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CONVERGENCE OF THE 2×2 GODUNOV METHOD FOR A GENERAL RESONANT NONLINEAR BALANCE LAW*

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Abstract. We introduce a generalized solution of the Riemann problem for a general resonant nonlinear balance law, and we prove the convergence of the 2×2 Godunov numerical method based on these solutions. In particular, we obtain generic conditions that guarantee a canonical structure for the elementary waves in the solution of the Riemann problem, and an interesting multiplicity of time-asymptotic wave patterns is observed and characterized.

Key words. Godunov method, resonance, shock waves, balance law, Reimann problem

AMS subject classification. 35L65

1. Introduction.

We study the initial value problem

$$\begin{aligned} &U_t + \mathcal{F}(U)_x = G, \\ &U(x,0) = U_0(x), \end{aligned} \tag{1}$$

for the system

$$\begin{aligned} &a_t = 0, \\ &u_t + f(a,u)_x = a'g(a,u), \end{aligned} \tag{2}$$

(where $a' \equiv a_x \equiv \frac{da}{dx}$) in a neighborhood of a state $U_* = (a_*, u_*)$ where the nonlinear wave speed $\lambda = f_u$ vanishes. Here, $U = (a,u), \mathcal{F} = (0,f), G = (0,a'g)$, and $a = a(x)$ is an inhomogeneous term that is treated as a variable (so that (2) is a system of two balance laws) to express the dependence of the time-asymptotic wave patterns on the inhomogeneity, and as a first attempt to model systems of this form.

In this paper we define a generalized solution of the Riemann problem for system (2), we isolate generic conditions on the functions f and g that guarantee a unique canonical structure to the elementary waves that appear in the solutions near U_*, and we construct the solution of the Riemann problem in a neighborhood of a state U_* under these generic conditions. Because of the inhomogeneous term in (2), there exists a multiplicity of solutions to the Riemann problem, (cf. [18], [19]), and here we introduce a new admissibilty criterion for Riemann problem solutions of (2). In the final section we prove the convergence of the 2×2 Godunov numerical method [3] based on these admissible solutions of the Riemann problem, and time-independent bounds on solutions as well as convergence (after extraction of a subsequence) is proved when $a(x)$ is an arbitrary function of bounded variation. Because of the appearance of a delta function in (2) when a is discontinuous, there is no classical weak formulation of (2) when a is discontinuous, but in the final section we prove that the limits of the converging approximate solutions generated by the 2×2 Godunov method do converge to weak solutions of (2) when a is Lipschitz continuous.

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Note that one can replace $f(a,u)$ by $f(a(x),u)$ in (2), in which case (2) reduces to a scalar conservation law with x-dependence. Our point of view here is to study (2) as a 2×2 nonstrictly hyperbolic system, and in this paper we characterize the generic structure of the time-asymptotic wave patterns (the solutions of the Riemann problem), and analyze the effects of implementing these elementary wave patterns in the 2×2 Godunov numerical method. The theory of scalar conservation laws does not apply to numerical methods which are based on the elementary waves for (2) when (2) is treated as a resonant nonlinear system. The presence of the coinciding wave speeds allows for oscillations to develop in solutions, and, in particular, this implies that the total variation of a solution $u(x,t)$ at time $t > 0$ can be arbitrarily large relative to both the total variation of the initial data $u(x,0)$ and the total variation of $a(x)$, the latter being a measure of the strength of the sources. This makes time-independent bounds on solutions that much more interesting in the presence of resonance. Moreover, when (2) is treated as a resonant nonlinear system, standing waves near points of coinciding wave speeds approximate rarefaction shocks of the scalar conservation law, and because of this, spurious oscillations can appear in numerical methods based on solutions of the Riemann problem, and entropy estimates from the scalar theory are difficult to obtain. Our interest in the Godunov method stems from the fact that the averaging step in the Godunov method has a regularizing effect in that it tends to *wipe out* the numerical oscillations that appear in the Riemann problem step of the method, and this was proven to be the case in [16] under the restriction $g \equiv 0$. In § 4 of this paper we obtain time-independent bounds on approximate solutions generated by the 2×2 Godunov method for (2), using the generalized solution of the Riemann problem which we establish in § 3. Thus we believe that (2) is of considerable interest because it is a canonical, nontrivial setting in which the phenomenon of oscillations in solutions of resonant nonlinear systems appears and can be analyzed (cf. [12], [14], [17], [20], [23]).

