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*MULTIPHASE FLOW MODELS WITH SINGULAR RIEMANN PROBLEMS**

ELI L. ISAACSON[1]
DAN MARCHESIN[2]
BRADLEY J. PLOHR[3]
J. BLAKE TEMPLE[4]

[1] Department of Mathematics, University of Wyoming
P. O. Box 3036, Laramie, WY 82071, USA
[2] Instituto de Matemática Pura e Aplicada and Pontifícia Universidade Católica
Rio de Janeiro, 22460 RJ, Brazil
[3] Dept. of Math. and of Dept. Appl. Math. and Statistics
State University of New York, Stony Brook, NY 11794-3651, USA
[4]Department of Mathematics, University of California–Davis
Davis, CA 95616, USA

ABSTRACT: *New phenomena occur in the solution of the Riemann problem for a system of two conservation laws that models three-phase flow in a porous medium. Both the development of a computer program for general* 2×2 *Riemann problems and techniques from bifurcation theory, geometry, and dynamical systems have been essential for the discovery and understanding of the complex features of the model. Using these methods, we construct the global solution for one such model.*

KEY WORDS: singular Riemann problems, multiphase flow, system of conservation laws.

RESUMO: *MODELO DE ESCOAMENTO MULTIFÁSICO COM PROBLEMAS DE RIEMANN SINGULARES. Novos fenomenos ocorrem na solução do problema de Riemann para um sistema de duas leis de conservação que modela escoamento tri-fásico em um meio poroso. Tanto o desenvolvimento de um programa de computador para problemas de Riemann gerais* 2×2*, como técnicas da teoria de bifurcação, geometria e sistemas dinâmicos, tem sido essenciais para a descoberta e entendimento de aspectos complexos do modelo. Usando estes métodos, construimos a solução global para um destes modelos.*

PALAVRAS-CHAVE: problemas de Riemann singulares, escoamento multifásico, sistemas de leis de conservação.

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1. INTRODUCTION

New mathematical phenomena have been discovered in the study of the wave structure in a system of two conservation laws that models three-phase flow in a porous medium. In understanding these phenomena and in constructing the Riemann problem solution, the interplay of computational experimentation and mathematical theory has been crucial. The purpose of this paper is threefold. The first is to summarize the new phenomena and the methods employed to construct nonlinear waves. The second is to present the solution of the Riemann problem for the case in which the fluids have equal viscosities. The final purpose is to give an overview of open questions concerning global Riemann problems.

The mechanism that generates new phenomena is the presence of an isolated hyperbolic singularity or umbilic point: an isolated point interior to state space at which the characteristic speeds coincide. That is, the system fails to be strictly hyperbolic. In fact, the Jacobian matrix of the system is a multiple of the identity at this point, and all directions are characteristic. The usual construction of the solution of the Riemann problem for classical (i.e., strictly hyperbolic) systems, which relies on a local coordinate system of characteristic directions at every point, cannot be used near the umbilic point. Ordinarily, it is this coordinate system that guarantees the transversality of wave curves of different characteristic families. For a model in which the three fluids have equal viscosities, the rarefaction curves exhibit three-fold rotational symmetry about the umbilic point. This contrasts with classical systems in which the rarefaction curves form a rectangular coordinate system. Furthermore, the shock curves in this model have a nontrivial topological behavior. Therefore the wave curves, which are constructed using rarefaction and shock curves, have a complicated structure: transversality of wave curves of different families does not hold, and open regions of states fail to be covered by the classical construction of Riemann solutions.

One consequence is the need to use nonclassical transitional waves to complete the Riemann solution. These waves are not associated with a single characteristic family; rather, they are transitions between waves of the two characteristic families. These transitional waves may be continuous (rarefaction waves) or discontinuous (shock waves).

Lax's classical entropy criterion for determining admissibility of shock waves by counting characteristics has been successful for many hyperbolic systems. Although it has the advantage of simplicity, it does not resolve either the existence or uniqueness question for models such as the one under consideration. To construct physical solutions, therefore, we restore the influence of small parabolic terms that have been neglected in the formulation of the conservation law [6], [12]. Thus we admit those shock waves that can be obtained as the zero viscosity limit of traveling waves of the parabolic equation. This viscous profile criterion has advantages as well as drawbacks: it is physically motivated, and the solution of the Riemann problem exists and is unique for models with isolated umbilic points (see §3); however, the criterion is difficult to enforce in numerical computations, and for some conservation laws it is insufficient to insure that solutions are unique (see, e.g., [32], [1]).

Our work has focused on finding the Riemann solution for a system of conservation laws arising in oil reservoir engineering; the model is described in §2. This system has a unique interior state at which the characteristic speeds coincide. In order to solve this particular problem, we have developed a computer program that solves general 2×2 Riemann problems. This program has been essential for discovering and understanding the complex features exhibited by solutions of this highly nonlinear model. The solution for equal fluid parameters is presented in §6. The structure of the solution is topologically similar to that obtained in [40] where a simpler problem containing an umbilic point is solved. However, the detailed wave structure of the solution in §6 is more complex.

The failure of strict hyperbolicity for various models of three-phase flow has been proved [37], [42], [31], [45] using topological arguments. For example, it occurs in Stone's model, which is used commonly in petroleum engineering [8]. This model exhibits an elliptic region, i.e., a region in which the characteristic speeds are complex [42]. In general, perturbations of a system having an isolated umbilic point produce systems having bounded elliptic regions [36], [34]. However, the isolated umbilic point is preserved by the large perturbations of the model caused by the inclusion of gravitational effects [31], [45]. These effects arise from the different densities of the various fluids. We consider the umbilic point as suggestive of an elliptic region [34], and therefore expect the solution of this model to shed light on standard engineering multiphase flow models.

The analysis of this problem in a neighborhood of the umbilic point has proved to be fruitful. A summary of the resulting theory of Riemann problems for quadratic polynomial flux functions in two variables is found in §3. The wave curves associated to both characteristic families, as well as the manifolds on which they bifurcate, are described in §4. In §5 we describe the viscous profile criterion. We also discuss the novel transitional shock waves that appear in this problem; see [21] for more discussion. All of the admissible waves are used in §6 to construct the Riemann solution for the model. Details of the solution are presented in several cases that illustrate new features and persisting difficulties. Conclusions are drawn in §7, and remaining problems in the area are listed.

