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A COMPARISON OF CONVERGENCE RATES FOR GODUNOV'S METHOD AND GLIMM'S METHOD IN RESONANT NONLINEAR SYSTEMS OF CONSERVATION LAWS*

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Abstract. We obtain time independent bounds on derivatives, prove convergence, and establish a rate of convergence for Godunov's numerical method as applied to the initial value problem for a resonant inhomogeneous conservation law which is treated as a 2×2 nonstrictly hyperbolic system. We compare the results with a corresponding analysis of Glimm's method and see that our analysis gives equivalent (sharp) convergence rates in the strictly hyperbolic setting, but an improvement is seen in Godunov's method over Glimm's method in the nonstrictly hyperbolic resonant regime. The 2×2 Glimm and Godunov methods are the only methods for which we can obtain time independent bounds on derivatives; these bounds represent a purely nonlinear phenomenon because there are no corresponding time independent bounds for the linearized equations which blow up at a linear rate in time.

Key words. Godunov method, Glimm method, resonance, conservation law, convergence rate

AMS subject classifications. 35, 65, 76

1. Introduction. We consider the initial value problem for the 2×2 resonant nonlinear system of conservation laws

$$\begin{aligned} &a_t = 0, \\ &u_t + f(a,u)_x = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &a(x,0) = a_0(x) \equiv a(x), \\ &u(x,0) = u_0(x), \end{aligned} \tag{2}$$

where $u \in R$, $a \in R$, and we let $U = (a, u)$. This is the special case of an $n \times n$ resonant nonlinear system as introduced in [10], [11]. System (1) is a system with the two wave speeds $\lambda_0(U) = 0$ and $\lambda_1(U) = \partial f/\partial u$. Here we consider the problem of the convergence of the 2×2 Godunov method for solutions of (1) taking values in a neighborhood of a state U_*, where we assume (these assumptions were introduced in [10], [11])

$$\lambda_1(U_*) = 0, \tag{3}$$

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$$\frac{\partial}{\partial u}\lambda_1(U_*) \neq 0, \tag{4}$$

$$\frac{\partial}{\partial a}f(U_*) \neq 0. \tag{5}$$

Condition (3) states that the wave speeds λ_0 and λ_1 coincide at U_*, condition (4) states that the nonlinear family of waves is genuinely nonlinear at U_*, and condition (5) is a nondegeneracy condition introduced in [11] that determines a canonical structure to the nonlinear wave curves in a neighborhood of the state U_*. The initial value problem is more complicated when wave speeds coincide, and one can show that unlike the strictly hyperbolic problem, in the resonant problem (1), the total variation of the solution at time $t > 0$ (a measure of the strength of the nonlinear waves present) cannot be bounded by the total variation of the initial data in this setting, even for sufficiently weak solutions. Moreover, the process by which solutions decay to time asymptotic wave patterns is correspondingly more interesting and more complicated than in the strictly hyperbolic case [17], [7]. In this paper we obtain time independent bounds on derivatives, prove convergence, and establish a rate of convergence, for Godunov's numerical method as applied to solutions of (1) and (2) in a neighborhood of such a state U_*. The fact that such time independent bounds exist at all is a purely nonlinear phenomenon and is surprising in that the resonant linearized system blows up. The analysis in [22] carries over directly to problems (1) and (2), thus providing a corresponding convergence theorem for Glimm's numerical method.

We prove that the 2×2 Godunov method converges to a weak solution of (1) and (2) (modulo extraction of a subsequence) by showing that the total variation of any approximate solution of (1) and (2) at time $t > 0$ is bounded by the initial total variation when the total variation is measured under the singular transformation $\Psi : (a, u) \to (a, z)$, which was introduced in [22] (see also [8]). We sharpen the analysis in [22] that gives a rate of convergence for Glimm's method, and we use the improved estimates to conclude that the best rate of convergence of the 2×2 Godunov method implied by our analysis is

$$R(U_{\Delta x}, \phi) = O(\Delta x^{1/(1+p)}), \tag{6}$$

where $U_{\Delta x} \equiv (a_{\Delta x}, u_{\Delta x})$ denotes the Godunov approximate solution, p is the order of contact between the wave curves at the points of coinciding wave speeds, and R denotes the residual of the weak solution

$$R(U, \phi) \equiv \int_0^{\infty} \int_{-\infty}^{+\infty} (U\phi_t + F(U)\phi_x)\, dx\, dt + \int_{-\infty}^{+\infty} U_0(x)\phi_0(x)\, dx, \tag{7}$$

where $\phi(x, t)$ is any smooth test function. We can compare this with the rate of convergence of Glimm's method in this problem, which by our methods (sharpening [22]) is given by

$$\int_{\Pi[0,1]_{ij}} R(U_{\alpha\Delta x}, \phi)^2\, d\alpha = O(\Delta x^{1/(1+2p)}). \tag{8}$$

Here, $U_{\alpha\Delta x}$ denotes the approximate solution generated by the Glimm scheme, where $\alpha \in A \equiv \Pi[0, 1]_{ij}$ denotes the random sequence which determines the choice of sampling in Glimm's method. (The rate originally obtained in [22] was $\Delta x^{1/(3+2p)}$ which is not correct when $p = 0$. The improvement here is in estimate (37) of Proposition 3, which refines the estimate obtained in Proposition 6.2 of [22].) Note that when $p = 0$, both of the rates in (8) and (6) reduce to the known rate $O(\Delta x)$ for the convergence of these methods in the presence of a total variation bound in the conserved quantities. (In fact, condition (4) forces $p = 2$ at the state U_*, but we can consider the more general case $p \neq 2$ as well.) We attempt to make a meaningful comparison of these rates as follows: the error estimate in (8) is given in terms of the integral of $R(U_{\alpha\Delta x}, \phi)^2$ over the measure space of sampling sequences. Now estimate (8) can be interpreted as saying that the convergence rate of $R(U_{\alpha\Delta x}, \phi)^2$, averaged over all equidistributed sequences (a set of measure one in A), is $O(\Delta x)$, suggesting that the average rate of convergence of $R(U_{\alpha\Delta x}, \phi)$ is $O(\Delta x^{1/2})$. However, the best equidistributed sequences (a set of measure zero in A), give an improvement in convergence over the average by essentially a power of two (ignoring logarithms); thus we expect that in the best cases, the convergence of $R(U_{\alpha\Delta x}, \phi)$ in Glimm's method is $O(\Delta x)$.[1] Thus when $p = 0$, the comparison of the rates (7) and (8) should be valid, suggesting that (7) and (8) may provide a fair basis of comparison when $p > 0$. We believe that there is an improvement in the convergence of the Godunov method over the Glimm method for $p > 0$ due to the fact that the averaging in the Godunov method regularizes the oscillations that can occur in both methods at the states of coinciding wave speeds (see [22]). Specifically, the Glimm method is designed to give an accurate time evolution of elementary waves through an exact calculation of wave interactions. The problem here is that the coordinate system of wave curves is singular at the points of coinciding wave speeds, and thus the solution for smooth initial data which is transverse to the wave curves is not well approximated by elementary waves, and so the piecewise constant approximation scheme generates oscillatory waves that are exactly time evolved by the Glimm method. The averaging process in the Godunov method regularizes these oscillations at each time step.