The conditions on the functions f and g that give the generic structure of elementary waves near U_* are that f and g be smooth and satisfy:

$$f_u(U_*) = 0, \tag{3}$$

$$f_a(U_*) \neq 0 \quad \text{(we assume } f_a(U_*) < 0\text{)}, \tag{4}$$

$$g(U_*) - f_a(U_*) \neq 0 \quad \text{(we assume } g(U_*) - f_a(U_*) > 0\text{)}, \tag{5}$$

$$f_{uu}(U_*) \neq 0 \quad \text{(we assume } f_{uu}(U_*) < 0\text{)}, \tag{6}$$

and

$$g_u(U_*) \neq 0. \tag{7}$$

The structure is qualitatively different for $g_u > 0$ and $g_u < 0$, and a complete description of the structure of the elementary waves (shock waves and rarefaction waves), in a neighborhood of U_* is given in both cases. In the case $g \equiv 0$, the system (2) reduces to

$$u_t + f(a,u)_x = 0, \tag{8}$$

and the canonical structure of the waves presented in [10] for system (8) is obtained from the ones given here in the limit $g \to 0$. Note that when $a \equiv$constant, so that $a' \equiv 0$, the solutions of (2) solve the equation (8), which is a scalar conservation law because a is fixed.

The structure of asymptotic wave patterns is understood in terms of the solutions of the Riemann problem, i.e., the initial value problem with initial data given by the jump discontinuity

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0. \end{cases} \tag{9}$$

The presence of the inhomogeneous term $a'g$ in (2) introduces nonuniqueness of solutions of the Riemann problem. This nonuniqueness results from a multiplicity of time-asymptotic wave pattens to which solutions will decay as $t \to \infty$ for given states U_L and U_R at $x = -\infty$ and $x = +\infty$, respectively. Analogous multiplicities are observed in nozzle flow and in shallow water flow down a ramp [1] (see also [2], [4]–[6], [9], [11], [19]). Our procedure is to construct a nonlinear functional F that generalizes the functional F defined in [22]. We use this functional as an entropy condition to pick out a set of admissible solutions of the Riemann problem. This functional assigns an "F-value" to every nonlinear wave, and we choose the admissible solutions of the Riemann problem as the ones that minimize the total variation in a as well as the "F-value" of the waves in the solutions. (Our admissible solutions are in general not unique.) We use this minimization property to show that the sum of the F-values of the waves in a solution at a given time level is monotone decreasing in time in the approximate solutions generated by the 2×2 Godunov method based on the admissible solutions of the Riemann problem given here. In particular, this implies the stability of the scheme (in the total variation norm defined in terms of the singular coordinate system of wave curves), as well as the compactness of the approximate solutions. Since the functional F measures the strength of a wave, our results demonstrate that a finite amount of wave strength is generated in solutions of (2). This helps explain why, as waves interact due to the nonlinearity of wave speeds, we expect solutions to decay to time-asymptotic wave patterns. In particular, we observe that, unlike linear equations that blow up in the presence of resonance, we expect the general resonant nonlinear system with source terms to exhibit decay instead of blowup.

Note that there is no general weak formulation of (2) when a' is discontinuous, and so our solutions of the Riemann problem are actually generalized weak solutions of (2). In § 4 we show that the solutions generated by the 2×2 Godunov method are bounded and converge (modulo extraction of a subsequence) and are weak solutions of (2) in the case when a is Lipschitz continuous. Also, it is interesting to note that the solutions of the Riemann problem in general contain four elementary waves, whereas at most three were observed in [10] when $g = 0$. Equation (6) implies that (8) is genuinely nonlinear for each fixed a in a neighborhood of a_* when u is in a neighborhood of U_*. For definiteness, we assume that

$$f_{uu}(U_*) < 0, \tag{10}$$

to be consistent with [10]. A nonlinear hyperbolic wave is a solution of the Riemann problem for the scalar conservation law (8) in which a is constant.

