phi)\|_2^2 \leqslant 0(1)\{\mathrm{Const}(\varepsilon)^2 r + \varepsilon\}, \qquad (6.29)$$

where

$$\mathrm{Const}(\varepsilon) = 0(1)\varepsilon^{-k-1}.$$

We can maximize the rate of convergence in (6.29) by choosing $\varepsilon = r^{1/(3+2k)}$ and so conclude that

$$\|D(r,a,\phi)\|_2^2 = 0(1)r^{1/(3+2k)}$$

gives the best rate of convergence implied by our analysis.

Appendix 1

Here we prove the following two propositions:

PROPOSITION 5.1. *Let $J' = scs$ take u_L to u_R. Then $F(J') \geqslant F(J)$, where $J = [u_L, u_R]$.*

PROPOSITION 5.2. *Let $J' = c_1 s c_2$ take u_L to u_R. Then $F(J') \geqslant F(J)$, where $J = [u_L, u_R]$.*

These propositions are true basically because the Riemann problem solution always takes the "weaker" c-wave (a c-wave along which s decreases) whenever three waves appear in the solution. A rigorous proof involves a study of cases, and in order to reduce the number of cases, we define the "addition" and "interchange" of two waves. First, if $J = s_1 s_2$ takes u_L to u_R, define the addition of s_1 and s_2 to be the unique s-wave $\bar{s}$ such that $\bar{J} = \bar{s}$ takes u_L to u_R. If $J = c_1 c_2$, where c_i lie on the same side of the transition curve T, define the addition of c_1 and c_2 to be the unique c-wave $\bar{c}$ such that $\bar{J} = \bar{c}$ takes u_L to u_R; and if c_i lie on opposite sides of T with $c(u_L) \leqslant c(u_R)$ [resp. $c(u_L) \geqslant c(u_R)$], define $\bar{J} = \bar{c}\bar{s}$ [resp. $\bar{J} = \bar{s}\bar{c}$] to be the unique I-curve of this form that takes u_L to u_R. With these definitions it is easy to check that $F(\bar{J}) \geqslant F(J)$ (see Fig. 15). We now define two instances in which we can interchange the order of s- and c-waves on an I-curve J without changing the value of $F(J)$. If $J = sc$ [resp. cs] takes u_L to u_R and a "parallelogram" of s-waves and c-waves can be drawn between u_L and u_R on one side of T (as indicated in Fig. 16), then we define the