We are only able to obtain the time independent bounds on derivatives for the 2×2 Glimm and Godunov methods, and have not succeeded in obtaining these bounds directly by any other 2×2 method (including the Lax–Friedrichs scheme), nor for any of the scalar methods that apply to the system $u_t + f(a(x), u)_x = 0$. Moreover, because of the resonant behavior near the state U_*, the total variation of u at any time cannot be bounded by the total variation of the initial data when a is only of bounded variation, making it difficult to obtain estimates for the derivatives of u directly at each fixed time. Indeed, (3)–(5) imply that the linearized equations for (1) at the state U_* are given by

$$
\begin{aligned}
a_t &= 0, \\
u_t + f_a(U_*)a_x &= 0,
\end{aligned}
\tag{9}
$$

and thus it is easy to see that the solution in the linearized case is $u(x,t) = -f_a(U_*)a'(x)t + u_0(x)$. Thus in the linear problem, u and all x-derivatives of u blow up at a linear rate. The results here and in [22] confirm that solutions of the nonlinear problem satisfy time independent bounds on derivatives when $u_0(\cdot)$ and $a(\cdot)$ are of bounded variation, and this shows that the nonlinear problem is much more regular than the linearized problem. The methods of Oleinik [19] and Kruzkov [13],

[1] Proving this rigorously is an open problem. The conclusion is suggested by the convergence properties of a single shock. For a rigorous result in this direction, see [24, Lem. 3.3], where, in the strictly hyperbolic case, $R(U_{\alpha\Delta x}, \phi)$ is shown to be at least $O(\Delta x^{1/3})$ for all α except on a set of measure $\leq O(\Delta x^{1/3})$.

which apply to the inhomogeneous scalar conservation law $u_t + f(a(x), u)_x = 0$ when $a \in C^2$, yield at best Gronwall-type exponential growth bounds on the total variation in u. Thus it has not been shown that the entropy solutions of Oleinik and Kruzkov satisfy the time independent bounds that we are obtaining via the 2×2 methods. This appears to be a subtle problem for arbitrary $a(\cdot)$ of bounded variation, because the argument of Kruzkov is easy to apply only when based on a total variation bound on u, and the established total variation bound on z does not imply estimates on the total variation of u when $a(\cdot)$ is not smoother than C^1. Counterexamples show that there is no bound on the rate of blowup of u in the total variation norm that is based on the C^1-norm of a alone. The issue of uniqueness, continuous dependence, and blowup will be addressed in a subsequent paper, where we will obtain the correspondence of the entropy solutions and the solutions generated by the 2×2 Godunov method by establishing the sharp result that the blowup in the total variation of u is sublinear when the total variation of u_0 and of da/dx is finite. The averaging in Godunov's method is used to advantage, and no one to date has obtained a total variation estimate for the conserved variable u based on Glimm's method. This will establish, for the first time, the entropy conditions of Kruzkov and the time independent bounds on z-derivatives simultaneously in solutions generated by the same numerical method (Godunov's method), thus giving the first proof that the classical solutions of the scalar, resonant inhomogeneous equation $u_t + f(a(x), u)_x = 0$ blow up in the x-derivatives of the conserved variables just like the linearized problem, but satisfy uniform bounds for all time on the derivatives as measured under a singular transformation that is only meaningful in the nonlinear problem. (See also [23] where exponential growth bounds on u are obtained under similar assumptions via an upwind scheme that applies to the polymer equations (see (10) below, an Eulerian formulation of (1) and (2)).)

A special case of a resonant system of form (1) is obtained in the Lagrangian formulation of the polymer equations (see [5] and [22])

$$\begin{aligned} &s_t + f(s, c)_x = 0, \\ &(cs)_t + (cf(s, c))_x = 0. \end{aligned} \tag{10}$$

System (10) is the Eulerian formulation of a model for the polymer flooding of an oil reservoir, a two-phase, three-component flow in which s represents the saturation of the aqueous phase, c represents the concentration of polymer within this phase, and $g = f/s$ plays the role of the particle velocity of water, $0 \le s, c \le 1$. The relevant constitutive assumption is that $g(\cdot, c)$ is positive, convex down, and $g(0, c) = 0 = g(1, c)$ [5]. These assumptions imply that the wave speeds $\lambda_0 = 0$ and $\lambda_1 = s^2 g_s$ coincide on a curve in state space. The equivalence of system (10) with system (1) is obtained through the Lagrangian map defined by specifying $\xi(x, t)$ through the solution dependent mapping

$$\frac{\partial \xi(x, t)}{\partial t} = g(\xi(x, t), t),$$

$$\xi(x, 0) = \int_0^{\xi} \frac{1}{s_0(x)}\, dx.$$

The Lagrangian map takes system (10) to the equivalent system [11]

$$c_t = 0, \tag{11}$$
$$\left(\frac{1}{s}\right)_t - g(s,c)_\xi = 0,$$

which is of form (1) under the identification $a = c$, $u = 1/s$, and $x = \xi$. Since the state space and wave curves are the same for system (10) and (11), the analysis of Glimm's method for (1) or (11) is equivalent to that given in [22] for (10), but the analysis of Godunov's method is fundamentally different. Indeed, we are able to obtain the time independent bounds for Godunov's method in the Lagrangian formulation essentially because the contact discontinuities move at zero speed, and thus the process of averaging at each time step does not involve states on the contact discontinuities in (11) as it does in the Eulerian formulation (10). As a final comment, we note that in this paper we address the problem of solving the initial value problem for (1) by Godunov's method locally in a neighborhood of a state U_* where (3)–(5) hold. However, the analysis applies globally to the problem (11) under the appropriate constitutive assumptions.

It is important to note that (6) and (8) give rates of convergence which depend only on the total variation of the initial data $U_0(x) = (a_0(x), u_0(x))$ as measured under the singular transformation Ψ. There is no corresponding time independent convergence rate provided by the methods of Kruzkov [13] and Oleinik [19] since their methods do not provide time independent bounds on derivatives and apply only when $a(x)$ is sufficiently smooth. A study of (1) when a is of bounded variation is equivalent to studying the time asymptotic wave patterns when a is smooth.