2. Preliminaries. To begin, we first show that system (2) has standing wave solutions that can be rescaled into discontinuities; thus, these waves can be treated as elementary waves as in the theory of hyperbolic conservation laws.

Let $(a(x), u(x))$ be a standing wave (i.e., time-independent) solution of (2). Then

$$\frac{d}{dx} f = a'g,$$

which is equivalent to

$$f_a da + f_u du = g da.$$

We rewrite this as

$$(f_a - g)da + f_u du = 0. \tag{11}$$

The nondegeneracy assumption (5) implies that $f_a - g \neq 0$ in a neighborhood of U_*, and therefore (11) is equivalent to the autonomous ODE

$$\frac{da}{du} = \frac{f_u}{g - f_a}. \tag{12}$$

This equation has a unique solution through each point in a neighborhood of U_* in the (a, u)-plane. Thus, for any solution $a = a_s(u)$ of (12) and any smooth function $\varphi(x)$, the curve $u = \varphi(x), a = a_s(\varphi(x))$ is a standing wave solution of (2). Moreover, if $a_L = a_s(u_L)$ and $a_R = a_s(u_R)$, then the standing wave discontinuity

$$U(x,t) = \begin{cases} (a_L, u_L) & \text{if } x < 0, \\ (a_R, u_R) & \text{if } x > 0, \end{cases} \tag{13}$$

is obtained as a limit of smooth solutions; specifically, if $\varphi_\epsilon(x) \to \varphi_0(x)$ where

$$\varphi_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

then $U_\epsilon = (a_s(\varphi_\epsilon(x)), \varphi_\epsilon(x)) \to U(x,t)$. We view the standing wave discontinuities defined in (13) as elementary waves (in fact, contact discontinuities) for system (2). In § 3, we construct the general solution of the Riemann problem for (2) (in a neighborhood of U_*) in the class of standing waves and nonlinear hyperbolic waves. In § 4, we prove the convergence of the 2×2 Godunov method based on this generalized solution of the Riemann problem.

The standing wave curves are solutions of (11). Note that for a standing wave,

$$\frac{da}{du} = 0 \text{ if and only if } f_u = 0. \tag{14}$$

Moreover, if $da/du = 0$, then

$$\frac{d^2a}{du^2} = \frac{f_{uu}}{g - f_a} < 0. \tag{15}$$

Thus, $d^2a/du^2 < 0$ in a neighborhood of U_*.

DEFINITION 1. *The* transition curve $\mathcal{T}$ *associated with system* (2) *is the set*

$$\mathcal{T} = \{(a, u) : f_u = 0\}. \tag{16}$$

Since $f_{uu} \neq 0$, the implicit function theorem implies that (in a neighborhood of U_*) $\mathcal{T}$ is a smooth curve passing through U_*, which we denote by

$$u = u_{\mathcal{T}}(a). \tag{17}$$

The curve $\mathcal{T}$ comprises the states near U_* for which the nonlinear wave speed is zero.

By (14) and (15), the standing wave curves $u \mapsto (a_s(u), u)$ (in a neighborhood of U_*) are convex down, cross $\mathcal{T}$ transversally, and maximize a on $\mathcal{T}$. (See Fig. 1.)

