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Shock waves in general relativity - a generalization of the Oppenheimer–Snyder model for gravitational collapse

§1. In 1939, J.R. Oppenheimer, and H. Snyder, [4], published the first paper on gravitational collapse. In this pioneering paper, they gave a mathematical model based on spherically symmetric solutions of the Einstein gravitational field equations, and they provided rigorous results which supported the conclusion that "black holes" could form from gravitational collapse of massive stars. Their method was to use the covariance of the equations to match two solutions across an interface, thereby providing the first example of a solution of the Einstein equations having interesting dynamics. However, the O.S. model requires the simplifying assumption that the pressure be identically zero. In this paper, we describe a generalization of the O.S. model in which the pressure is non-zero. In our case, we replace the boundary surface of the star in the O.S. model by a shock-wave interface across which mass and momentum are transported. We also present a general theory for matching two solutions of the Einstein field equations at arbitrary shock-wave interfaces, across which the metric is only Lipschitz continuous; i.e. smooth surfaces across which the metric derivatives have a jump discontinuity. We apply this general theory to explicitly construct shock wave interfaces in spherically symmetric solutions of the Einstein equations, and these provide a natural generalization of the O.S. model to non-zero pressure. We also briefly discuss some interesting mathematical problems associated with Lipschitz continuous solutions of the Einstein gravitational field equations. Complete details can be found in [7].

§2. In this section we shall review the basic notions in Einstein's Theory of General Relativity; for a more complete discussion, see, e.g., [1,3,8].

We consider a 4-dimensional space time manifold with metric tensor $g = (g_{ij})$ having signature $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. The Einstein field equations are

$$G = kT , \tag{1}$$

where G denotes the Einstein tensor, and T is the stress-tensor, the source of the gravitational field. Here $k = 8\pi G/c^4$, where G is Newton's gravitational constant, and c is the speed of light. In a given coordinate system $x = \left(x^0, x^1, x^2, x^3\right)$, $x^0 = ct$, the metric tensor has components $g_{ij} = g_{ij}(x)$, which defines a 4×4 symmetric matrix. In this coordinate system, (1) takes the form (in components),

$$G_{ij} = kT_{ij} , \tag{2}$$

where

$$G_{ij} = R_{ij} - \frac{1}{2} Rg_{ij} , \tag{3}$$

is the Einstein tensor and R_{ij} and R denote, respectively, the Ricci tensor, and the scalar curvature obtained from the Riemann curvature tensor for the metric g. The Riemann curvature tensor is given by

$$R^i_{jk\ell} = \Gamma^i_{jk,\ell} - \Gamma^i_{j\ell,k} + \Gamma^i_{\sigma\ell}\, \Gamma^\sigma_{jk} - \Gamma^i_{\sigma k}\, \Gamma^i_{j\ell} , \tag{4}$$

and R_{ij} and R are obtained by contractions:

$$R_{ij} = R^\sigma_{i\sigma j} , \qquad R = R^\sigma_\sigma$$

Here we use Einstein's summation convention, whereby summation is assumed (from 0 to 3) over repeated up-down indices. The notation ",i" denotes differentiation with respect to the variable x^i, and all indices run from 0 to 3. The Γ^i_{jk} denote the Christoffel symbols for the metric connection determined by g and are defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{\sigma i}\left[-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\right] .$$

We recall that the Γ^i_{jk} define the geodesics associated to the metric as determined by the equations,

$$\ddot{x}^i = \Gamma^i_{jk}\, \dot{x}^j\, \dot{x}^k , \quad i = 0, 1, 2, 3 . \tag{5}$$

The raising and lowering of indices is done via the metric. For example,

$$G^i_j = g^{i\sigma} G_{\sigma j} , \quad T_{ij} = g_{i\sigma} T^\sigma_j ,$$

where g^{ij} denotes the inverse of g_{ij}.