2. Preliminaries. We now construct solutions of (1) and (2) by the 2×2 Godunov scheme. We first review the solution of the Riemann problem and the construction of the singular transformation Ψ as they apply to system (1) near a state U_* satisfying (3)–(5). We rewrite system (1) as

$$U_t + F(U)_x = 0, \tag{12}$$

where $U = (a, u)$ and $F(U) = (0, f(u))$. Nonstrictly hyperbolic systems of form (12) were previously discussed in [12], [5], [7]–[11], but the construction of Ψ first given in [22] applies to the Eulerian formulation (10) of (12), and we now translate those results into the setting of a resonant inhomogeneous system. We use the notation set out in [11]. The Riemann problem, denoted $[U_L, U_R]$, is the initial value problem for initial data given by a jump discontinuity

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0. \end{cases} \tag{13}$$

Because (12) is nonstrictly hyperbolic at $U = U_*$, there are in general three waves that solve the Riemann problem as follows: the eigenvalues of (12) are $\lambda_0(U) \equiv 0$ and $\lambda_1(U) = \partial f/\partial u$. In [11] it is shown that the assumptions (3)–(5) imply that $\lambda_1(U) = 0$ defines a smooth curve Γ (named the transition curve) in U-space passing through the state $U = U_*$ in a direction transverse to the u-axis. Moreover, the matrix dF has the form

$$dF = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{14}$$

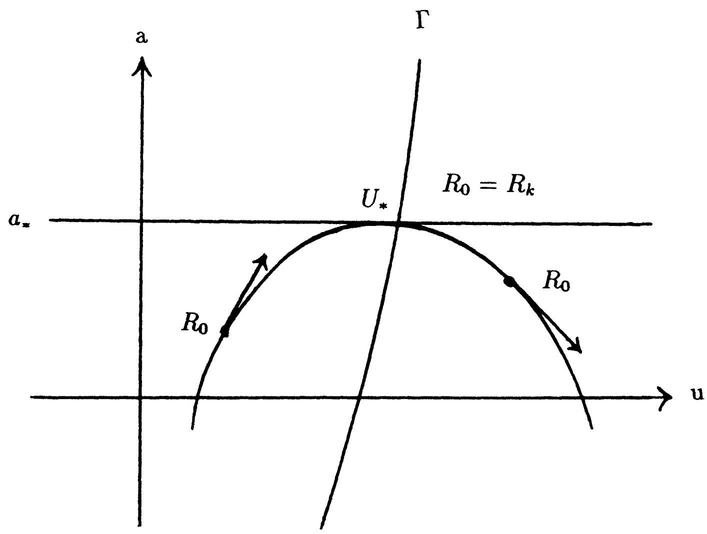

FIG. 1. *Generic quadratic tangency of integral curves.*

at each state $U \in \Gamma$. The wave curves for (12) are the integral curves of the eigenvector fields R_0 and R_1 associated with λ_0 and λ_1. The 1-wave curves are given by $a = \bar{a}$, $\bar{a}$ constant, and 1-waves are determined by solutions of the scalar conservation law $u_t + f(\bar{a}, u)_x = 0$. The 0-wave curves are given by $f = \text{const}$ Because of (3)–(5), in a neighborhood of U_*, $f = \text{const}$ defines a smooth curve of nonzero curvature which is tangent to the curves $a = \text{const}$ only at the states $U \in \Gamma$ in the au-plane; and the transition curve Γ intersects the 0-wave and 1-wave curves transversally at U_*. Without loss of generality and in order to be consistent with [22] and [11], we assume that $f_u u < 0$ and $f_a < 0$, so that the curves $f = \text{const}$ are convex down in a neighborhood of U_* (see Lemma 3.1 in [11]) as diagramed in Fig. 1.

Thus, let $\mathbf{B}$ denote a neighborhood of U_* bounded above by an integral curve of R_0 and below by an integral curve of R_1, such that the integral curves of R_0 are convex down in $\mathbf{B}$, and such that each integral curve intersects the transition curve Γ transversally at a unique point in $\mathbf{B}$. Assume further that $\partial f/\partial a \neq 0$ in $\mathbf{B}$. Our assumptions (3)–(5) imply that such an open set $\mathbf{B}$ exists in a neighborhood of U_*. In [11] it is shown that for U_L and $U_R \in \mathbf{B}$, the Riemann problem $[U_L, U_R]$ is solved uniquely by at most three elementary waves: a negative speed 1-wave followed by a zero speed 0-wave followed by a positive speed 1-wave (as seen moving from left to right in the xt-plane). The condition that a 0-wave cannot cross the transition curve Γ serves as an entropy condition [5], [12]. The solution diagrams for the solution of the Riemann problem are reproduced in Figs. 2 and 3 for the two cases when U_L lies to the left and the right of the transition curve Γ. The following proposition holds [11], [22].

PROPOSITION 1. *Let* $\mathbf{B}$ *denote a neighborhood of* U_* *bounded above by an integral curve of* R_0 *and below by and integral curve of* R_1*, such that the integral curves of* R_0 *are convex down in* $\mathbf{B}$*, and such that each integral curve cuts the transition curve* Γ *transversally in* $\mathbf{B}$*. Then* $\mathbf{B}$ *is an invariant region for Riemann problems in the sense that if* $U_L, U_R \in \mathbf{B}$*, then all intermediate states in the solution of the Riemann problem* $[U_L, U_R]$ *are also in* $\mathbf{B}$.

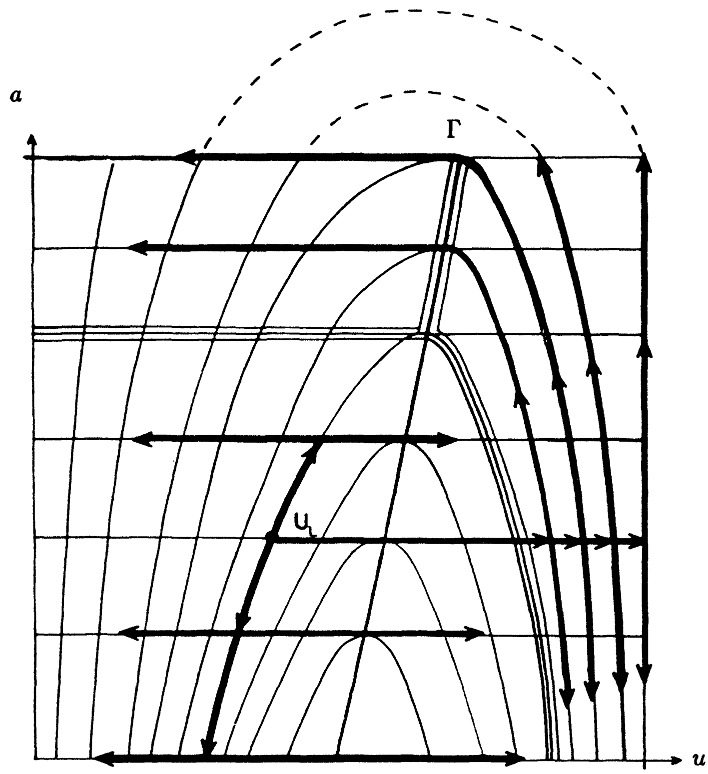

FIG. 2. *The Riemann problem for U_L left of* Γ.