We now define the 0-speed shock curve corresponding to a given standing wave curve. By our choice of signs ($f_{uu} < 0$ and $g - f_a > 0$), the entropy shock waves (see [21]) for the scalar conservation laws (8) jump always from left to right in the (x,t)-plane and (a,u)-plane simultaneously; thus, by the Rankine–Hugoniot jump relation for shocks,

$$s[u] = [f],$$

the 0-speed shocks ($s = 0$) cross $\mathcal{T}$ from left to right at a constant value of f. By (6) and (14), the curves $f =$ constant (near U_*) also are convex down in the (a,u)-plane, taking a maximum value of a on $\mathcal{T}$. (See Figs. 1 and 2.) Since $f_a < 0$ and $f_u = 0$ on $\mathcal{T}$, f decreases with increasing a along $\mathcal{T}$.

Now, for a given standing wave $a = a_s(u)$ and a given state (a,u) on the standing wave to the left of $\mathcal{T}$ (i.e., $u < u_{\mathcal{T}}(a)$), define $\bar{u}$ and $\tilde{u}$ such that the states $(a,\bar{u})$ and $(a,\tilde{u})$ lie to the right of $\mathcal{T}$ at the same a-level and lie, respectively, on the same standing wave curve and constant f curve as the state (a,u). That is, $\bar{u}$ satisfies

$$a_s(\bar{u}) = a_s(u), \tag{18}$$

and $\tilde{u}$ satisfies

$$f(a,\tilde{u}) = f(a,u). \tag{19}$$

DEFINITION 2. *Let $a = a_s(u)$ be a standing wave. Then (assuming $f_{uu} < 0$), the 0-speed shock curve corresponding to standing wave a_s is the curve*

$$\{\tilde{u} : f(a,\tilde{u}) = f(a,u) \text{ where } u \le u_{\mathcal{T}}(a) \text{ and } \tilde{u} \ge u_{\mathcal{T}}(a)\}.$$

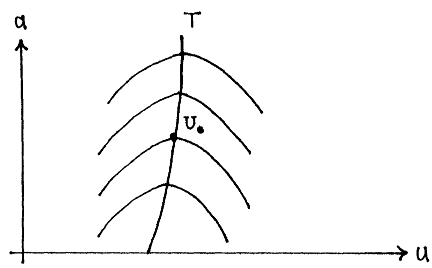

FIG. 1. *The qualitative shape of both the standing wave curves and the curves of constant f.*

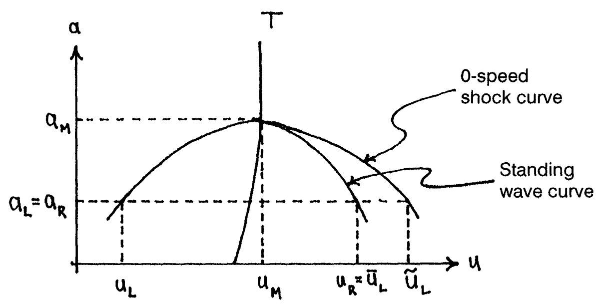

FIG. 2.

(*When* $f_{uu} > 0$, *we change to* $u \geq u_{\mathcal{T}}(a)$ *and* $\tilde{u} \leq u_{\mathcal{T}}(a)$.)

LEMMA 1. *If* $g_u < 0$, *then for each standing wave* $a = a_s(u)$, *the corresponding* 0*-speed shock curve lies to the right of the standing wave curve in the* (a,u)*-plane. That is, if* (a,u) *satisfies* $a = a_s(u)$ *with* $u < u_{\mathcal{T}}(a)$, *then*

$$f(a,\bar{u}) < f(a,\tilde{u}) = f(a,u). \tag{20}$$

If $g_u > 0$, *then the corresponding* 0*-speed shock curve lies to the left of the standing wave curve in the* (a,u)*-plane. That is,*

$$f(a,\bar{u}) > f(a,\tilde{u}) = f(a,u). \tag{21}$$

Proof. Let $a = a_s(u)$ denote a given standing wave, assume that $a_L = a_s(u_L)$ where (a_L, u_L) lies to the left of $\mathcal{T}$, and let $\bar{u}_L$ and $\tilde{u}_L$ satisfy (18) and (19), respectively. Let $a_M = a_s(u_M)$, where (a_M, u_M) is the unique point where the standing wave crosses $\mathcal{T}$. (See Fig. 2.) Also, let $a_L(u)$ and $a_R(u)$ denote the restrictions of $a_s(u)$ to $u_L \leq u \leq u_M$ and $u_M \leq u \leq u_R = \bar{u}_L$, and let $u_L(a)$ and $u_R(a)$ denote their inverses. Then