2. A MODEL FOR THREE-PHASE FLOW

For several years we have been studying the Riemann problem for a system of two conservation laws that models three phase flow in a porous medium. This model approximates the flow of oil, water, and gas in a petroleum reservoir. The importance of such problems was emphasized in 1941 by Leverett and Lewis [28]. The two-phase scalar problem for oil and water was solved in 1942 in the classical work of Buckley and Leverett [4], who established the formation of saturation shock waves, or oil banks, as the mechanism responsible for oil recovery in petroleum reservoirs.

A solution of a model of three-phase flow is the aim of our work. The simplified model represents the conservation of mass of each fluid combined with Darcy's force law. Compressibility, capillarity, and gravity effects are neglected. With appropriate boundary conditions, the model is represented in one spatial dimension by the system of conservation laws

$$u_t + f(u,v)_x = 0 \ ,$$
$$v_t + g(u,v)_x = 0$$

with $f = U/D$, $g = V/D$, and $D = U + V + W$. Here, U, V, and W are the phase mobilities of oil, water, and gas, respectively; they depend on the saturations u, v, w and the viscosities a, b, c of the three fluids. The saturations are non-negative and sum to unity; hence, $w = 1 - u - v$.

Laboratory measurements are qualitatively consistent with the approximations $V = V(v) = v^2/b$, $W = W(w) = w^2/c$, and $U = U(u,v)$ where U depends only weakly on v [8]. Stone's model, which is used commonly in petroleum engineering, consists of a special choice of the dependence of the mobility U on the saturations u and v. This model exhibits an elliptic region [2]. In our simplified model, we take $U = U(u) = u^2/a$, and instead of an elliptic region there exists an umbilic point. The umbilic point is easily determined: requiring the Jacobian matrix $\partial(f,g)/\partial(u,v)$ to be a multiple of the identity matrix leads to equality of the derivatives $U' = V' = W'$, which can be proved to occur at precisely one interior point [31]. Umbilic points also occur at the corners of the triangle; the Riemann problem near the corners has been studied by Schaeffer and Shearer [35].

The solution for the parameter choice $a = b = c$ is presented in §6.

3. WAVES NEAR REGULAR OR SINGULAR POINTS

In this section we discuss the waves appearing in solutions of Riemann problems. The classical construction of Lax for strictly hyperbolic systems is presented in §3a, and the new features that arise near an umbilic point are described in §3b.

3a. The Classical Solution

There is a beautiful construction by Lax [27] of the solution of the Riemann problem for nearly constant data for systems of any number of equations. The classical theory applies to systems of conservation laws which are strictly hyperbolic and genuinely nonlinear. That is, the Jacobian matrix $D\mathbf{f}$ of the flux $\mathbf{f} = (f, g)$ has distinct real eigenvalues $\lambda_1 < \lambda_2$ corresponding to right eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$, and each eigenvalue varies monotonically along the integral curves of the corresponding eigenvector field. These integral curves give rise to rarefaction waves, which are smooth solutions $\mathbf{u} = (u, v)$ that depend only on the ratio x/t and satisfy $\lambda(\mathbf{u}(x,t)) = x/t$. A shock wave consists of two constant states $\mathbf{u}_-$ and $\mathbf{u}_+$ separated by a discontinuity traveling with speed $s = s(\mathbf{u}_-, \mathbf{u}_+)$; the states are related by the Rankine-Hugoniot jump condition $s\,(\mathbf{u}_+ - \mathbf{u}_-) = \mathbf{f}(\mathbf{u}_+) - \mathbf{f}(\mathbf{u}_-)$. For a given state $\mathbf{u}_-$, the set of states $\mathbf{u}_+$ satisfying the jump condition forms the Hugoniot locus $\mathcal{H}(\mathbf{u}_-)$ of $\mathbf{u}_-$, which parameterizes shock waves. This locus consists of two branches that emanate from $\mathbf{u}_-$ in the directions of the eigenvectors. We call this locus the Hugoniot curve to stress that it is a one-dimensional object, even for systems with an arbitrary number of equations.

The wave curves associated with each characteristic field are the main ingredients in the construction of the solution of the Riemann problem in the classical case. Each of the two wave curves through a state $\mathbf{u}_-$ consists of two portions joined smoothly at $\mathbf{u}_-$: the rarefaction curve and the shock curve of the corresponding field. The rarefaction curve is the portion of the integral curve of the eigenvector field through $\mathbf{u}_-$ along which the eigenvalue increases. The shock curve is the portion of the Hugoniot curve through $\mathbf{u}_-$ that is tangent to the rarefaction curve and along which the shock speed s decreases. In the strictly hyperbolic case, the (right) eigenvector fields, and hence their integral curves, cross transversally. Therefore the wave curves, which are tangent to these integral curves, also intersect transversally. Thus the solution of the Riemann problem is obtained by following the first wave curve from a given left state to a middle state and then following the second wave curve from this middle state to a given right state. Loosely speaking, the wave curves form a coordinate system for the construction of the Riemann problem solution.

3b. Near the Umbilic Point

For the model described in §2, the rarefaction curves are depicted in Figure 3.1 (compare [36]). Rarefaction curves corresponding to the smaller characteristic speed are drawn as double lines, while those for the larger speed are single lines. The arrows on the curves indicate the direction of increasing characteristic speed. It is clear that there is no longer a coordinate system throughout a neighborhood of the umbilic point, although there is one in a neighborhood of any other point.

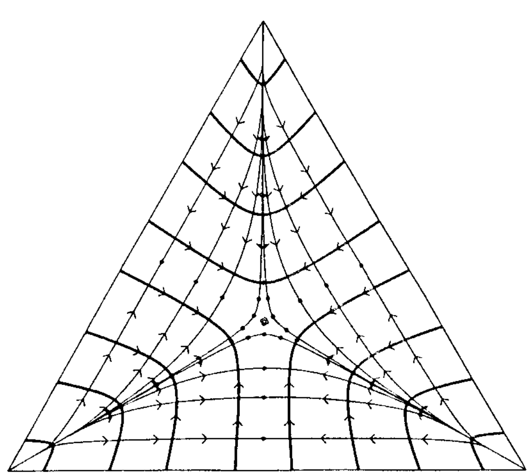

Fig. 3.1. Rarefaction curves.

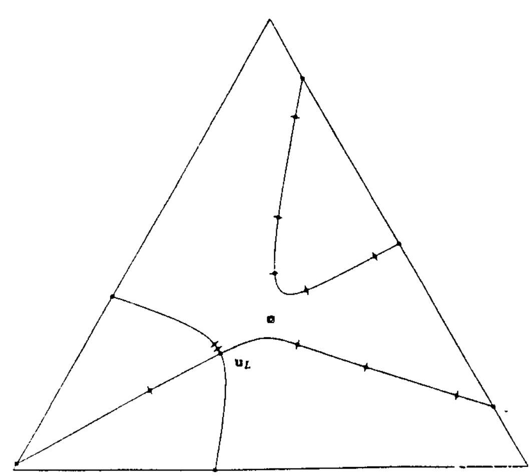

Fig. 3.2. A Hugoniot curve.