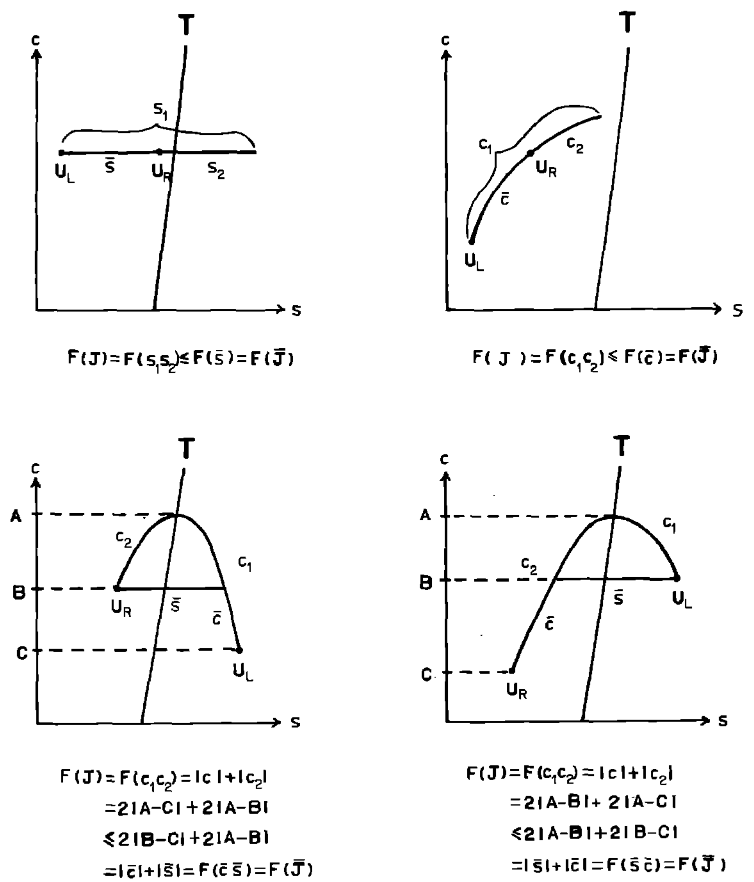


FIG. 15. Examples of the addition of two waves.

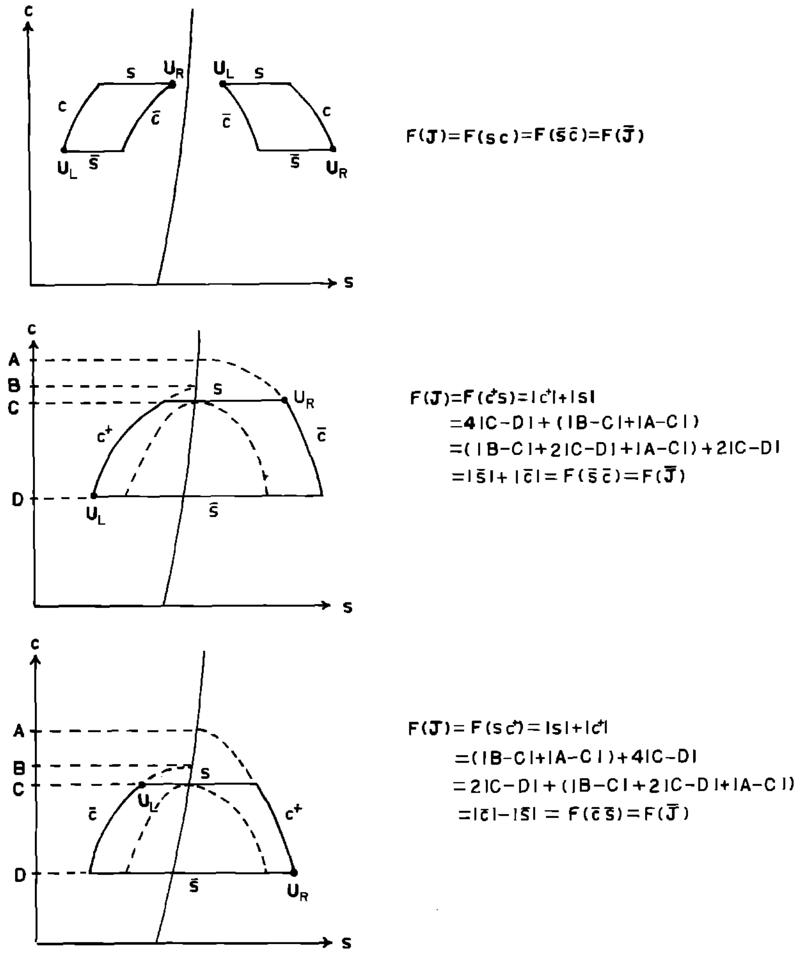


FIG. 16. Examples of the interchange of two waves.

interchange of J to be the I-curve $\bar{J} = \bar{c}\bar{s}$ [resp. $\bar{s}\bar{c}$] that takes u_L to u_R along the sides of this parallelogram opposite the sides of J. When such a parallelogram cannot be drawn, we can only define the interchange of J in certain cases; i.e., let c^+ [resp. c^-] denote a c-wave that moves toward increasing s [resp. decreasing s], and let c_L [resp. c_R] denote a c-wave that lies to the left [resp. right] of T. Then if $J = c_L^+ s$ [resp. sc_R^+], we define the interchange of J to be the unique I-curve $\bar{J} = sc$ [resp. $\bar{c}\bar{s}$] which takes u_L to u_R (see Fig. 16). If J equals sc or cs and takes u_L to u_R, and J has an interchange $\bar{J}$, then it is easy to check that the Riemann problem $[u_L, u_R]$ must be either J or $\bar{J}$. Moreover, Fig. 16 verifies that $F(J)$ equals $F(\bar{J})$. We