$$\begin{aligned}
[f] &= f(a_s(u_R),u_R) - f(a_s(u_L),u_L) \\
&= \int_{u_L}^{u_R} \frac{d}{du} f(a_s(u),u)\ du \\
&= \int_{u_L}^{u_R} \frac{da_s}{du} g(a_s(u),u)\ du \\
&= \int_{u_L}^{u_M} \frac{da_s}{du} g(a_s(u),u)\ du + \int_{u_M}^{u_R} \frac{da_s}{du} g(a_s(u),u)\ du \\
&= \int_{a_s(u_L)}^{a_s(u_M)} g(a,u_L(a))\ da + \int_{a_s(u_M)}^{a_s(u_R)} g(a,u_R(a))\ da \\
&= \int_{a_s(u_L)}^{a_s(u_M)} \{g(a,u_L(a)) - g(a,u_R(a))\}\ da,
\end{aligned} \tag{22}$$

where we have used $a_s(u_R) = a_L = a_s(u_L)$. Therefore, if $g_u < 0$, then $g(a,u_L(a)) > g(a,u_R(a))$ for $a \neq a_s(u_M)$, so that by (22) we obtain

$$\begin{aligned}
0 < [f] &= f(a_s(u_R),u_R) - f(a_s(u_L),u_L) \\
&= f(a_L,\bar{u}_L) - f(a_L,u_L) \\
&= f(a_L,\bar{u}_L) - f(a_L,\tilde{u}_L).
\end{aligned}$$

This, together with the assumption $f_a < 0$, implies that the 0-speed shock curve lies above and to the right of the standing wave curve $a_s(u)$ (see Fig. 2). The proof for the case $g_u > 0$ is similar. □

3. The Riemann problem. We now construct the solution of the Riemann problem (2) for arbitrary left state U_L and right state U_R in a neighborhood of U_*. The solution consists of standing wave discontinuities and nonlinear hyperbolic waves, as in the classical theory of conservation laws [13], [21], [8]. The presence of the inhomogeneous term $a'g$ in (2) introduces a nonuniqueness of solutions of the Riemann problem.

In Figs. 3–6 we give the admissible solutions of the Riemann problem. The nonlinear waves satisfy the standard entropy condition for the scalar conservation law (8); cf. [21]. As a further entropy condition, we require that solutions of the Riemann problem have a total variation in a (along the standing wave curves in the solution) no larger than $|a_L - a_R|$. (This ensures that no additional variation in $a(x)$ is introduced into numerical solutions based on these solutions of the Riemann problem.) In the case of an individual standing wave discontinuity, this entropy condition implies that a is monotone in a smooth standing wave approximation of the discontinuity. An equivalent requirement is that standing wave discontinuities do not cross $\mathcal{T}$ [8].

Even with these entropy conditions there is a multiplicity of solutions of the Riemann problem. To obtain the compactness of solutions generated by the 2×2

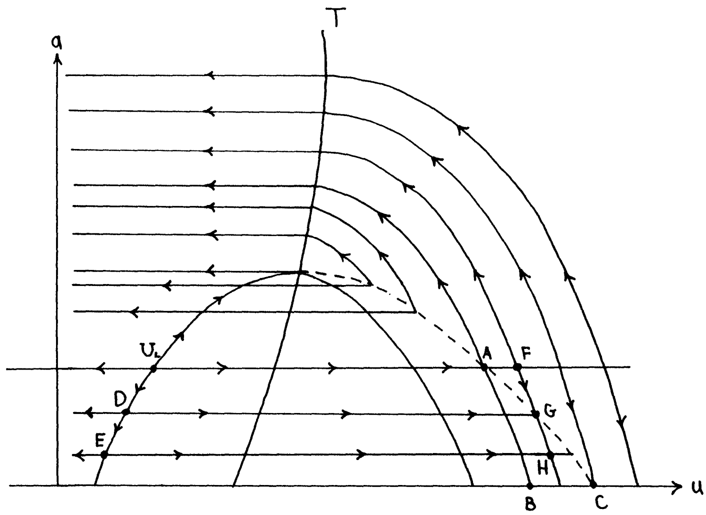

FIG. 3. $g_u < 0, u_L < u_T(a_L)$.