Another new feature is that the Hugoniot curves can have detached branches. This is indicated in Figure 3.2 for the Hugoniot curve of a state $\mathbf{u}_L$, at which the two primary branches cross. States on the curve at which the shock speed coincides with a characteristic speed of a state on either side of the discontinuity are marked; these states are important for constructing wave curves and viscous profiles. In other models [22], the Hugoniot curve may have closed loops. The Hugoniot curve also changes topology as its origin crosses a certain locus, the bifurcation manifold, which is defined in the next section. (See Figure 3.3.) As $\mathbf{u}_L$ approaches the bifurcation manifold, the detached branch of the Hugoniot curve approaches the local branch. This behavior, which has no analogue in scalar conservation laws, makes even more difficult the task of finding a coordinate system in which to determine the Riemann problem solution.

Generically, the behavior of rarefaction and shock curves near an isolated umbilic point is dictated by the second order Taylor expansion terms of the fluxes $\mathbf{f}$. It is natural to study the Riemann problem for models with homogeneous quadratic polynomial flux functions. Such homogeneous quadratic models depend on a two parameter family, and the Riemann solutions exhibit four different kinds of behavior [36]. Solutions of Cases I-IV with a special symmetry are described in [40], [22], [24], [23]. General solutions for Cases II–IV appear in [39], and for Case I in [17], [38]. The differences among the several cases is related to the existence and relative position of certain special manifolds (see §4).

Within a neighborhood of the umbilic point, our flow model falls into Cases I and II. There is a one to one correspondence between the position of the umbilic point and the viscosity ratios $a : b : c$ in our model. For each of the two cases, the possible locations of the umbilic point is shown in Figure 3.4. The isolated umbilic point becomes a bounded elliptic region when linear terms are added to homogeneous quadratic fluxes [34]. The solution for such perturbed Cases I and II are studied in [18] and [19] using the Lax entropy criterion; in these cases, solutions are not unique.

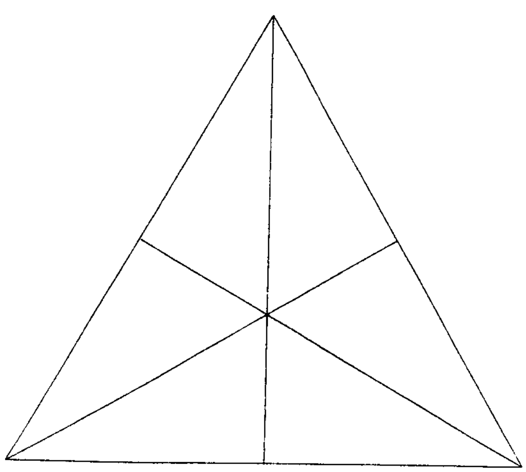

Fig. 3.3. The bifurcation manifold.

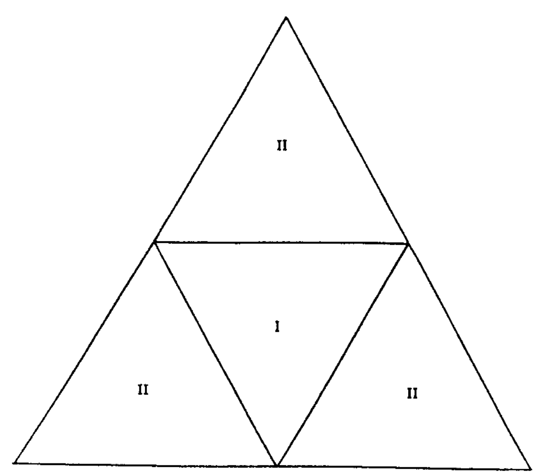

Fig. 3.4. Cases for umbilic positions.

Rarefaction curves do not enter the elliptic region. When they intersect its boundary, the 1- and 2-rarefaction curves have the same direction: generically they form a cusp, but at special points they join smoothly. In the latter case, a new phenomenon happens, which is described geometrically in [34]: the 2-rarefaction curves can continue as 1-rarefaction curves, forming what we call transitional rarefaction waves [21]. Another type of transitional wave, transitional shock waves, is described in §5. Such transitional waves are a nonclassical feature. In our model with umbilic points, there is one transitional rarefaction curve in Case II.

A shock branch cannot start at a point within the elliptic region; thus shock curves have more complicated shapes when elliptic regions are present [18]. More work is needed to understand the solution of the Riemann problem for systems with nonhomogeneous quadratic fluxes, as such systems possess elliptic regions.

Quadratic Riemann problems have proved to be amenable to analysis. The region where this analysis holds is the center-most part of Figure 4.5; it represents an exceedingly small part of the area of interest in our model. It is clear that computer experiments, as well as topological methods, play an important role in understanding the global solution of the model.

4. WAVE CURVES

In this section, we describe briefly the basic ingredient necessary for the generalization of the classical construction of the Riemann problem solution utilized in our computer program. Reference [11] contains further details. This ingredient is the wave curve, a representation of certain coherent sequences of invariant waves as points in the space of possible states. Since they usually have many branches, wave curves are not actual curves; this terminology is used because they are one-dimensional.

In physical space, the solution consists of a sequence of rarefaction fans, discontinuities, and constant states; these elementary waves are grouped into waves that belong to the first family (1-waves), to the second family (2-waves), or constitute transitional waves. In all the cases we have studied so far, the solution of the Riemann problem consists of either: (1) the left state, 1-waves, a constant state, 2-waves, and the right state; or (2) the left state, 1-waves, a constant state, a transitional wave, a constant state, 2-waves, and the right state. These solutions obey the geometrical constraint that speeds in physical space increase from left to right.