The Einstein tensor G satisfies the condition div $G = 0$, where div denotes the covariant divergence, defined in terms of the covariant derivative ∇ of the metric connection for g ; i.e.,

$$\nabla_k G^i_j \equiv G^i_{j;k} = G^i_{j,k} + \Gamma^i_{\sigma k} G^\sigma_j - \Gamma^\sigma_{jk} G^i_\sigma ,$$

so that

$$(\text{div } G)_j \equiv G^i_{j;\sigma} = G^\sigma_{j,\sigma} + \Gamma^\sigma_{\sigma\tau} G^\nu_j - \Gamma^\sigma_{j\tau} G^\tau_\sigma .$$

We remark that div $G = 0$ holds identically, and thus it follows that for solutions of (1) we must have div $T = 0$. The point is that div $G = 0$ is a geometric identity, independent of the Einstein equations (and holds as a consequence of the Binanchi identities; see [1,3,8]), while div $T = 0$ relies on both the identity div $G = 0$ as well as in the Einstein equations (1). For example, in §3, below, we shall consider the case of a "perfect fluid"; here the stress-energy tensor takes the form

$$T_{ij} = (p + \rho c^2)u_i u_j + p g_{ij} , \tag{6}$$

where p denotes the pressure, $u = (u_i)$ denotes the 4-velocity of the flu particle, and ρ denotes the mass-energy density, as measured in a frame of reference moving with the fluid particle. We assume too that the gas is barotropic; i.e. p is a function of ρ alone; $p = p(\rho)$. Thus in this case, div $T = 0$ gives 4 additional equations which hold on solutions of (1). These equations reduce to the Euler equations for compressible fluids (which express the conservation of energy and momentum), when g is taken to be the flat Minkowski metric, $g_{ij} = \eta_{ij} = \text{diag}(-1, 1, 1, 1)$; see [1,3,6,8].

It is well-known that shock-wave discontinuities form in solutions of the compressible Euler equations, [5,6]. In this case, the Rankine-Hugoniot jump conditions

$$[T^i_j]n_i = 0 \tag{7}$$

expresses the weak formulation of the conservation of energy and momentum across shocks. (Here $n = (n_i)$ is the normal vector to the shock surface, and the square bracket denotes the jump across the discontinuity.) In the development below, we will generalize the OS model for gravitational collapse by matching two (metric) solutions of the Einstein equations (1) in a Lipschitz continuous manner. We were unable to verify the jump conditions (7) directly because these involve the fluid variables in (6), and a direct verification involves using $\mathrm{div}\ T = 0$, which is not an identity, and so cannot be handled without involving the full Einstein equations (1). However, we will show below how to bypass this problem, via a general theorem ([7]), showing that (7) follows as a geometric identity from the corresponding identity $\mathrm{div}\ G = 0$, together with geometric constraints on the second fundamental form defined on the shock surface. The second fundamental form $K : T_\Sigma \longrightarrow T_\Sigma$ on a co-dimension-one surface Σ with normal vector field n embedded in an ambient Riemannian space with metric tensor g_{ij}, is a tensor field defined on Σ in terms of the metric, and describes how Σ is embedded in the ambient space. Here T_Σ denotes the tangent space of Σ, and K is defined by

$$K(X) = -\nabla_X n \ , \qquad X \in T_\Sigma \ . \tag{8}$$

If the metric is only Lipschitz continuous on Σ, K is determined separately from the metric on each side. In the next section, we give necessary and sufficient conditions, (the Israel conditions), for conservation to hold at a Lipschitz continuous shock interface, these being given in terms of geometric conditions on the jump in K across the surface. We conclude that the physical conservation laws are a consequence of geometrical constraints, built a priori into the Einstein tensor, together with geometrical constraints describing how the shock surface is embedded in the ambient space time manifold.

§3. In extending the OS model for gravitational collapse, the following three metrics are relevant; (c.f [8]). First we have *Schwarzschild metric*:

$$ds^2 = -\left(1 - \frac{2GM}{\bar{r}}\right)d\bar{t}^2 + \left(1 - \frac{2GM}{\bar{r}}\right)d\bar{r}^2 + \bar{r}^2\, d\Omega^2 \;; \tag{S}$$

where M is a constant and

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 ,$$

is the standard metric on S^2. The metric (S) describes the gravitational field due to a ball in $\mathbb{R}^3$, with no matter outside (empty space).

Next there is the *Robertson-Walker metric*:

$$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2\, d\Omega^2\right] \;; \tag{RW}$$

where k is a constant, and $R(t)$ is called the cosmological scale factor. The (RW) metric describes a homogeneous, isotropic universe: no preferred point, no preferred direction.