We now define the singular transformation $\Psi{:}(a,u) \to (a,z)$ by specifying the function $z = z(a,u)$. For a given $U = (a,u)$, let $U_\Gamma(U)$ be the point where the 0-wave curve through U intersects the transition curve Γ, so that $U_\Gamma \in \Gamma$ and $f(U_\Gamma) = f(U)$. Then U_Γ is well defined in a neighborhood of U_*, and we can define

$$z(a,u) = \begin{cases} a_\Gamma - a & \text{if } U \text{ lies to the right of } \Gamma, \\ a - a_\Gamma & \text{if } U \text{ lies to the left of } \Gamma. \end{cases} \tag{15}$$

Since $a \le a_\Gamma$ in a neighborhood of U_* due to the convexity of the integral curves of R_0, we conclude that the mapping

$$\Psi : (a,u) \to (a,z)$$

is $1-1$ and onto in a neighborhood of U_*, and is regular except at Γ where the Jacobian vanishes. We let $\mathrm{Var}_{az} U_0$ denote the total variation of U_0 in the variables a and z, etc. Now let γ denote an arbitrary elementary wave, i.e., a 0-wave or a 1-wave. Following [22], we define the strength $|\gamma|$ of the elementary wave γ having left state U_L and right state U_R as follows:

$$|\gamma| = \begin{cases} |z(U_L) - z(U_R)| & \text{if } \gamma \text{ is a 1-wave}, \\ 2|z(U_L) - z(U_R)| & \text{if } \gamma \text{ is a 0-wave and } u_L > u_R, \\ 4|z(U_L) - z(U_R)| & \text{if } \gamma \text{ is a 0-wave and } u_L \le u_R. \end{cases} \tag{16}$$

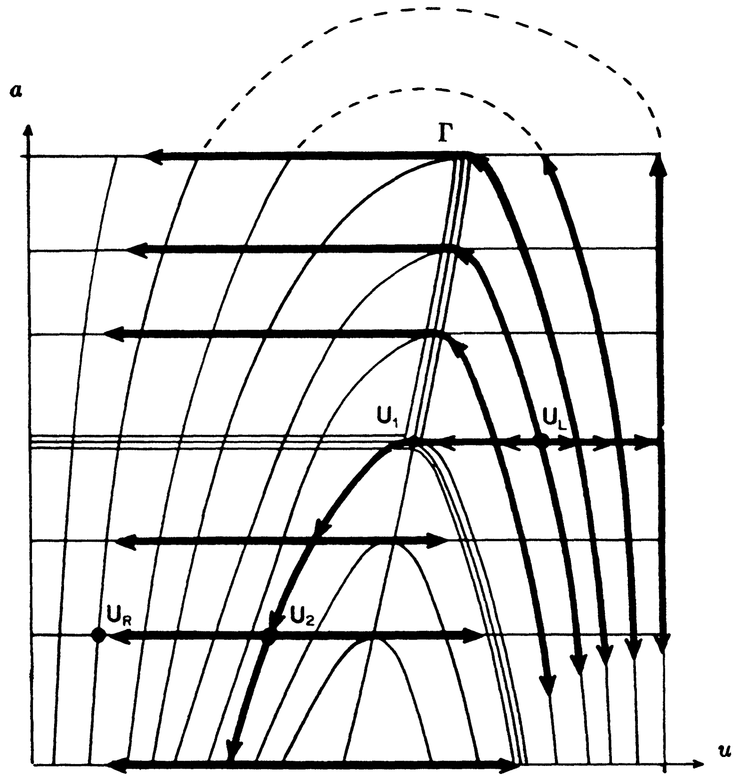

FIG. 3. *The Riemann problem for U_L right of* Γ.

We say that $\gamma_1, \ldots, \gamma_n$ is a connected sequence of elementary waves taking U_L to U_R if $U_L^1 = U_L$, $U_R^n = U_R$ and $U_R^i = U_L^{i+1}$, $i = 1, \ldots, n-1$, where U_L^i and U_R^i denote the left and right states of the wave γ_i. For such a sequence $\gamma_1, \ldots, \gamma_n$, define

$$F(\gamma_1, \ldots, \gamma_n) \equiv \sum_{i=1}^{n} |\gamma_i|. \tag{17}$$

We use the following proposition which was obtained in [22] for the Eulerian problem (10), but which is valid for the Lagrangian problems (11) or (1) because the Lagrangian transformation that takes (10) to (11) preserves the weak equations and thus the structure of the wave curves in state space. For a proof we refer to [22], [11].

PROPOSITION 2. *There exists a neighborhood* $\mathbf{B}$ *of* U_* *such that if* U_L, $U_R \in \mathbf{B}$, *then*

$$F([U_L, U_R]) \leq F(\gamma_1, \ldots, \gamma_n),$$

where $\gamma_1, \ldots, \gamma_n$ *is any connected sequence of elementary waves in* $\mathbf{B}$ *taking* U_L *to* U_R.

Let p denote the order of contact between the integral curves of R_0 and R_1 at $U \in \Gamma$. According to Lemma 3.1 in [11], the assumption that λ_1 is genuinely nonlinear at U_* (assumption (4)) implies that $p = 2$. However, it follows directly from [22] that Propositions 1 and 2 apply equally well under the assumptions that the region $\mathbf{B}$ is a neighborhood of U_* bounded above in au-space by an integral curve of R_0 and below by an integral curve of R_1, that the integral curves of R_0 are convex down in $\mathbf{B}$ and intersect Γ uniquely and transversally in $\mathbf{B}$, and that $\partial f/\partial a \neq 0$ in $\mathbf{B}$, but with the weaker assumption that the integral curves of R_0 and R_1 make pth order contact at Γ. Thus we can treat the case $p > 2$ by relaxing assumption (4) and assuming that our problem (1) is posed in such a region $\mathbf{B}$. We make this assumption from here on out. Finally, for technical reasons, we assume that $\mathbf{B}$ is chosen sufficiently small so that $|da/du| < 1$ along each integral curve of R_0 in $\mathbf{B}$. All of our above assumptions hold for $\mathbf{B}$ in a neighborhood of a point U_* of interest.

3. The Godunov scheme. We now apply the Godunov scheme to construct the approximate solutions $U_{\Delta x}(x,t)$ to the Cauchy problem (1), (2) for arbitrary initial data $U_0(x)$ taking values in the neighborhood $\mathbf{B}$ of U_*, and satisfying $\mathrm{Var}_{az}U_0(\cdot) < \infty$. In the next section we will establish a rate of convergence of the method in terms of the order of contact p of the 0- and 1-wave curves at the transition curve Γ. First we discretize $R \times [0,\infty)$ by spacial mesh length Δx and time mesh length Δt such that

$$\frac{\Delta t}{\Delta x} = \Lambda \leq \frac{1}{2} \sup_{(a,u)\in\mathbf{B}} \{|\lambda_0|, |\lambda_1|\}, \tag{18}$$

and we let $x_n = n\Delta x$, $t_j = j\Delta t$ so that (x_n, t_j) denotes the mesh points of the approximate solution. Define

$$S_i = \{(x,t) : t_i \leq t < t_{i+1}\}.$$

The approximate solution $U_{\Delta x}$ generated by the Godunov scheme is defined as follows [3], [4]: to initiate the scheme at $n = 0$, define

$$U_j^0 \equiv U_{\Delta x}(x,0) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U_0(x)\,dx, \quad x_j < x < x_{j+1}. \tag{19}$$

Assuming that $U_{\Delta x}(x,t)$ has been constructed for $(x,t) \in \bigcup_{i=0}^{n-1} S_i$, then we define $U_{\Delta x}$ in S_n as the solution of (1) with the initial values

$$U_j^n \equiv U_{\Delta x}(x,t_n) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U(x,t_n-)\,dx, \quad x_j < x < x_{j+1}. \tag{20}$$