Wave curves in our problems differ from classical wave curves in several respects. First, they are represented in state space by three types of elementary segments, consisting not only of shock waves and rarefaction waves as in the classical case, but also of composite waves, which are shock waves adjacent to rarefaction waves: the final states $\mathbf{u}$ satisfy

$$\mathbf{u} \in \mathcal{H}(\mathbf{u}') \quad \text{with} \quad \lambda_i(\mathbf{u}') = s(\mathbf{u}, \mathbf{u}') ,$$

where $\mathbf{u}'$ ranges in a rarefaction segment. These waves appear when the problem is not genuinely nonlinear [29] and [33]. Second, in each wave curve there are many such elementary segments. Each elementary segment must stop whenever its wave speed attains an extreme, and the type of elementary segment that follows is determined by simple rules. Third, since Hugoniot curves possess nonlocal (i.e., detached) branches, wave curves also have complicated shapes; e.g., they may have disconnected parts or branching points. One consequence is that our computer program cannot use only continuation algorithms to find the wave curves. Rather, global searching algorithms are employed.

The continuation rules for wave curves are justified by the Bethe-Wendroff theorem [47], as applied to determine the qualitative behavior of the wave speed along a wave curve. This analysis is conveniently performed using wave speed diagrams, which generalize to systems Oleĭnik's convex envelope construction for scalar conservation laws [33]. Using these techniques, the stability of wave curves, with respect to perturbations of left state, can be established [11]. We have implemented the continuation algorithm in a computer program that constructs both the local and nonlocal branches of wave curves.

A typical wave speed diagram is shown in Figure 4.1. The horizontal axis corresponds to a parameterization of the wave curve, and the vertical axis is speed. The solid lines are the two characteristic speeds, while the dashed (resp. crossed) curves are the propagation speeds of shock waves (resp. composite waves). The example shows the speeds for the 2-wave curve starting at a state $\mathbf{u}_L$. Near this state the curve consists of rarefaction and shock waves, as usual. The rarefaction segment ends when the characteristic speed reaches a maximum, at which point the wave curve follows a segment of composite waves. Points on the composite correspond to points on the rarefaction at the same speed; these points, which work back along the rarefaction segment, are indicated by dots. The composite segment ends when its speed coincides with the faster characteristic speed on the right, and is followed by another rarefaction segment, whose speed eventually maximizes, leading to another composite segment. This composite segment ends when the corresponding rarefaction points have reached the beginning of the segment; then the wave curve continues with a new composite segment based on the previous rarefaction segment. Finally, this last composite wave ends when its speed coincides with the faster characteristic speed on the left, where the wave curve becomes a shock segment.

The wave curve corresponding to the wave speed diagram is shown in Figure 4.2. Again, solid, dashed, and crossed curves represent rarefaction, shock, and composite segments, respectively. Also shown is a disconnected branch of the wave curve. Waves on a wave curve are physical only if they obey appropriate admissibility criteria. If such criteria are not satisfied, spurious solutions to the Riemann problem can arise. We avoid these solutions by admitting only those shock waves that possess viscous profiles (see §5). In Figure 4.2, the shock waves employed to construct the nonlocal branch are not admissible in this sense, so that this branch must be discarded.

The Bethe-Wendroff theorem says that generically the speed $s(\mathbf{u}', \mathbf{u})$ of a shock or a composite based at $\mathbf{u}'$ has an extremum at a state $\mathbf{u}$ on a segment of a wave curve if and only if $s = \lambda_i(\mathbf{u})$ for $i = 1$ or $i = 2$, provided that $\mathbf{l}_i(\mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}') \neq 0$; here, $\mathbf{l}_i(\mathbf{u})$ is a left eigenvector of the Jacobian matrix. Using the Bethe-Wendroff theorem, we have proved that certain loci play a fundamental role in determining the nature of a wave curve: these are the bifurcation manifold, the inflection manifold, the hysteresis manifold, and the double contact manifold. These loci might have boundaries, self-intersections, self-tangencies, and other singularities, but we call them manifolds to emphasize that they

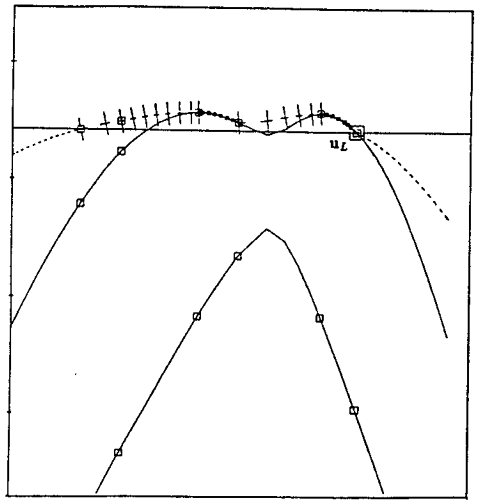

Fig. 4.1. A wave speed diagram for the local branch.

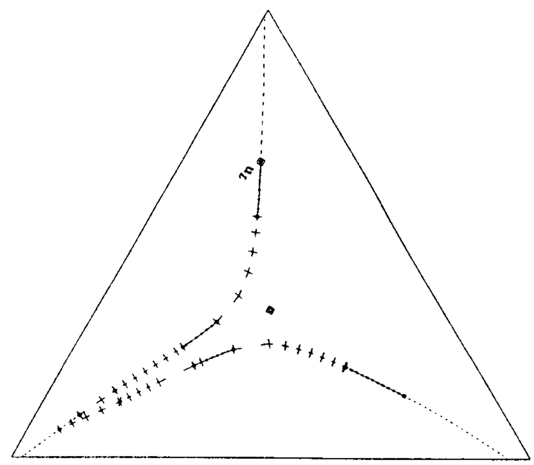

Fig. 4.2. A wave curve near the center of the domain.

have dimension $n - 1$ for systems with n equations. Several other manifolds, which are based on these four manifolds, are important in proving the stability of wave curves (see [11]); in the model studied here, however, they are of secondary importance.

The bifurcation manifold is where shock curves change topology. A state $\mathbf{u}$ belongs to the bifurcation manifold for family i when there is a state $\mathbf{u}' \neq \mathbf{u}$ such that

$$\mathbf{u}' \in \mathcal{H}(\mathbf{u}) \quad \text{with} \quad \lambda_i(\mathbf{u}') = s(\mathbf{u}, \mathbf{u}') \quad \text{and} \quad \mathbf{l}_i(\mathbf{u}') \cdot (\mathbf{u}' - \mathbf{u}) = 0 \; .$$

Note that the Bethe-Wendroff theorem does not apply on a bifurcation manifold.