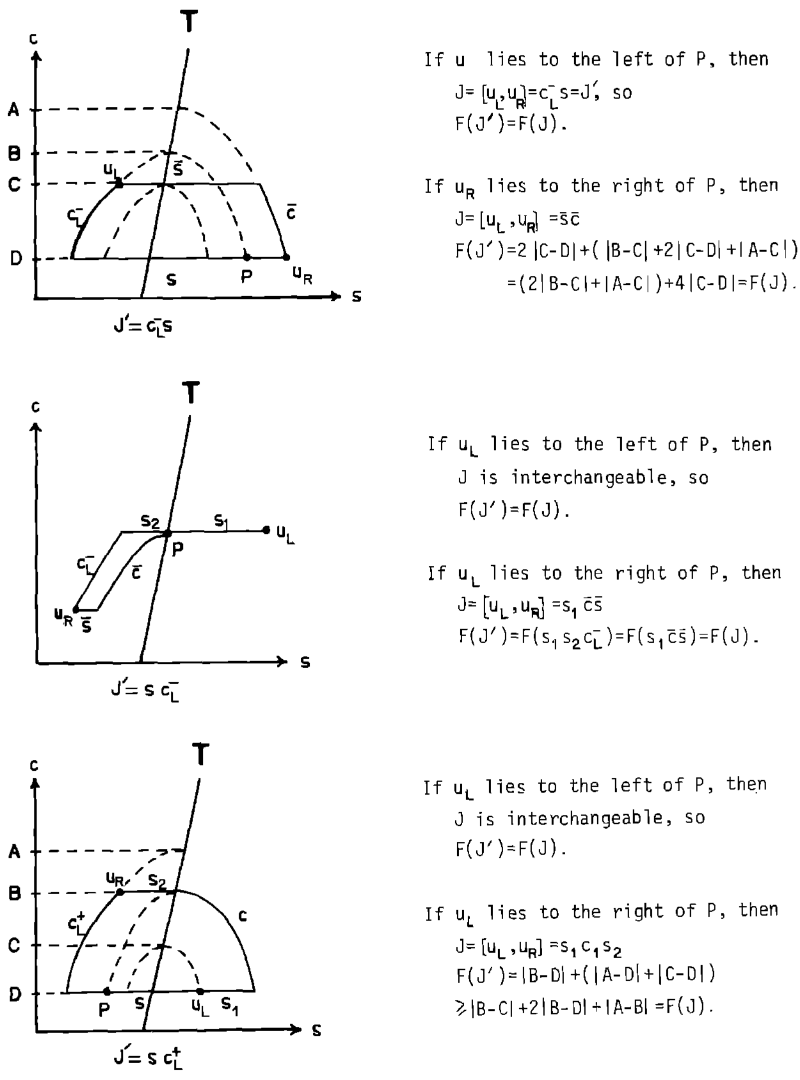
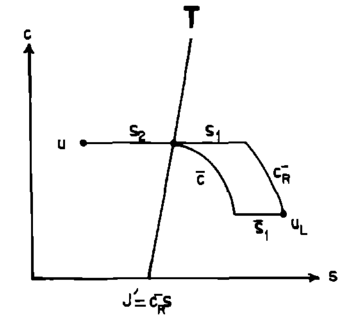


FIG. 17. The proof that $F(J') \geqslant F(J)$ for the six cases of Proposition 5.0.

now prove the following:

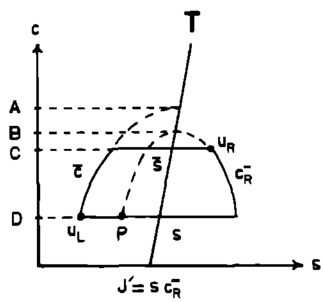
PROPOSITION 5.0. *If $J' = sc$ or $J' = cs$ and J' takes u_L to u_R, then $F(J') \geqslant F(J)$, where $J = [u_L, u_R]$.*