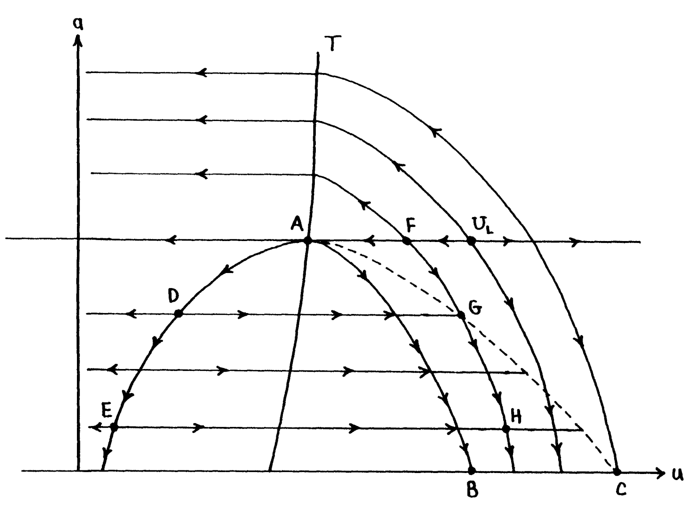

FIG. 4. $g_u < 0, u_L > u_T(a_L)$.

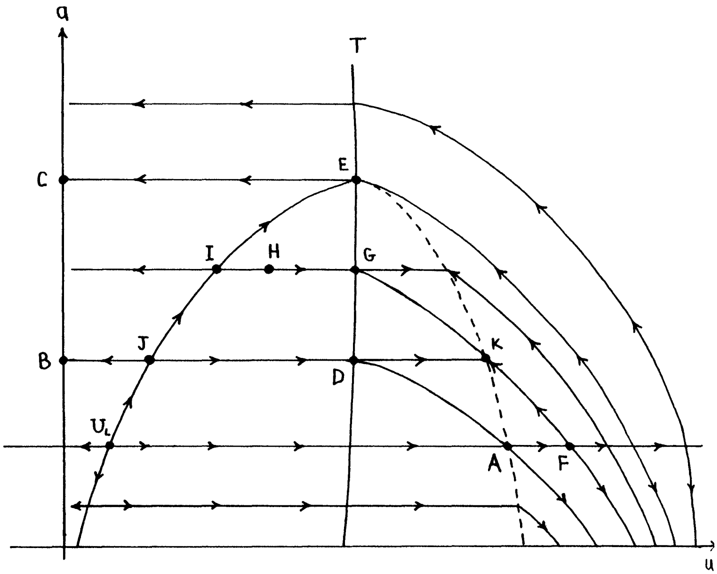

FIG. 5. $g_u > 0, u_L < u_T(a_L)$.