The inflection manifold is named by analogy with scalar conservation laws. It is the manifold where genuine nonlinearity is lost: the eigenvalue does not vary monotonically along a rarefaction curve near an inflection point. Thus rarefaction curves stop at this manifold. At a state $\mathbf{u}$ on the inflection manifold,

$$\nabla\lambda_i(\mathbf{u}) \cdot \mathbf{r}_i(\mathbf{u}) = 0 \; .$$

The hysteresis manifold contains states of a composite segment joined to the end of a rarefaction segment by a nonlocal shock wave. Thus a state $\mathbf{u}$ lies on this manifold when there is a state $\mathbf{u}'$ such that

$$\mathbf{u} \in \mathcal{H}(\mathbf{u}') \quad \text{with} \quad \lambda_i(\mathbf{u}') = s(\mathbf{u}, \mathbf{u}') \quad \text{and} \quad \nabla\lambda_i(\mathbf{u}') \cdot \mathbf{r}_i(\mathbf{u}') = 0 \; ,$$

for all $\mathbf{u}' \neq \mathbf{u}$ on the inflection manifold. This manifold is important for constructing nonlocal composite branches.

The double contact manifold consists of states $\mathbf{u}$ for which there is a state $\mathbf{u}'$ such that the shock wave joining $\mathbf{u}$ and $\mathbf{u}'$ has speed coincident with the characteristic speeds at both its sides. (This is called a double contact discontinuity.) The states $\mathbf{u}$ and $\mathbf{u}'$ satisfy

$$\mathbf{u}' \in \mathcal{H}(\mathbf{u}) \quad \text{with} \quad \lambda_i(\mathbf{u}) = s(\mathbf{u}, \mathbf{u}') = \lambda_j(\mathbf{u}') \ ,$$

where the families i and j may be different. Composite segments end at the double contact manifold.

Because of the presence of the boundary of the physical region, another manifold plays a role in the model problem, which we call an interior boundary contact. It is the manifold of states $\mathbf{u}$ joined to points $\mathbf{u}'$ on the boundary by characteristic shock waves, namely

$$\mathbf{u} \in \mathcal{H}(\mathbf{u}') \quad \text{with } \mathbf{u}' \text{ on the boundary and} \quad \lambda_i(\mathbf{u}) = s(\mathbf{u}, \mathbf{u}') \ .$$

A boundary contact wave occurs when a junctions between wave segments coincides with the boundary.

The manifolds are defined implicitly by nonlinear maps from $\mathbf{R}^p$ to $\mathbf{R}^{p-1}$, where p is as large as four. Their computation involves solving such nonlinear equations by triangulating a cube in $\mathbf{R}^p$. We have written a general-purpose program to accomplish this, and have computed the manifolds for equal viscosities $a = b = c = 1$. Figures 3.3 and 4.3–6 show the projections of the manifolds into state space.

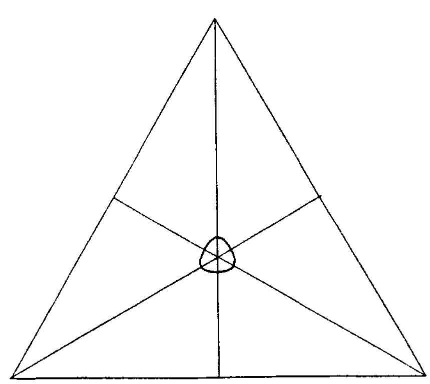

Fig. 4.3. Inflection manifold.

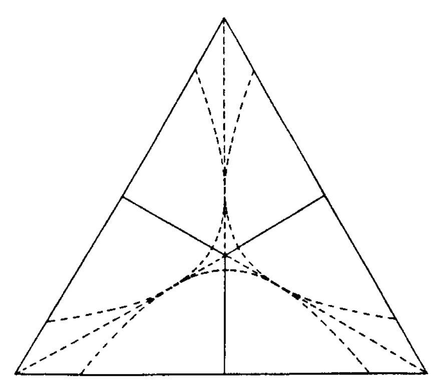

Fig. 4.4. Hysteresis manifold.

The construction of wave curves usually yields many branches and multiple solutions for Riemann problems. To avoid this difficulty, we consider a shock wave to be admissible

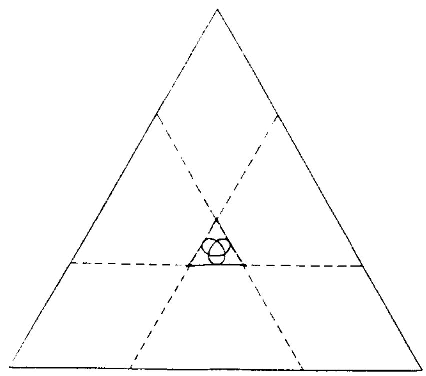

Fig. 4.5: Double contact manifold.

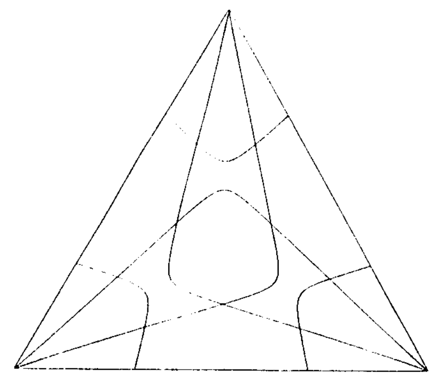

Fig. 4.6: Boundary contact manifold.

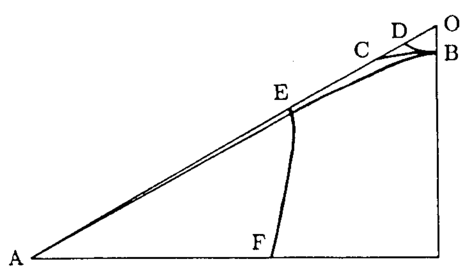

Fig. 4.7: Boundaries for 1-wave curves.

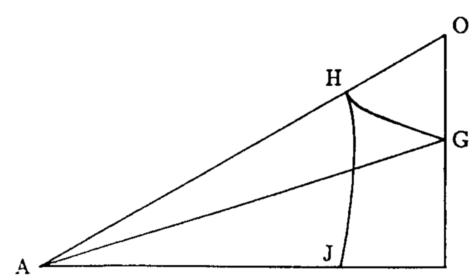

Fig. 4.8: Boundaries for 2-wave curves.

only if it has a viscous profile (as discussed in §5). Therefore only the solid parts of the manifolds drawn in the figures play a role, since the dashed parts involves non-admissible shock waves. For the same reason, the second branch in the wave curve of Figure 4.2 is also discarded.