In other words, at each time t_n, a piecewise constant approximation $U_{\Delta x}(x,t_n)$ is obtained by taking the arithmetic averages of $U_{\Delta x}(x,t_n-)$ at each interval of the mesh, so that the solution in S_n can be constructed by solving the Riemann problems $[U_j^n, U_{j-1}^n]$ posed at each point of discontinuity $(x_j,t_n), j \in Z$. The Courant–Friedrichs–Lewy (CFL) restriction (18) ensures that the Riemann problem solutions in each S_n do not interact before time t_{n+1} [22], and thus, at each time level t_n+, the Godunov approximate solution determines a sequence of connected elementary waves taking $U_{\Delta x}(-\infty,t_n+)$ to $U_{\Delta x}(+\infty,t_n+)$ in the sense of §3. (Indeed, we will show that $\mathrm{Var}_{az}U_{\Delta x}(\cdot,t) \leq 4\mathrm{Var}_{az}U_0(\cdot)$, from which it follows that $U_{\Delta x}(-\infty,t_n+) = U_0(-\infty) \equiv U_{-\infty}$ and $U_{\Delta x}(+\infty,t_n+) = U_0(+\infty) \equiv U_{+\infty}$.) Since the solution of the Riemann problem in general consists of three waves, we can label these waves as $J_n \equiv \{\gamma_\sigma^n\} : \sigma \in \{j-1/3, j, j+1/3\}, \{j \in Z, n \in Z\}$. Here we let $\gamma_{j-1/3}^n, \gamma_j^n, \gamma_{j+1/3}^n$ label the three waves emanating from the mesh point (x_j,t_n) from left to right in the xt-plane, respectively (see Fig. 3). Similarly, we label the right and left states of the elementary waves so that $\gamma^n_{j-1/3}$ is the 1-wave that solves the Riemann problem $[U^n_{j-1}, U^n_{j-2/3}]$, γ^n_j is the 0-wave that solves the Riemann problem $[U^n_{j-2/3}, U^n_{j-1/3}]$, and $\gamma^n_{j+1/3}$ is the 1-wave that solves the Riemann problem $[U^n_{j-1/3}, U^n_j]$, as labeled in Fig. 3. We now prove the following theorem.

THEOREM 1. *The functional F (defined in* (17)), *evaluated on the sequences of elementary waves defined at each time level in the Godunov approximate solution $U_{\Delta x}$, is nonincreasing in time; i.e.,*

$$F(J_{n+1}) \le F(J_n).$$

Proof. Since $U^{n+1}_j \equiv (a^{n+1}_j, u^{n+1}_j)$ is obtained by averaging $U_{\Delta x}(x, t_{n+1-})$ over the interval (x_j, x_{j+1}) as defined in (19), we have that

$$a^{n+1}_j = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} a^n_j \, dx = a^n_j$$

for all $j \in Z$ and

$$\text{Min}\{u^n_{j-1/3}, u^n_j, u^n_{j+1/3}\} \le u^{n+1}_j \le \text{Max}\{u^n_{j-1/3}, u^n_j, u^n_{j+1/3}\}. \tag{21}$$

Thus it follows from Proposition 2 that (cf. Fig. 3)

$$F[U^{n+1}_j, U^{n+1}_{j+1}] \le F[U^{n+1}_j, U^n_{j+1/3}] + F[U^n_{j+1/3}, U^n_{j+2/3}] + F[U^n_{j+2/3}, U^{n+1}_{j+1}]. \tag{22}$$

Summing both sides of inequality (22) over all integers and regrouping we obtain

$$\begin{aligned} F(J_{n+1}) &= \sum_{j\in Z} F[U^{n+1}_j, U^{n+1}_{j+1}] \\ &\le \sum_{j\in Z}\{F[U^{n+1}_j, U^n_{j+1/3}] + F[U^n_{j+1/3}, U^n_{j+2/3}] + F[U^n_{j+2/3}, U^{n+1}_{j+1}]\} \\ &= \sum_{j\in Z}\{F[U^n_{j-1/3}, U^{n+1}_j] + F[U^{n+1}_j, U^n_{j+1/3}] + F[U^n_{j+1/3}, U^n_{j+2/3}]\}. \end{aligned} \tag{23}$$

By (17),

$$F[U^n_{j-1/3}, U^{n+1}_j] + F[U^{n+1}_j, U^n_{j+1/3}] = |z^n_{j-1/3} - z^{n+1}_j| + |z^{n+1}_j - z^n_{j+1/3}|. \tag{24}$$

But for fixed a, the variable z is a strictly increasing function of u, thus it follows from (21) that

$$\text{Min}\{z^n_{j-1/3}, z^n_j, z^n_{j+1/3}\} \le z^{n+1}_j \le \text{Max}\{z^n_{j-1/3}, z^n_j, z^n_{j+1/3}\}; \tag{25}$$

and thus from (25) and simple inequalities for real numbers we have

$$\begin{aligned} |z^n_{j-1/3} - z^{n+1}_j| + |z^{n+1}_j - z^n_{j+1/3}| &\le |z^n_{j-1/3} - z^n_j| + |z^n_j - z^n_{j+1/3}| \\ &= F[U^n_{j-1/3}, U^n_j] + F[U^n_j, U^n_{j+1/3}]. \end{aligned} \tag{26}$$

Therefore, substituting (24) and (26) into (24) gives

$$\begin{aligned} F(J_{n+1}) &= \sum_{j\in Z} F[U^{n+1}_j, U^{n+1}_{j+1}] \\ &\le \sum_{j\in Z}\{F[U^n_{j-1/3}, U^n_j] + F[U^n_j, U^n_{j+1/3}] + F[U^n_{j+1/3}, U^n_{j+2/3}]\} \\ &= \sum_{j\in Z} F[U^n_j, U^n_{j+1}] = F(J_n), \end{aligned} \tag{27}$$

thus completing the proof of Theorem 1.

By the same argument given in [22], we can conclude from Theorem 3 that $\mathrm{Var}_{az}U_{\Delta x}$ is uniformly bounded in Δx and Δt; and similarly we can obtain that the approximate solutions $W_{\Delta x} \equiv (a_{\Delta x}, z_{\Delta x})$ are L^1 Lipschitz continuous in time. Therefore, by a standard compactness argument [2], there exists a subsequence $W_{\Delta x_k}(x,t)$ that converges boundedly almost everywhere in the upper half xt-plane. The continuity of Ψ^{-1} and the fact that by Proposition 1, $U_{\Delta x}(x,t) \in \mathbf{B}$ for all (x,t), imply that $U_{\Delta x_k}$ converges boundedly almost everywhere to a function U. By a general theorem due to Lax and Wendroff [15], U must be a weak solution of the Cauchy problem (1) in the sense that

$$R(U,\phi) = \lim_{\Delta x_k \to 0} R(U_{\Delta x_k}, \phi) = 0.$$

We thus obtain the following result regarding the convergence of the Godunov numerical method in the resonant regime.