Finally, we have the *interior Schwarzschild metric*:

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \left(1 - \frac{2GM(\bar{r})}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \;; \tag{IS}$$

this metric describes the interior of a star. Here the following equations hold:

$$B'/B = \frac{-2\bar{p}'}{\bar{p} + \bar{\rho}} , \tag{9}$$

$$-\bar{r}^2\, \bar{p}'(\bar{r}) = GM(\bar{r})\bar{\rho}(\bar{r}) \left[1 + \frac{\bar{p}}{\bar{\rho}}\right]\left[1 + \frac{4\pi\bar{r}^3\bar{p}}{M}\right]\left[1 - \frac{2GM(\bar{r})}{\bar{r}}\right]^{-1} , \tag{10}$$

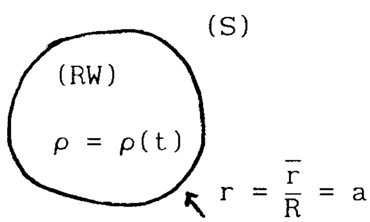
$$M'(\bar{r}) = 4\pi\bar{r}^2\bar{\rho}(\bar{r}) , \tag{11}$$

where $\bar{p}$ is the pressure and $\bar{\rho}$ is the density. Equation (10) is known as the "fundamental equation of Newtonian astrophysics, with general relativistic corrections supplied by the last 3 factors"; cf. [8]. Note that (11) implies that $M(\bar{r})$ is the "total mass inside the sphere of radius $\bar{r}$."

Now all 3 of these metrics solve the Einstein equations (1), where T is the stress-energy tensor of a perfect fluid, given by (6). We further assume that we have "co-moving coordinates", so the velocity vector is given by

$$(u_i) = \left(-\sqrt{g_{00}}, \; 0, 0, 0 \right) .$$

Now let's consider the O.S. problem, ([4]). Namely, we define a coordinate mapping, $(t, r) \longrightarrow (\bar{t}, \bar{r})$ such that the (RW) metric is equal to the (S) metric along a 3-dimensional shock surface. The solution is, (c.f. [4]), to take $p \equiv 0$, then $\bar{r} = Rr$ and the matching occurs at the "surface of the star", $r = a$; c.f. Figure 1



$$M = \frac{4\pi}{3} \rho(o) a^3$$

$$k = \frac{2MG}{a^3}$$

Figure 1

One shows, [4], that $R(t)$ vanishes at some finite time T_0. Thus ([4]), "a fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite density in the finite time T_0". This is the celebrated OS result, and it gave the first example of gravitational collapse. Note that there is no pressure inside the star to possibly prevent collapse.

Now we consider the following problem, ([7]). Namely, we define a coordinate mapping $(t, r) \longrightarrow (\bar{t}, \bar{r})$ such that the (RW) metric is equal to the (IS) metric along a 3-dimensional shock surface. We show, ([7]), that there exists a shock-wave solution for arbitrary equations of state $p = p(\rho)$ in (RW) and $\bar{p} = \bar{p}(\bar{\rho})$ in (IS).[1] The shock surface is given by

$$M(\bar{r}) \;=\; \frac{4\pi}{3} p(t) \bar{r}^3 \tag{12}$$

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[1] A non-degeneracy condition must be checked, but is valid e.g. for $p = \sigma^2 \rho$.

nd defines $\bar{r} = \bar{r}(t)$ along the shock, (provided that $(\rho - \bar{\rho}) \neq 0$; this s valid if $d\bar{\rho}/d\bar{r} < 0$.)

Now let's examine (12) more closely. First consider the lhs. Fix $_0$; then $M(\bar{r}_0)$ is the total mass inside the ball of radius $\bar{r}_0$, were he (IS) solution continued to values of $\bar{r} < \bar{r}_0$. Now consider the rhs.)efine $\tilde{M}$ to be the total mass inside a ball of radius $\bar{r}$ at time t in he (RW) solution:

$$\tilde{M} = \int_0^{\bar{r}} 4\pi\rho(t)\, s^2 ds = \frac{4\pi}{3}\, \rho(t)\, \bar{r}^3 .$$

'hus (12) implies that $M(\bar{r})$ is the total mass inside the ball of adius $\bar{r}$, at a fixed time t in the (RW) metric. We conclude that 12) expresses a *global* conservation of mass principle.