The wave curve based on $\mathbf{u}_L$ suffers bifurcation only if one of its segments does. Therefore the wave curve does not bifurcate unless $\mathbf{u}_L$ lies on a manifold defined by conditions related to the inflection, bifurcation, hysteresis, or double contact manifolds or physical boundary. Bifurcation also occurs if the wave curve contains certain exceptional points, defined precisely in [11]; the wave curves drawn backward from exceptional points give rise to additional bifurcations. The manifolds for which the wave curve based on $\mathbf{u}_L$ bifurcates are called $\mathbf{u}_L$-boundaries. For homogeneous flux functions these boundaries coincide with the four basic manifolds [22], [24], [23], [39], [17], [38].

The $\mathbf{u}_L$-boundaries for the 1-wave curves and 2-wave curves are drawn in Figures 4.7 and 4.8, respectively. Because of the threefold and reflection symmetry, it suffices to draw them in one-sixth of the domain triangle. The wave curves with $\mathbf{u}_L$ in each of the regions shown in the figures have the same topology and consist of the same sequence of segments. In Figure 4.7 the curves (AG) and (GH) are sections of the 1-boundary contacts from Figure 4.6. The curve (HJ) is the 1-rarefaction ending at H; it plays a role because the 1-wave curve for a state $\mathbf{u}_L$ to the left of (HJ) has only a local branch, while to the right of (HJ) it has also a nonlocal branch (Figure 6.1). In Figure 4.8, where the $\mathbf{u}_L$-boundaries for family 2 are drawn, (EF) is a 2-boundary contact manifold, (BC) is a 2-double contact manifold, (BD) is a 2-inflection manifold, and (AB) is a 2-rarefaction curve ending at B. The curves (AG) and (GH) in Figure 4.7 also affect the structure of 2-wave curves, although they have been suppressed in Figure 4.8. Therefore there are five different types of 1-wave curves, corresponding to $\mathbf{u}_L$ in each region of Figure 4.7, and ten types of 2-wave curves, corresponding to the regions of Figure 4.8. Examples of these wave curves will be described in §6; in §5 we discuss our criterion for choosing appropriate branches of wave curves.

It is an important open problem to understand how these manifolds are affected by changes in the parameters specifying the system of conservation laws. This amounts to studying the bifurcation of the general Riemann problem as the model is varied.

5. VISCOUS PROFILES

It has long been known that the set of shock waves must be restricted by using physical considerations, thereby preventing nonuniqueness of solutions of the Riemann problem. In the case of multiphase flow, a small diffusive term modeling capillary pressure effects is restored in the hope of obtaining unique solutions. Therefore we have used the entropy condition of vanishing viscosity: a shock joining $\mathbf{u}_-$ to $\mathbf{u}_+$ must be the limit, as the positive parameter ϵ tends to zero, of traveling wave solutions of the parabolic equation

$$\mathbf{u}_t + \mathbf{f}_x = \epsilon\,(D(\mathbf{u})\mathbf{u}_x)_x$$

with the boundary conditions $\mathbf{u}(-\infty) = \mathbf{u}_-$ and $\mathbf{u}(+\infty) = \mathbf{u}_+$. Since the traveling wave $\mathbf{u}$ is a smooth function of $\xi = (x - st)/\epsilon$, this equation can be integrated once to yield the dynamical system

$$D(\mathbf{u})\mathbf{u}_\xi = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_-) - s(\mathbf{u} - \mathbf{u}_-)\ ,$$

for which both $\mathbf{u}_-$ and $\mathbf{u}_+$ are singularities of the vector field. A traveling wave solution corresponds to an orbit connecting these two states and is called a viscous profile of the shock wave.

For classical systems it has been shown that there are connecting orbits for weak Lax shock waves [9], [5]. For 1-shock waves, these orbits connect a repelling node to a saddle point; for 2-shock waves, they connect a saddle point to an attracting node. These types of connections, even when extended to include strong shock waves and limiting cases, are insufficient to complete the solution of the Riemann problem in our model. It is necessary to introduce a new kind of discontinuity whose profile is the connection between two saddle points of the vector field. Such a discontinuity appears already in a homogeneous quadratic model [40]. Because this discontinuity can be preceded by a 1-wave and followed by a 2-wave, it is not associated to any classical family. We refer to it as a transitional shock wave, because its role is similar to that of the transitional rarefaction waves described in §3. From the viewpoint of dynamical systems, the orbits for these transitional shock waves are structurally unstable, in contrast to those for classical shock waves.

Transitional waves are either transitional shock, rarefaction, or composite waves. Allowing for these additional waves, the existence of a solution of the quadratic models is guaranteed [40], [17], [38] and [21] in the case when $D = I$. In [17], a detailed study of the associated dynamical systems and their singularities is made.

For general constant D, the nature of the transitional shock waves is very different from that for the degenerate case $D = I$. This is discussed in [21]. It is also shown there for Case II of the quadratic models that different global solutions of the Riemann problem exist, depending on which of the two entropy criteria is used, the characteristic criterion of Lax or the viscous profile criterion. The solutions are different because the viscous profile criterion disallows those Lax shock waves that do not have profiles and admits those transitional waves that do.

We remark that the viscosity criterion is not sufficiently sharp to guarantee uniqueness. In fact, there are systems having multiple solutions all of whose shock waves have viscous profiles (see, e.g., [32], [1]). Thus additional physical criteria must be invoked to reduce further the class of admissible shock waves.

6. RIEMANN SOLUTION FOR THE MODEL

In this section we give a brief description of the solution of the general Riemann problem for the model with equal viscosity ratios and $D = I$. We describe the 1-, 2-, and transitional wave curves arising from $\mathbf{u}_L$ in some of the regions depicted in Figures 4.7 and 4.8. In the following figures, solid curves comprise points representing rarefaction waves, with arrows indicating the direction of increasing speed. Shock waves and composite waves are represented as points on dashed curves and crossed curves, respectively. We refer to these curves as rarefaction, shock, and composite segments.

Wave curves of the first family for $\mathbf{u}_L$ and $\mathbf{u}_{L'}$ in two of the regions of Figure 4.7 are shown in Figure 6.1. The wave curve for $\mathbf{u}_{L'}$ consists of a rarefaction segment $(L'b)$, a shock segment $(L'a)$, and a composite segment (bc). A composite wave in (bc) is a rarefaction wave from $\mathbf{u}_{L'}$ to a point in $(L'b)$ adjoined by a shock wave from this point to the point in (bc); the shock wave is characteristic on its left side. If $\mathbf{u}_{L'}$ is moved above (AG), a new shock segment appears near the boundary of the triangle. This segment is (gh) in the wave curve for $\mathbf{u}_L$; g lies on the boundary when L' lies on (AG). If $\mathbf{u}_{L'}$ is moved to the right of (HJ), its wave curve develops a nonlocal branch. Thus for $\mathbf{u}_L$, the local branch of the 1-wave curve is $(eLfgh)$, and the nonlocal branch is (nop). The curve (no) is a composite segment based on (fL), while (op) is a shock segment. When $\mathbf{u}_L$ is moved below (GH), the shock segment (op) disappears and (no) extends to the boundary; if $\mathbf{u}_L$ lies on (GH), the point o is on the boundary.