THEOREM 2. *The 2×2 Godunov numerical method converges boundedly and pointwise almost everywhere (modulo extraction of a subsequence) to a weak solution of the resonant nonlinear system (1.1) for arbitrary initial data in $\mathbf{N}$ which is of bounded variation in a and z. Moreover, the solution is regular in the sense that it remains in $\mathbf{N}$ for all time, is uniformly Lipschitz continuous in the L^1-norm generated in az-space, and satisfies the time independent estimate*

$$\mathrm{Var}_{az}U(\cdot,t) \le 4\mathrm{Var}_{az}U_0(\cdot). \tag{28}$$

In particular, (28) is a time independent bound on the derivatives of the solution. Note that because $\partial z/\partial u = 0$ on Γ, $\mathrm{Var}_{az}(U) \le \mathrm{Var}_{au}(U)$, but (28) does not provide a bound on the total variation of the conserved quantities U. Indeed, counterexamples [22] show that, just as in the linearized problem (6), the $\mathrm{Var}_{au}(U(\cdot,t))$ cannot be bounded uniformly by $\mathrm{Var}_{au}(U_0)$ for solutions of the resonant nonlinear problem (1).

4. Rate of convergence. In §3 we showed that a subsequence of the Godunov approximate solutions $U_{\Delta x_k}$ converges globally to a weak solution U under the assumptions of Theorem 2. In this section we give a rate of convergence of the residual $R(U_{\Delta x}, \phi)$ which applies to any sequence of mesh lengths $\Delta x \to 0$. We prove the following theorem.

THEOREM 3. *Assume that the integral curves of R_0 and R_1 make pth order contact at each point $U \in \Gamma \bigcap \mathbf{B}$ in the au-plane, and assume that U_0 satisfies the assumptions of Theorem 2 ($\mathrm{Var}_{az}(U_0) < \infty$, $U_0(x) \in \mathbf{B}$). Let $U_{\Delta x}(x,t)$ be the approximate solution generated from U_0 by Godunov's method for the resonant nonlinear system (1). Then the following error estimate holds:*

$$D(\Delta x, \phi) \equiv |R(U_{\Delta x}, \phi)| \le C\Delta x^{1/(p+1)}, \tag{29}$$

where C is a constant depending only on $\mathrm{Var}_{az}(U_0)$, $\| \phi \|_{C^1}$, *and* $d \equiv \mathrm{diam}(\mathrm{Supp}\{\phi\})$.

Proof. Since $U_{\Delta x}(x,t)$ is an exact weak solution of (1) in each strip $S_n \equiv \{(x,t) : t_n \le t < t_{n+1}\}$, $n \in Z^+$, the following estimate holds:

$$D(\Delta x, \phi) = \left| \sum_{n,j} \int_{x_j}^{x_{j+1}} \phi(x,t_n)[U_{\Delta x}(x,t_n-) - U_{\Delta x}(x,t_n)]\, dx \right|$$

$$= \left| \sum_{n,j} \left\{ \int_{x_j}^{x_{j+1}} \phi(x_j, t_n)[U_{\Delta x}(x, t_n-) - U_{\Delta x}(x, t_n)]\, dx \right.\right.$$

$$\left.\left. + \int_{x_j}^{x_{j+1}} (\phi(x, t_n) - \phi(x_j, t_n))[U_{\Delta x}(x, t_n-) - U_{\Delta x}(x, t_n)]\, dx \right\} \right|$$

$$= \left| \sum_{n,j} D_{nj} \right|, \tag{30}$$

where

$$D_{nj} = \int_{x_j}^{x_{j+1}} (\phi(x, t_n) - \phi(x_j, t_n))[U_{\Delta x}(x, t_n-) - U_{\Delta x}(x, t_n)]\, dx, \tag{31}$$

and we have used (20). But since $U_{\Delta x} = (a_{\Delta x}, u_{\Delta x})$, we can estimate

$$|D_{nj}| \le \| \phi \|_{C^1} \Delta x \int_{x_j}^{x_{j+1}} |U_{\Delta x}(x, t_n-) - U_{\Delta x}(x, t_n)|\, dx$$

$$\le C(\Delta x)^2 \mathrm{Var}_{au}\{U_{nj}\}, \tag{32}$$

where $U_{nj} \equiv U_{nj}(x)$ denotes the restriction of the function $U_{\Delta x}(x, t_n-)$ to the interval $(x_j, x_{j+1}]$, and $\mathrm{Var}_{au}\{U_{nj}\}$ denotes the total variation of $U_{nj}(x)$ in terms of the variables a and u over the interval $(x_j, x_{j+1}]$. It follows from (31) and (30) that

$$D(\Delta x, \phi) \le C(\Delta x)^2 \sum_{(n,j)\in\Theta} \mathrm{Var}_{au}\{U_{nj}\}, \tag{33}$$

where Θ denotes the set of mesh points such that $\phi \neq 0$ for some $x_j \le x \le x_{j+1}, t = t_n$, and C is a constant depending only on $\|\phi\|_{C^1}$. We now estimate (33) in terms of $\mathrm{Var}_{az}\{U_{nj}\}$. To this end, denote by $S(\epsilon)$ the strip in the au-plane consisting of all points whose u-distance from Γ is strictly less than ϵ. Because we assumed at the outset that $\mathbf{B}$ was chosen small enough so that the integral curves of R_0 and R_1 intersect Γ uniquely and transversally in $\mathbf{B}$, we can conclude that an elementary wave in $\mathbf{B}$ can intersect the boundary of $S(\epsilon)$ at most twice inside $\mathbf{B}$ for ϵ sufficiently small. Now $U_{nj}(x)$ is a function of $x \in (x_j, x_{j+1}]$ whose image consists of states in the approximate solution $U_{\Delta x}(x, t_n-)$ that occurs at the interval $(x_j, x_{j+1}]$. Let $R \equiv R(\Delta x, \epsilon)$ denote the set of all indices $(n, j) \in \Theta$ which index the $U_{nj}(x)$ whose images lie strictly within $S(\epsilon)$, and let $\bar{R}$ denote the complement of R in Θ (cf. [22]). Thus we can partition the sum in (33) according to whether $(n, j) \in R$ or $(n, j) \in \bar{R}$, and write

$$D(\Delta x, \phi) \le C(\Delta x)^2 \sum_{(n,j)\in R} \mathrm{Var}_{au} U_{nj} + C(\Delta x)^2 \sum_{(n,j)\in \bar{R}} \mathrm{Var}_{au} U_{nj}. \tag{34}$$

The first sum is estimated as follows: for ϵ sufficiently small, the au-variation in any one Riemann problem solution, which is contained entirely within $S(\epsilon)$, is dominated by its variation in u (recall that we assume that $da/du < 1$ along the integral curves of R_0 in $\mathbf{B}$), and since there are at most three waves in $U_{\Delta x}(x, t_j-)$, $x \in (x_j, x_{j+1})$, this variation must be dominated by 5ϵ. Therefore we have

$$(\Delta x)^2 \sum_{(n,j)\in R} \mathrm{Var}_{au} U_{nj} \le \sum_{(n,j)\in R} 5\epsilon (\Delta x)^2. \tag{35}$$

But since $d = \operatorname{diam}\{\operatorname{Supp}\phi\}$, there are no more that $(d/\Delta x)^2$ nonzero terms in this sum, and so

$$(\Delta x)^2 \sum_{(n,j)\in R} \operatorname{Var}_{au} U_{nj} \leq 5d^2\epsilon. \tag{36}$$

We now estimate the second sum in (34) with the aid of the following proposition.