Proof. We do a case by case study depending on whether c is c^+ or c^-, whether c is c_L or c_R, and whether J' is sc or cs. This makes eight cases. But if $J' = c_L^+ s$ or sc_R^+, then J' is interchangable, and so $F(J') = F(J)$ since



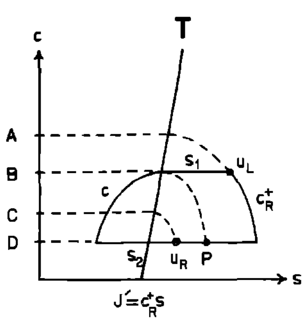
If u_R lies to the right of P, then J' is interchangeable, so $F(J') = F(J)$.

If u_R lies to the left of P, then $J = u_L, u_R = s_1 c\, s_2$, so $F(J') = F(c_R^- s_1 s_2) = F(s_1 c\, s_2) = F(J)$.



If u_L lies to the right of P, then $J = [u_L, u_R] = s\, c_R^- = J'$, so $F(J') = F(J)$.

If u_L lies to the left of P, then $J = [u_L, u_R] = \bar{c}\,\bar{s}$, so $F(J') = (|A-D| + |B-D|) + |B-C|) = F(J)$.



If u_L lies to the right of P, then J' is interchangeable, so $F(J') = F(J)$.

If u_L lies to the left of P, then $J = [u_L, u_R] = s_1 c\, s_2$

$F(J') = 4|B-D| + |A-C|$

$\geqslant |A-B| + 2|B-D| + (|B-D| + |C-D|) = F(J)$.

FIGURE 17 *Continued.*

$J = J'$ or $J = \bar{J}'$. This leaves six cases, and Fig. 17 verifies the proposition in each case.

Proof of Proposition 5.1. If sc or cS can be interchanged, then we can reduce the problem so that Proposition 5.0 applies; e.g., if $\bar{c}\bar{s}$ is an interchange for sc, then we can write

$$F(J') = F(scS) = F(\bar{c}\bar{s}S) \geqslant F(\bar{c}\bar{s}) \geqslant F(J),$$

where we have "added" $\bar{s}S$ and written $\bar{s}$. Therefore, assume that neither sc

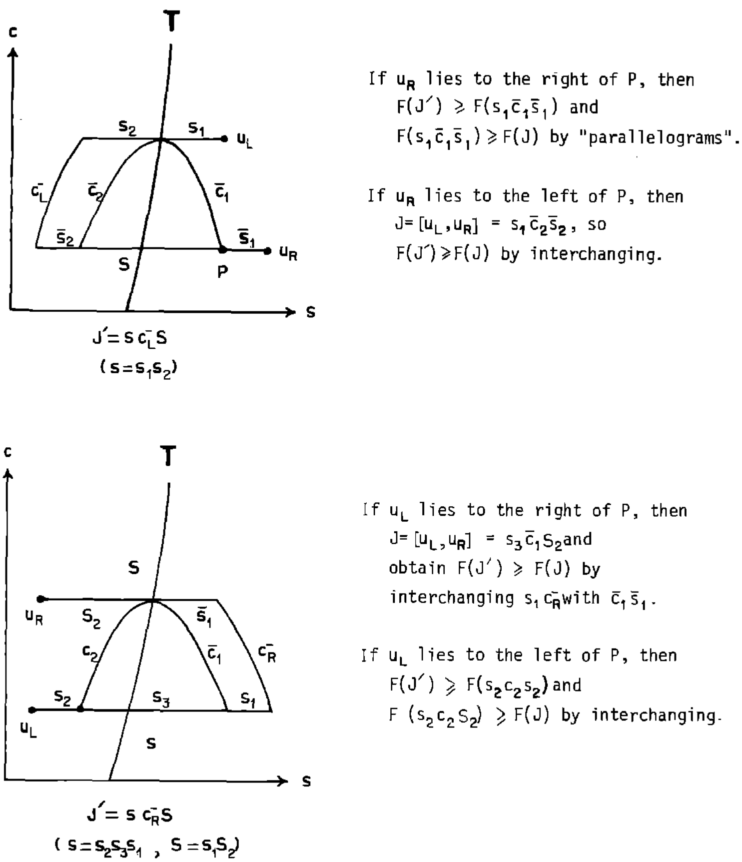
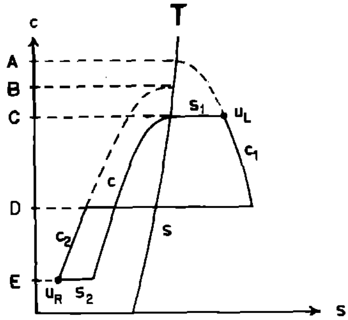