Next, differentiate (12) with respect to t to get the shock speed

$$s = \frac{d\bar{r}}{dt} = \frac{\dot{p}(t)\bar{r}}{3[\rho]} ,$$

vhere $[\rho]$ denotes $\bar{\rho} - \rho$. Now in the classical theory of shocks, ([5]), nly shocks which move into the fluid with lower pressure are stable; shocks moving into higher pressure are unstable (rarefaction shocks). So if $\bar{p}' > 0$, then the shock is stable if $s > 0$, and unstable if $s < 0$.

We next turn to an important problem; namely, if 2 metrics match ipschitz continuously across a surface Σ, does the weak form of onservation of energy and momentum hold across Σ ? That is, is $[T^{ij}]n_i = 0$; c.f. Figure 2.

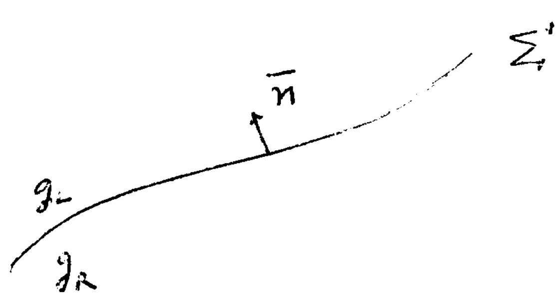

Figure 2

Now on solutions of (1), $\left[T^{ij}\right]n_i = 0$ iff

$$\left[G^{ij}\right]n_i = 0 \; . \tag{13}$$

That is, assume that div $G = 0$ holds on each side of Σ, and g is Lipschitz continuous across Σ. We pose the question: Does G satisfy the *weak form* of div $G = 0$; namely does (13) hold. The answer is no, in general. But, in our problem, the Lipschitz continuity of g reduces the 2 conditions (13) to the one condition

$$\left[T^{ij}\right]n_i n_j = 0. \tag{14}$$

To understand why this is so, we must first discuss the theory of jump conditions in general relativity.

Thus, suppose we are given a surface Σ across which a metric $g = g_L \cup g_R$ is only Lipschitz continuous (c.f. Figure 2). Recall from (8) the definition of the extrinsic curvature K; in our case K is determined separately from the metric g on each side of Σ. In [7], the following theorem was proved.

Theorem (Israel). If the extrinsic curvature K is continuous across Σ, then conservation holds; i.e. (13) is valid.

We apply this result to prove the following theorems, (c.f. [7]):

Theorem 2. For spherically symmetric matrices, the extrinsic curvature K is continuous across Σ if and only if $\left[T^{ij}\right]n_i n_j = 0$.

In fact, we prove that for *spherically symmetric* matrices, (e.g. (S), (RW) or (IS)), the smoothness problem and the problem of conservation are related. To see this, we first construct a *Gaussian normal* coordinate system, $\left(w^1, \cdots, w^n\right)$ associated with a surface Σ in a neighborhood N of a point P_0 on Σ. Namely, suppose g has y-components g_{ij}. By a smooth coordinate transformation, we may assume that Σ is defined in N by $y^n = 0$. For $P \in \Sigma \cap N$, let $\gamma_p(0) = P$, $\dot{\gamma}_p(0) = \bar{n}$, where S is arclength, and $\bar{n}$ is the normal vector. Define the w^n coordinate in N as the “distance from Σ” as follows (c.f. Figure 3): If $\gamma_p(s) = Q$, set $w^n(Q) = s$. Define

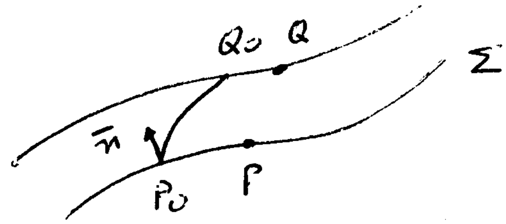

Figure 3

w^i coordinates, $(i = 1, \cdots, n-1)$ by $w^i(P) = y^i(P)$, and define w^i in N by taking w^i to be constant along each $\gamma_p(s)$; i.e.

$$w^i(Q) = w^i(P) \quad \text{iff} \quad Q = \gamma_p(s) \quad \text{for some} \quad s \quad \text{and} \quad P \in \Sigma \ .$$

The coordinates $\left(w^1, \cdots, w^n\right)$ are called *Gaussian normal coordinates.* Note that w is only C^1 related to y because the geodesics normal to Σ are, in general, only C^1 curves since the Γ^i_{jk} can, in general, have jump discontinuities on Σ if the metric match only Lipschitz continuously on Σ. We now have the following theorem ([7]).