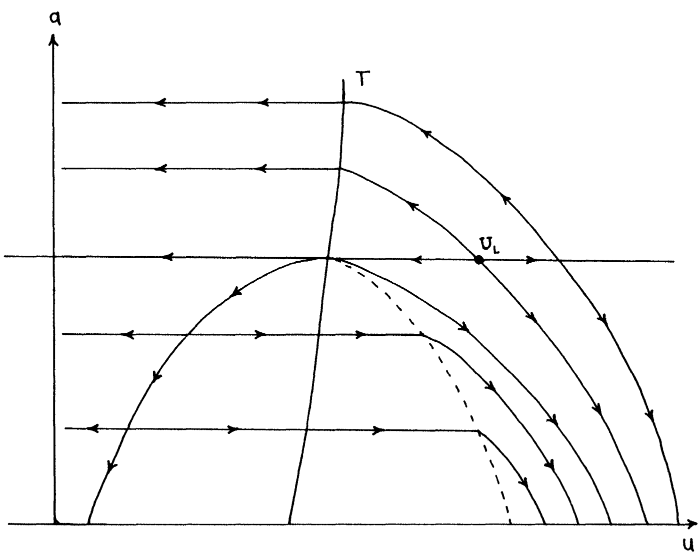

FIG. 6. $g_u > 0, u_L > u_T(a_L)$.

Godunov method (based on these admissible solutions of the Riemann problem) in the next section, we impose as a further condition that the functional F defined below (in analogy with the functional introduced in [22]) be minimized on the waves in the admissible solution of the Riemann problem. This final condition still does not lead to uniqueness. However, any further choice does not affect the compactness of the approximate solutions. We note also that the F-value of admissible solutions depends continuously on U_L and U_R even though the waves change discontinuously in the (x,t)-plane. This reflects an interesting instability in the time asymptotics of solutions of (2).

We define F and the "singular" coordinate z in analogy with the quantities defined in [22]. The coordinate z is based on the singular coordinate system of nonlinear hyperbolic wave curves (a = constant) and standing wave curves as observed in the (a,u)-plane. (The system is singular since the curves are tangent on $\mathcal{T}$.) For each point (a,u), let $(a_{\mathcal{T}}, u_{\mathcal{T}})$ denote the unique point where the standing wave curve through (a,u) crosses $\mathcal{T}$. Now define the singular coordinate z by

$$z(a,u) = \operatorname{sgn}(u - u_{\mathcal{T}})|a - a_{\mathcal{T}}|$$

and the strength $|\gamma|$ of the elementary wave γ by

$$|\gamma| = \begin{cases} |z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a nonlinear wave,} \\ 2|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a standing wave with } u_R < u_L, \\ 4|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a standing wave with } u_R > u_L \end{cases} \tag{23}$$

[22], [18]. For a Riemann problem solution consisting of a sequence of elementary waves $\gamma_1, \ldots, \gamma_n$, define

$$F[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|. \tag{24}$$

The solutions of the Riemann problem that minimize F among all sequences of elementary waves taking U_L to U_R are diagrammed in Figs. 3–6 for the cases $g_u < 0, g_u > 0$, and U_L to the left of $\mathcal{T}$, U_L to the right of $\mathcal{T}$. The cases $g_u < 0$ and $g_u > 0$ are qualitatively different because of the location of the 0-speed shock curve. To read the diagrams, start at U_L and follow the arrows to an arbitrary state U_R. The wave curves traversed then give the elementary waves in the solution of the Riemann problem going from left to right in the (x,t)-plane. It is easy to verify that in the limit as g tends to zero, these diagrams reduce to those for the resonant homogeneous system

$$a_t = 0, \qquad u_t + f(a,u)_x = 0$$

[10]. In this sense, the present analysis generalizes that for $g = 0$.

In Figs. 3–6, the solid convex down curves denote standing wave curves, and the dotted curve to the right of $\mathcal{T}$ denotes the 0-speed shock curve corresponding to the standing wave curve through U_L. In Figs. 3 and 4, the dotted line falls to the right of the standing wave curve through U_L because $g_u < 0$. Similarly, in Figs. 5 and 6, it falls to the left because $g_u > 0$. (The proof that F is minimized on these admissible solutions follows by the argument given in [22] and [7] where the relevant geometry is the same.) We discuss the multiplicity of solutions in Figs. 3–5 below. In Fig. 6, solutions are unique.

In each of Figs. 3–5, there is a region of right states U_R for which there are multiple solutions of the Riemann problem that minimize the total variation in a. In the interior of the region there are always three solutions, while at the boundary the multiplicity reduces to two solutions. We expect that all solutions of the Riemann problem represent possible time-asymptotic states to which solutions of the Cauchy problem can decay. Note also that some solutions are composed of four elementary waves, whereas there are at most three waves when $g = 0$ [10].