FIG. 18. The proof that $F(J') \geqslant F(J)$ for the two cases of Proposition 5.1.

nor cS can be interchanged, so that in particular $c = c^-$. We check two cases, depending on whether c^- lies to the left or right of the transition curve T. For the first case, assume c^- lies to the left of T. Then s must cross T (otherwise we can interchange s and c^+ by constructing a "parallelogram"). This yields an I-curve diagrammed in Fig. 18, where it is verified that $F(J') \geqslant F(J)$.

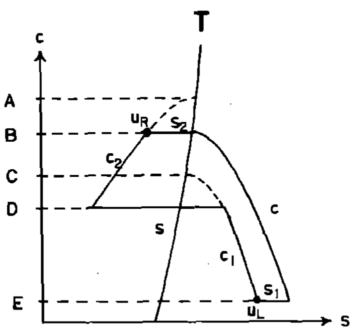
For the second case, assume c^- lies to the right of T. Then S crosses T (otherwise we can interchange c^- and S by constructing a "parallelogram"). This yields an I-curve diagrammed in Fig. 18, where again it is verified that $F(J') \geqslant F(J)$. This completes the proof of Proposition 5.1.

Proof of Proposition 5.2. Again, if $c_2 s$ or sc_2 is interchangeable, then we can interchange and add waves until Proposition 5.1 applies; e.g., if $\bar{c}_2 \bar{S}$ is



$$J' = c_1 s c_2 \,,\quad J = s_1 c s_2$$
$$F(J') = 4|C-D| + (|A-D| + |B-D|) + |D-E| \geqslant |A-C| + 2|C-E| + |B-C| = F(J).$$

The case when C decreases along J .



$$J' = c_1 s c_2, \quad J + s_1 c s_2 .$$
$$F(J') = 2|D-E| + (|C-D| + |A-D|) + 4|B-D| \geqslant |B-C| + 2|B-E| + |A-B| = F(J)$$

The case when C increases along J .

FIG. 19. The proof that $F(J') \geqslant F(J)$ for the two cases of Proposition 5.2.

an interchange for sc_2, then we can write

$$F(J') = F\left(c_1 \bar{c}_2 \bar{S}\right) \geqslant F(\bar{s}\bar{c}\bar{S}) \geqslant F(J),$$

where we have taken $c_1\bar{c}_2$ to "add" to $\bar{s}\bar{c}$ ($\bar{c}\bar{s}$ is similar). Therefore, assume that $c_1 s$ and sc_2 are not interchangeable, which implies that J' crosses T (for otherwise we can construct a parallelogram and interchange). Also, note that if the variable c increases along one c-wave and decreases along the other c-wave on J', then it is easy to check that $F(J') \geqslant F(cs)$ or $F(J') \geqslant F(sc)$ where either cs or sc takes u_L to u_R; and so Proposition 5.0 implies that $F(J') \geqslant F(J)$. Therefore, assume that the variable c either increases along both c_1 and c_2 or decreases along both c_1 and c_2. This makes four cases depending on whether J' crosses T from left to right or from right to left, and on whether c increases or decreases along J'. But if J' crosses T from left to right, then a quick check shows that either $c_1 s$ or sc_2 is interchangeable. This leaves only two cases to check: J' crosses T from right to left and c either increases or decreases along J'. Figure 19 verifies that $F(J') \geqslant F(J)$ in these two cases, and so completes the proof of Proposition 5.2.