In Figure 6.2 we show the 2-wave curves for $\mathbf{u}_L$ and $\mathbf{u}_{L'}$ in two of the regions of Figure 4.8. (For clarity, Figure 6.2 and 6.3 have not been drawn to scale.) The wave curve for $\mathbf{u}_{L'}$ is $(aL'bcd)$; we have chosen $\mathbf{u}_{L'}$ to lie below the line (AG) from Figure 6.1. If $\mathbf{u}_{L'}$ is moved to the left of (EF), the point c reaches the boundary and the shock segment (cd) disappears. The 2-wave curve $(eLfghijkl)$ for a point $\mathbf{u}_L$ above (AB) has the same general shape, but other segments arise because of proximity to the umbilic point. The curves (eL) and (kl) are shock segments; (Lfg) and (hi) are rarefaction segments; and (gh), (ij), and (jk) are composite segments based on (gf), (ih), and (fL), respectively. The point f lies on the double contact manifold (BC); it corresponds to the point h on the trefoil in Figure 4.5. The 2-wave curve through $\mathbf{u}_L$ does not extend to the boundary because shock waves cease to be admissible at the points e and l. As $\mathbf{u}_L$ is moved down in Figure 6.2, l reaches the boundary precisely when $\mathbf{u}_L$ is on the curve (GH), while e reaches the boundary when $\mathbf{u}_L$ is on the curve (AG).

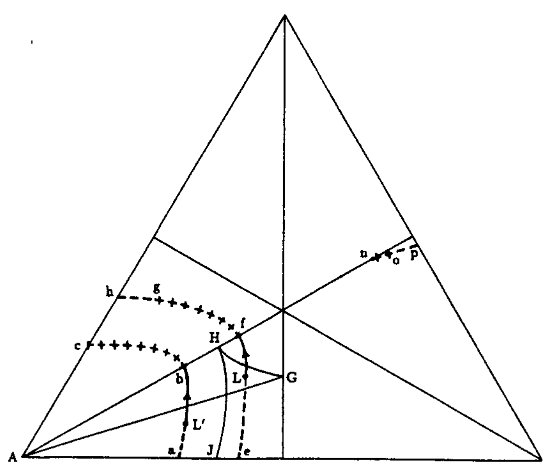

Fig. 6.1. 1-wave curves.

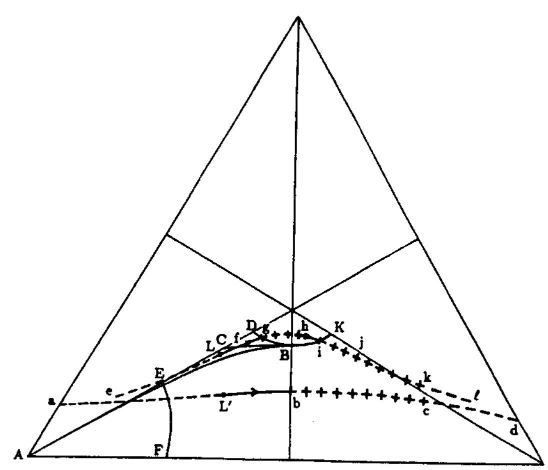

Fig. 6.2. 2-wave curves.

Transitional waves occur for points $\mathbf{u}_L$ on the bifurcation manifold. The different cases depend on whether $\mathbf{u}_L$ lies in (AE), (EH), (HC), (CD), or (DO) (see Figures 4.7 and 4.8). In Figure 6.3 we show the transitional waves for $\mathbf{u}_L$ in (HC). The segment (mn) consists of admissible crossing shock waves, while (hm) consists of crossing composite waves based on (CL). A crossing composite wave consists of a crossing shock wave preceded by adjacent 2-rarefaction wave; this is a new wave structure not found in any of the quadratic models described in §3. If the point $\mathbf{u}_L$ is moved past H, the point n moves to the boundary, and if $\mathbf{u}_L$ crosses C, the points h and m coalesce.

Also shown in Figure 6.3 is the 2-wave curve; $(eLCDhijkl)$ is obtained as the limit of the wave curve $(eLfghijkl)$ of Figure 6.2 as $\mathbf{u}_L$ tends to the bifurcation manifold. Similarly, the limit of the 1-wave curve through $\mathbf{u}_L$ in Figure 6.1 consists of local shock segments from the bifurcation manifold to the boundary, together with a nonlocal shock segment, which extends from n to q in Figure 6.3. If $\mathbf{u}_L$ in Figure 6.1 is moved along the rarefaction curve toward f, then the point n remains fixed, while o tends to n and p tends to q.

We are now ready to solve the general Riemann problem. As much as possible, we use the classical construction: first follow a 1-wave curve from the state $\mathbf{u}_L$ to a state $\mathbf{u}_M$, and then follow the 2-wave curve from $\mathbf{u}_M$ to the state $\mathbf{u}_R$. To complete the solution for all possible pairs $(\mathbf{u}_L, \mathbf{u}_R)$, however, transitional waves are required. We show an example of the construction in Figure 6.4; $\mathbf{u}_L$ is as in Figure 6.1. (If $\mathbf{u}_L$ lies to the left of (HJ), the segment (np) in Figure 6.4 disappears and (hn) extends to the boundary.)

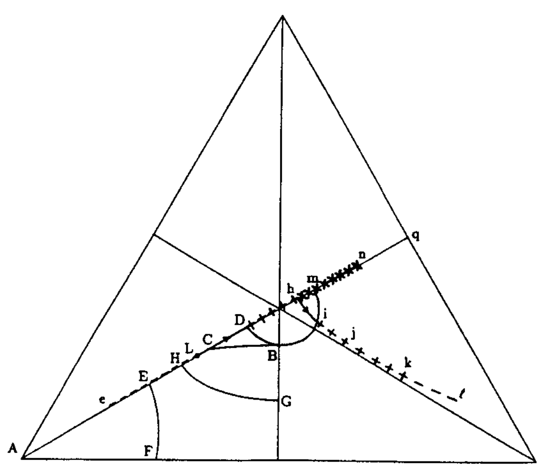

Fig. 6.3. Transitional wave curves.