PROPOSITION 3. *If the function $U_{nj}(x)$ is not entirely contained within $S(\epsilon)$ and the curves $f = \text{const}$ and $a = \text{const}$ make pth order contact an Γ, then*

$$\operatorname{Var}_{au} U_{nj} \leq O(1)\epsilon^{-p} \operatorname{Var}_{az}\{U_{nj}\} + 5\epsilon. \tag{37}$$

Assuming Proposition 3 (whose proof we postpone until after the proof of Theorem 3), we complete the proof of Theorem 3 as follows: first obtain (the summations are restricted to $(n,j) \in \Theta$ if not indicated)

$$\begin{aligned} C(\Delta x)^2 \sum_{(n,j)\in \bar{R}} \operatorname{Var}_{au}\{U^n_{\Delta x}\} &\leq C(\Delta x)^2 \sum_{(n,j)\in \bar{R}} \{O(1)\epsilon^{-p}\operatorname{Var}_{az}\{U_{nj}\} + 5\epsilon\} \\ &\leq O(1)(\Delta x)^2 \sum_{n,j} \epsilon^{-p}\operatorname{Var}_{az}\{U_{nj}\} + C(\Delta x)^2 \sum_{n,j} 5\epsilon \\ &= O(1)(\Delta x)^2 \sum_{n} \epsilon^{-p}\operatorname{Var}_{az} U^n_{\Delta x} + C(\Delta x)^2 \sum_{n,j} 5\epsilon \\ &\leq O(1)(\Delta x \epsilon^{-p} + \epsilon). \end{aligned} \tag{38}$$

Here the fact that the total variation of $U_{\Delta x}(x,t)$ in a and z is bounded, and that there are only $(d/\Delta x)^{-1}$ nonzero terms in the summation over n and $(d/r)^{-2}$, nonzero terms in the summation over n and j are used. It now follows from (34), (36), and (38) that

$$D_{\Delta x} \leq O(1)(\Delta x \epsilon^{-p} + \epsilon). \tag{39}$$

We can minimize the right-hand side of (39) by choosing $\epsilon = \Delta x^{1/(p+1)}$, and so conclude that

$$D_{\Delta x} \leq O(1)(\Delta x)^{1/(p+1)}.$$

This completes the proof of Theorem 3, once we give the proof of Proposition 3.

Proof of Proposition 3. We first consider the case $U_{nj}(x,t)$ is entirely contained in $\bar{S}(\epsilon)$ (the complement of $S(\epsilon)$ in $\mathbf{B}$). For this case we prove that

$$\left|\frac{\partial z(a,u)}{\partial u}\right| \geq O(1)\epsilon^p$$

for all $(a,u) \in \bar{S}(\epsilon)$. We recall that the mapping $\Psi : (a,u) \to (a,z)$ is a $1-1$ smooth function defined on $\mathbf{B}$, which is regular except at points on Γ. Moreover, by assumption, the $f = \text{const}$ curves (the integral curves of R_0) describe the convex down function of u in $\mathbf{B}$, which take a maximum value of a on Γ. Thus, let

$$a_f(u) \equiv a(f,u)$$

denote the value of a which corresponds to given values of f and u, so that for fixed f, $a_f(\cdot)$ describes the $f = \text{const}$ curve as parameterized by u. The curve $a_f(\cdot)$ is smooth, convex down, and takes a maximum value of a at the unique value of $u = u^\Gamma \equiv u^\Gamma(a, f)$ where $(a_f(u^\Gamma), u^\Gamma) \in \Gamma$. The assumption that the $f = \text{const}$ and $a = \text{const}$ curves make pth order contact on Γ in $\mathbf{B}$ means that there exists a constant M_1, depending only on the function $f(a, u)$ and the set $\mathbf{B}$, such that

$$a_f^{(i)}(u^\Gamma) = 0, \quad i = 1, \ldots, p, \tag{40}$$

$$\frac{1}{M_1} \leq a_f^{(p+1)}(u^\Gamma) \leq M_1 \tag{41}$$

for all f. Here, $a_f^{(i)}$ denotes the ith derivative of the function a_f with respect to the variable u, and we restrict the values of all variables to those defined on $\mathbf{B}$. In particular, this implies that

$$\frac{\partial^i f}{\partial u^i}(a, u) = 0, \quad i = 1, \ldots, p, \tag{42}$$

$$\frac{1}{M_2} \leq \frac{\partial^{p+1} f}{\partial u^{p+1}}(a, u) \leq M_2 \tag{43}$$

for all $(a, u) \in \mathbf{B}$ since we have made the assumption that $\partial f / \partial a(a, u) \neq 0$ in $\mathbf{B}$. To see this, note that if we partial differentiate the identity

$$a = a(f(a, u), u) \tag{44}$$

with respect to u holding a fixed, we obtain

$$\frac{\partial a}{\partial f}\frac{\partial f}{\partial u} + a_f^{(1)}(u) = 0;$$

and so for $(a, u) \in \Gamma \bigcap \mathbf{B}$ we obtain $(\partial a / \partial f \neq 0)$,

$$\frac{\partial f}{\partial u} = 0. \tag{45}$$

Differentiating (44) twice with respect to u and using (45) yields

$$\frac{\partial^2 f}{\partial u^2} = 0$$

for all $(a, u) \in \Gamma$. Continuing we conclude that

$$\frac{\partial^i f}{\partial u^i} = 0, \quad i = 1, \ldots, p,$$

and

$$\frac{\partial a}{\partial f}\frac{\partial^{p+1} f}{\partial u^{p+1}} + a_f^{(p+1)}(u) = 0$$

for all $(a, u) \in \Gamma \bigcap \mathbf{B}$ from which (42) and (43) follow.

Now the function $z(a,u)$ defined in (15) can be equivalently given by

$$z(a,u) = \begin{cases} a_f(u^\Gamma) - a & \text{if } U \text{ lies to the right of } \Gamma, \\ a - a_f(u^\Gamma) & \text{if } U \text{ lies to the left of } \Gamma. \end{cases} \tag{46}$$

Consider, then, two points (a,u_1) and (a,u_2) in $\mathbf{B}$ which are outside of $S(\epsilon)$. Without loss of generality, assume $u_1 < u_2$ lies to the right of Γ. Let $f_1 \equiv f(a,u_1)$, $f_2 \equiv f(a,u_2)$ and let u_0, u_1^Γ, and u_2^Γ be such that (a,u_0), $(a_{f_1}(u_1^\Gamma),u_1^\Gamma)$, and $(a_{f_2}(u_2^\Gamma),u_2^\Gamma)$ all lie on Γ. It follows from (46), that

$$\begin{aligned} |z(a,u_2^\Gamma) - z(a,u_1^\Gamma)| &= |a_{f_2}(u_2^\Gamma) - a_{f_1}(u_1^\Gamma)| \\ &\geq -|a_{f_2}(u_2^\Gamma) - a_{f_2}(u_1^\Gamma)| + |a_{f_2}(u_1^\Gamma) - a_{f_1}(u_1^\Gamma)|. \end{aligned} \tag{47}$$