Appendix 2

In this section we supply the proof of Lemma 5.2. If the initial data $\psi(x)$ satisfy $\mathrm{Var}_{zc}\psi < \infty$, then

$$\int_{-\infty}^{\infty} |w_{ra}(x, t_2) - w_{ra}(x, t_1)| dx = 0(1)\{|t_2 - t_1| + r\}\mathrm{Var}_{zc}\psi.$$

Proof. This is a consequence of the Courant–Friedrichs–Lewy condition (4.1) which forces a uniform bound $1/M = r/s$ on the speed with which waves can propogate in the approximation solution $w_{ra}(x, t)$. That is, assuming that $t_2 > t_1$, it is easily verified that at any point x, $|w_{ra}(x, t_2) - w_{ra}(x, t_1)|$ is bounded by the variation of $w_{ra}(y, t_1)$ for y between $x - M|t_2 - t_1| - r$ and $x + M|t_2 - t_1| + r$. But at every time level, $w_{ra}(\cdot, t)$ is a function of bounded variation and so the absolute value of its distributional derivative is a measure whose mass on every x-interval is the total variation of $u_{ra}(\cdot, t)$ on that interval. Thus we can write

$$\int_{-\infty}^{\infty} |w_{ra}(x, t_2) - w_{ra}(x, t_1)| dx = 0(1) \int_{\infty}^{\infty} \int_{x - M|t_2 - t_1| - r}^{x + M|t_2 - t_1| + r} \left| \frac{dw_{ra}(y, t)}{dy} \right| dy\, dx$$

This is equal to an integral with respect to the product measure

$$\left| \frac{dw_{ra}(y, t)}{dy} \right| dy\, dx$$

over the convex region $-\infty < x < +\infty$, $|y - x| \leqslant M|t_2 - t_1| + r$. We continue to estimate by changing the order of integration:

$$\begin{aligned} &= 0(1) \int_{-\infty}^{\infty} \int_{y - M|t_2 - t_1| - r}^{y + M|t_2 - t_1| + r} dx \left| \frac{dw_{ra}(y, t)}{dy} \right| dy \\ &= 0(1)\{M|t_2 - t_1| + r\} \int_{-\infty}^{\infty} \left| \frac{dw_{ra}(y, t)}{dy} \right| dy \\ &= 0(1)\{|t_2 - t_1| + r\}\mathrm{Var}_{zc}\psi, \end{aligned}$$

where we have applied (5.6) in the last step. This completes the proof of Lemma 5.2.

ACKNOWLEDGMENTS

It is a pleasure to express my appreciation to James Glimm, Eli Isaacson, and Bradley Plohr for many helpful discussions.

REFERENCES

1. J. GLIMM, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697–715.
2. E. ISAACSON, Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery, Rockefeller University preprints.
3. B. KEYFITZ and H. KRANZER, A system of non-strictly hyperbolic conservation laws arising in elasticity theory, *Arch. Rat. Mech. Anal.* **72** (1980).
4. P. D. LAX, Hyperbolic systems of conservation laws, II, *Comm. Pure. Appl. Math*, **10** (1957) 537–566.
5. P. D. LAX, Shock waves and entropy, *in* "Contributions to Nonlinear Functional Analysis" (E. H. Zarantonello, Ed.), (pp. 603–634, Academic Press, New York, 1971.
6. D. W. PEACEMAN, "Fundamentals of Numerical Reservior Simulation," Elsevier, New York.
7. T. P. LIU and C. H. WANG, On a hyperbolic system of conservation laws which is not strictly hyperbolic, MRC Technical Summary Report 2184 March, 1981.
8. T. NISHIDA, Global solutions for an initial boundary value problem of a quasilinear hyperbolic system, *Proc. Japan Acad.* **44** (1968), 642–646.