Theorem 3. Assume g and $\bar{g}$ are two spherically symmetric metrics.

$$g: \quad ds^2 = -a(t,r)dt^2 + b(t,r)dr^2 + c(t,r)d\Omega^2 ,$$

$$\bar{g}: \quad d\bar{s}^2 = -\bar{a}(\bar{t},\bar{r})d\bar{t}^2 + \bar{b}(\bar{t},\bar{r})d\bar{r}^2 + \bar{c}(\bar{t},\bar{r})d\Omega^2 ,$$

that match Lipschitz continuously along a 3-dimensional shock Σ, and that there exists a smooth transformation $\psi : (t,r) \longrightarrow (\bar{t},\bar{r})$ such that the matrices agree on the surface $r = r(t)$. (We implicitly assume that θ and ϕ are continuous across Σ). Assume too that $c(t,r) = \bar{c}(\psi(t,r))$, and that the shock surface $r = r(t)$ is mapped to the surface $\bar{r} = \bar{r}(\bar{t})$ by $\psi(t, r(t)) = (\bar{t}, \bar{r}(\bar{t}))$. Finally, assume that the normal $\bar{n}$ to Σ is non-null, and that $\bar{n}(c) \neq 0$, where $n(c)$ denotes the derivative of c in the direction $\bar{n}$. Then the following are equivalent:

$$g \cup \bar{g} \text{ is}^2 \; C^{1,1} \text{ in Gaussian normal coordinates,} \tag{15}$$

$$\left[G^i_j\right] n_i = 0 \; , \tag{16}$$

$$\left[G^{ij}\right] n_i n_j = 0 \; , \tag{17}$$

$$[K] = 0 \; . \tag{18}$$

Here $[f] = \bar{f} - f$ denotes the jump in the quantity f across Σ, and K isthe 2nd fundamental form on the shock surfaces. (To contrast, for general Lipschitz continuous matrices, $\left[G^i_j\right] n_i = 0$ iff *both* of the following Israel jump relations hold, (c.f. [2,3,6]):

$$\left[(\mathrm{tr}\ k)^2 - \mathrm{tr}\left(k^2\right)\right] = 0, \quad \text{and} \quad \left[\mathrm{div}\ K - d(\mathrm{tr}\ k)\right] = 0 \; . \tag{19}$$

(In these formulas tr denotes the trace, div is the covariant divergence, and d denotes exterior differentiation on the surface.) We conclude then, that one condition (17), not two, must be imposed for conservation to hold, in our problem. Thus if $r(t)$, (where $\bar{r}(t) - R(t)r(t)$), denotes the shock surface, then a calculation gives

$$\left[T^{ij}\right] n_i n_j = 0 = (\bar{p} + \rho)\dot{r}^2 - (\bar{\rho}+\bar{p}) \frac{B}{A} \frac{(1 - kr^2)}{R^2} + (p-\bar{p}) \frac{1 - kr^2}{R^2} = 0. \tag{20}$$

(Here $A(r) = \left(1 - \dfrac{2GM(r)}{r}\right)^{-1}$, and B is given by (9)). As a check, for the OS model, $p = \bar{\rho} = 0$ so (20) becomes

$$\rho\dot{r}^2 + p \frac{(1 - kr^2)}{R^2} = 0 \; , \tag{21}$$

and hence if $\rho > 0$, then $p = 0 = \dot{r}$, so no solution is possible for non-zero pressure!