But

$$\begin{aligned} |a_{f_2}(u_2^\Gamma) - a_{f_2}(u_1^\Gamma)| &\leq O(1) a_{f_2}^{(p+1)} |u_2^\Gamma - u_1^\Gamma|^{p+1} \\ &\leq O(1)|z(a,u_2) - z(a,u_1)|, \end{aligned} \tag{48}$$

where we use the fact that Γ cuts the lines $u = \text{const}$ transversally at each point. Moreover,

$$\begin{aligned} |a_{f_2}(u_1^\Gamma) - a_{f_1}(u_1^\Gamma)| &= |a(f_2,u_1^\Gamma) - a(f_1,u_1^\Gamma)| \\ &\geq O(1)\left|\frac{\partial a}{\partial f}(f_1,s_1^\Gamma)(f_2 - f_1)\right|. \end{aligned} \tag{49}$$

But

$$f_2 - f_1 = f(a,u_2) - f(a,u_1) = O(1)\frac{\partial f}{\partial u}(a,u_1)(u_2 - u_1) \tag{50}$$

and

$$\begin{aligned} \frac{\partial f}{\partial u}(a,u_1) &= \frac{\partial f}{\partial u}(a,u_1) - \frac{\partial f}{\partial u}(a,u_0) \\ &= O(1)\frac{\partial^{p+1} f}{\partial u^{p+1}}(a,u_0)(u_1 - u_0)^p, \end{aligned} \tag{51}$$

whenever $|u_1 - u_0| < \bar{\epsilon}$ for some $\bar{\epsilon}$ sufficiently small. Now for $|u_1 - u_0| > \bar{\epsilon}$ we know that $|\partial z/\partial u(a,u_1)| > 0$, and thus by compactness we can obtain

$$\left|\frac{\partial z}{\partial u}(a,u_1)\right| \geq \text{Const}(\bar{\epsilon})\epsilon^p.$$

Thus we need only consider the case $|u_1 - u_0| < \bar{\epsilon}$. Then from (49)–(51) we have

$$|a_{f_2}(u_1^\Gamma) - a_{f_1}(u_1^\Gamma)| \geq O(1)\epsilon^p(u_2 - u_1), \tag{52}$$

since $|u_1 - u_0| > \epsilon$. Putting (52) and (48) into (47) gives

$$(1 + O(1))|z(a,u_2) - z(a,u_1)| \geq O(1)\epsilon^p(u_2 - u_1),$$

from which the desired result

$$\left|\frac{\partial z(a,u)}{\partial u}\right| \geq O(1)\epsilon^p \tag{53}$$

follows, the constant $O(1)$ being independent of ϵ. Thus, if the function U_{nj} takes values entirely within $\bar{S}(\epsilon)$, statement (53) implies that

$$\mathrm{Var}_{au}\{U_{nj}\} \leq O(1)\epsilon^{-p}\mathrm{Var}_{az}\{U_{nj}\}. \tag{54}$$

We now consider the case when $U_{nj}(x)$ takes values in both $S(\epsilon)$ and $\bar{S}(\epsilon)$. Since $U_{nj}(x) \equiv U_{\Delta x}(\cdot, t_n)$, $x \in (x_j, x_{j+1}]$, by construction, the values of U_{nj} lie on the connected sequence of elementary waves $\gamma^{n-1}_{j+1/3}, \gamma^{n-1}_{j+2/3}, \gamma^{n-1}_{j+1}$ (see Fig. 3). The three wave curves corresponding to these elementary waves make a continuous curve in the au state space, and, by assumption, this curve must meet the boundary of $S(\epsilon)$ at least once, and at most four times. Note here that we use the assumption that $\mathbf{B}$ is a small enough neighborhood of Γ, so that the 0- and 1-wave curves in $\mathbf{B}$ intersect Γ uniquely and transversally in $\mathbf{B}$, as well as the entropy condition that the 0-waves do not cross Γ. Thus we deduce that

$$\mathrm{Var}_{au}\{U_{nj}|_{S(\epsilon)}\} \leq 5\epsilon, \tag{55}$$

where $U_{nj}|_{S(\epsilon)}$ denotes the restriction of U_{nj} to the values it takes inside $S(\epsilon)$; and by (54) we have

$$\mathrm{Var}_{au}\{U_{nj}|_{\bar{S}(\epsilon)}\} \leq O(1)\epsilon^{-p}\mathrm{Var}_{az}\{U_{nj}\}. \tag{56}$$

Inequalities (55) and (56) give

$$\mathrm{Var}_{au}\{U_{nj}\} \leq O(1)\epsilon^{-p}\mathrm{Var}_{az}\{U_{nj}\} + 5\epsilon. \tag{57}$$

This completes the proof of Proposition 3.

Proposition 3 is a refinement of the inequality obtained by Temple [22]. There the respective inequality is

$$\mathrm{Var}_{au}\{U_{nj}\} \leq O(1)\epsilon^{-(p+1)}\mathrm{Var}_{az}\{U_{nj}\} + O(1)\epsilon. \tag{58}$$

The improvement in inequality (57) over (58) implies a better convergence rate for the approximate solutions $U_{\alpha\Delta x}$ generated by Glimm's method, as analyzed in [22]. In this case we define

$$D(\Delta x, \alpha, \phi) = \int_0^\infty \int_{-\infty}^{+\infty} (U_{\alpha\Delta x}\phi_t + F(U_{\alpha\Delta x})\phi_x)\,dx\,dt + \int_{-\infty}^{+\infty} U_0(x)\phi_0(x)\,dx. \tag{59}$$

Then the analysis in [22] using (57) in place of (58) gives the following result on the rate of convergence of Glimm's method.

THEOREM 4. *Assume that the integral curves of R_0 and R_1 make pth order contact at each point $U \in \Gamma \bigcap \mathbf{B}$ in the au-plane, and assume that U_0 satisfies the assumptions of Theorem 2* ($\mathrm{Var}_{az}(U_0) < \infty$, $U_0(x) \in \mathbf{B}$). *Let $U_{\alpha\Delta x}(x,t)$ be the approximate solution generated from U_0 by Glimm's method for the resonant nonlinear system* (1). *Then the following error estimate holds:*

$$\|D(\Delta x, \cdot, \phi)\|_2^2 \leq O(1)(\Delta x)^{1/(2p+1)}, \tag{60}$$

where $\|D(\Delta x, \cdot, \phi)\|_2^2 = \int_A D(\Delta x, \alpha, \phi)^2 d\alpha$, *and* $\alpha \in A$ *where* A *denotes the measure space product* $A = \prod_{n,j}[0,1]$, *each* $[0,1]$ *equipped with Lebesgue measure* [2].

As indicated above, the rates obtained in Theorems 1 and 2 are correct for the strictly hyperbolic case $p = 0$, thus indicating that they may provide a basis for comparing the convergence rates of the Glimm and Godunov methods in the nonstrictly hyperbolic case (1) as well.

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