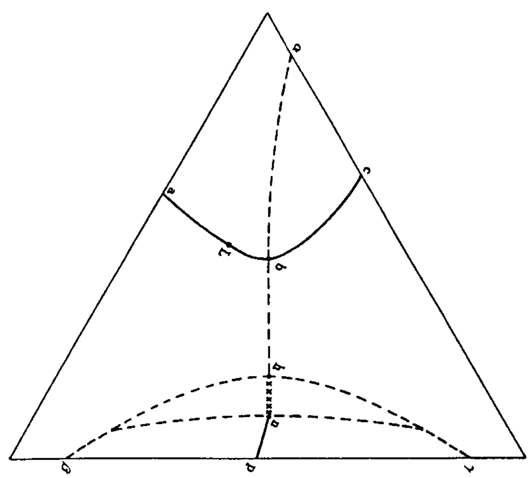

Fig. 6.4. Solution of the Riemann problem.

Given $\mathbf{u}_L$, construct the local 1-wave curve (abc) through $\mathbf{u}_L$; then, for any state $\mathbf{u}_M$ on (abc), construct the 2-wave curve through $\mathbf{u}_M$. This defines points $\mathbf{u}_R$ that can be reached by a local 1-wave curve from $\mathbf{u}_L$ to $\mathbf{u}_M$ followed by a 2-wave curve from $\mathbf{u}_M$ to $\mathbf{u}_R$. Notice that the speed of the wave from $\mathbf{u}_M$ to $\mathbf{u}_R$ must exceed the speed of the wave from $\mathbf{u}_L$ to $\mathbf{u}_M$. This geometric consistency restriction sometimes eliminates an end portion of a 2-wave curve. Thus the allowed 2-wave curves end at the dashed curves (αb), (βh), and $(h\gamma)$. In this way we obtain the solution for the given state $\mathbf{u}_L$ and any state $\mathbf{u}_R$ that lies to the left of $(\beta h\gamma)$. To obtain the solution for states $\mathbf{u}_R$ on the right of this curve, we use the transitional waves (hn) based on b as well as points on the nonlocal branch

(np) of the 1-wave curve; from these points on (hnp) we construct the 2-wave curves. Numerical experiments and the triple shock rule [22] show that the geometric consistency requirements force these 2-wave curves to stop precisely on $(\beta h\gamma)$. An explanation can be found in [43].

The general solution for the given $\mathbf{u}_L$ is now complete. The wave structure consists of one of the following sequences: (1) the left state, a 1-wave group, a constant state, a 2-wave group, and the right state; or (2) the left state, a 1-wave group, a constant state, a transitional wave, a constant state, a 2-wave group, and the right state. A transitional wave appearing in the solution is either a crossing shock wave or a composite of a 2-rarefaction wave and a crossing shock wave. For other states $\mathbf{u}_L$, the solution may be obtained by similar considerations. This solution is $L^1_{\rm loc}$-continuous in the Cauchy data $\mathbf{u}_L$, $\mathbf{u}_R$; the continuity may be verified by inspection. A justification for many features of the foregoing construction is found in [11], [43]. The latter reference also treats three-phase flow models with only two equal fluid viscosities.

The solution just described has a direct consequence for petroleum reservoir simulation. The mechanism that allows the efficient recovery of oil in petroleum reservoirs is the formation of oil banks, or saturation shock waves. Because oil recovery is maximized by strong shock waves, it is important that numerical methods used in oil reservoir simulation be accurate for these waves. For the particular choice of physical parameters considered in this work, some of the largest shock waves are nonclassical: the nonlocal 1-family shock waves and the crossing shock waves.

We expect that standard numerical methods employed in oil reservoir simulation are inaccurate for nonclassical waves, for two reasons. First, many numerical schemes spread a strong shock wave across several mesh zones, replacing it with several weaker shock waves; but this approximation is not valid for nonlocal shock waves, which are non-contractable. (Schemes such as the random choice method [13] do not make this approximation.) Second, crossing shock waves are sensitive to the precise form of the diffusion term [12]. In contrast to classical shock waves, which are affected only by the overall magnitude of the viscosity, the asymptotic states in a crossing shock wave are dependent on the relative sizes of components of the viscosity matrix. Dissipative numerical schemes on coarse grids calculate transitional shock waves that correspond to the numerical viscosity, rather to than the physical viscosity. This indicates that better numerical methods are needed in such areas of application as multiphase flow, magnetohydrodynamics, and elasticity. The solutions of Riemann problems serve as test problems for developing such methods.

7. CONCLUSIONS

As many new questions as answers have been uncovered in the process of developing theory and computer algorithms to analyze the Riemann problem for a simple three-phase flow model. We feel that techniques in global analysis, dynamical systems, bifurcation theory, singularity theory, topology, and geometry will play an essential role in establishing the mathematical theory for this class of problems. We conclude by pointing out some open questions.

It has been established that many properties of the model do not depend upon the quadratic nature of the phase mobility functions. However, except for the example of fourth order [16], Riemann problem solutions for homogeneous polynomial flux functions of higher degree are not known. These problems might shed light on the global structure of solutions for systems with different behavior at infinity. Some issues pertaining to bifurcation properties of the associated dynamical system are studied in [30].

Bifurcation properties of the Riemann problem solution as a function of the initial data are partially understood [39] [11]. This includes the behavior of the boundary manifolds described in §4. The bifurcation properties are almost completely unknown as a function of parameters in the conservation laws. Varying the viscosities in the model [43], as well as including the effects of gravity [31] and [45], should prove illuminating.

The lack of uniqueness of Riemann problem solutions based on shock waves with viscous profiles and the strong dependence of these solutions on the particular viscous terms utilized [14], [15], [7], [21], [46] are serious indications that entropy and modeling issues in the context of nonstrictly hyperbolic conservation laws must be reconsidered from more fundamental points of view [6], [32]. A detailed study of the bifurcation of the dynamical system of §5 is important to determine the possible limitations of the viscous profile criterion [41].

The global Riemann problem solution for systems of more than two equations remains as a challenging problem. Elasticity and plasticity are important applications. Umbilic points were identified recently in models for elasticity [44] and for magnetohydrodynamics [3], [10]. A system of three equations arising in multiphase flow, which generalizes the model in [20], has been studied [43]. Even the general Riemann problem for two equations with several umbilic points, coincidence curves [26], or elliptic regions [25] poses interesting geometrical and topological questions. Stone's model provides a suitable context in which to study many of these issues [2], [8], and [42].

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