Now in [7], the idea is to fix $\bar{\rho}, \bar{p}$, and $M(\bar{r})$ in the (IS) solution, and to obtain ordinary differential equations for $R(t)$ and $\rho(t)$:

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[2] $C^{1,1}$ is the class of C^1 functions with Lipschitz continuous derivatives.

$$\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - k \tag{22}$$

$$p = \frac{-d}{dt} (\rho R^3) \,/\, 3r^2\dot{R} \,. \tag{23}$$

Note that if $p = p(\rho)$, this gives two equations for ρ and R and determines a unique solution; thus conservation, (20), need not hold. However, we take a different approach and allow p to be free, taking (20) to be an extra constraint. We find that on the shock surface Σ, we obtain an *autonomous* system of ode's for (r,R):

$$\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - k \tag{24}$$

$$(\bar{p} + \rho)\dot{r}^2 - (\bar{p} + \bar{\rho}) \frac{B}{A} \frac{(1 - kr^2)}{R^2} + (p - \bar{p}) \frac{(1 - kr^2)}{R^2} = 0. \tag{25}$$

Here the equations (12), (23) and $\bar{r} = Rr$ are used to eliminate p and ρ in terms of r and R.

It remains to study the system of equations (24), (25); in particular to show that $p > 0$, in the large; we have proved this locally. The system (24) can be used to study gravitational collapse; in particular it is interesting to see if non-zero pressure can prevent collapse.

We have outlined a procedure by which solutions with interesting dynamics can be constructed out of simple solutions, since the matched solutions dynamically evolve from one solution to (a coordinate transformation of) another solution as the shock surface propagates. This procedure leads us also to the construction of new cosmological models.

We remark that many interesting problems arise from Lipschitz continuous but not $C^{1,1}$ metrics. In particular we have an example showing that there exist Lipschitz continuous shock waves which satisfy the Israel jump relations (19) across a shock interface, but cannot be transformed to a C^1 metric. For such metrics, can we interpret the jump conditions, $[T^i_j]n_i = 0$, as representing conservation of energy and momentum? The answer is yes, only in "locally Lorenztian" frames; i.e. frames where $g_{ij}(P) = 0$, so $\Gamma^i_{jk}(P) = 0$. In these frames the divergence theorem holds up to higher order corrections. But for general Lipschitz continuous metrics, can one transform to a locally Lorentzian frame? (The answer is that one can for C^1 metrics; in particular for spherically symmetric metrics). For general Lipschitz continuous metrics, there can be δ-function sources on the shock surface, because as g_{ij} is only Lipschitz continuous, the second derivatives, $g_{ij,k\ell}$, can be δ-functions. Then the Einstein tensor G_{ij}, and hence the stress-energy tensor T_{ij}, can also contain δ-functions. Such δ-function sources can serve as sources of energy and momentum, and hence "physical" conservation can fail to hold. It is thus an issue for Lipschitz continuous metrics, as to how $[T^i_j]n_i = 0$ is to be interpreted; in particular when does this equality imply conservation? (For spherically symmetric metrics, in particular for our extension of the O.S. case, we show that a $C^{1,1}$ matching can be achieved in some frame, so there are no δ-function sources in this frame, and we are able to construct locally Lorentzian frames; hence $[T^i_j]n_i = 0$ *can* be interpreted as conservation of energy and momentum.) As discussed in [7], the introduction of Lipschitz continuous metrics also raises interesting new questions of existence, uniqueness, and admissibility for weak solutions of the Einstein field equations (1).

References

1. Adler, R., M. Bazin, and M. Schiffer, Introduction to General Relativity (2nd ed.), McGraw-Hill, New York, 1975/

2. Israel, W., "Singular Hypersurfaces in General Relativity", *Il Nuovo Cimento*, Vol. XLIVB, N.1, 1966, 1-14.

3. Misner, C., K. Thorne, and J. Wheeler, Gravitation, W.H. Freeman and Co., New York, 1973.

4. Oppenheimer, J.R., and H. Snyder, "On Continued Gravitational Contraction", *Phys. Rev.*, 56, (1939), 455-459.

5. Smoller, J., "Shock Waves and Reaction-Diffusion Equations", Springer-Verlag, New York, 1983.

6. Smoller, J., and B. Temple, "Global Solutions of the Relativistic Euler Equations", *Comm. Math. Phys.*, (in press).

7. Smoller, J. and B. Temple, "Shock-Wave Solutions of the Einstein Equations", (to appear).

8. Weinberg, S., Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley and Sons, New York, 1972.

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