Discussion of Fig. 3 [$g_u < 0$; U_L to the left of $\mathcal{T}$]. A multiplicity of solutions occurs when U_R lies within the interior of the region ABC, e.g., $U_R = H$. The three solutions are: $U_L \to F \to H$, $U_L \to D \to G \to H$, and $U_L \to E \to H$. (Here, e.g., $U_L \to F$ denotes the elementary shock wave taking U_L on the left to F on the right. Since F lies to the right of the 0-speed shock curve (the dotted line), and since $f_{uu} < 0$, $U_L \to F$ is a shock wave of negative speed.) All of these solutions have the same F-value.

Discussion of Fig. 4 [$g_u < 0$; U_L to the right of $\mathcal{T}$]. A multiplicity of solutions that minimize the total variation in a (but do not necessarily minimize F) occurs when U_R lies within the interior of the region ABC, e.g., $U_R = H$. The three solutions are: $U_L \to F \to H$, $U_L \to A \to E \to H$, and $U_L \to A \to D \to G \to H$. The F-value is minimized only on the first of these, and thus in this case there is a unique admissible solution except on the boundary where the two solutions have the same F-value.

Discussion of Fig. 5 [$g_u > 0$; U_L to the left of $\mathcal{T}$]. A multiplicity of solutions that minimize the total variation in a occurs when U_R lies within the interior of the region CEADB, e.g., $U_R = H$. The three solutions are: $U_L \to I \to H$, $U_L \to F \to G \to H$, and $U_L \to J \to K \to G \to H$. In this case minimizing the F-value picks out the unique solution $U_L \to I \to H$, except on the boundary where again the two solutions have the same F-value.

Discussion of Fig. 6 [$g_u > 0$; U_L to the right of $\mathcal{T}$]. In this case the solution that minimizes the total variation in a is unique.

The next result follows directly from the diagrams and implies an L^∞ bound on approximate solutions generated by this class of Riemann problems.

PROPOSITION 1. *All admissible Riemann problem solutions lie on the convex side of the outer of the two standing wave curves through U_L and U_R. In particular, the convex side of each standing wave curve is an invariant region for admissible solutions of the Riemann problem.*

4. The generalized Godunov method. Let $U_{\Delta x}(x,t)$ denote an approximate solution of the Cauchy problem (2), (8) generated by the Godunov scheme, for initial data $U_0(x)$ taking values in a neighborhood $\mathbf{B}$ of U_* where the solution of the Riemann problem has been constructed in the previous sections. (We use the notation of [15], and [16].) To construct the approximate solutions, first discretize $R \times [0,\infty)$ by spacial mesh length Δx and time mesh length Δt such that

$$\frac{\Delta x}{\Delta t} = \lambda, \tag{25}$$

where

$$\lambda \equiv 2 \sup_{(a,u)\in\mathbf{B}} \left\{ \left| \frac{\partial f}{\partial u} \right| \right\}. \tag{26}$$

We let $x_n = n\Delta x$, $t_j = j\Delta t$ so that (x_n, t_j) denote the mesh points of the approximate solution. Define

$$S_i = \{(x,t) : t_i \le t < t_{i+1}\}.$$

The approximate solution $U_{\Delta x}$ generated by the Godunov scheme is defined as follows [15]: to initiate the scheme at $n = 0$, define

$$U_j^0 \equiv U_{\Delta x}(x,0) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U_0(\xi)d\xi, \qquad x_j < x < x_{j+1}. \tag{27}$$

Assuming that $U_{\Delta x}(x,t)$ has been constructed for $(x,t) \in \bigcup_{i=0}^{n-1} S_i$, then we define $U_{\Delta x}$ in S_n as the solution of (2) with the initial values

$$U_j^n \equiv U_{\Delta x}(x,t_n) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U_{\Delta x}(\xi,t_n-)d\xi, \qquad x_j < x < x_{j+1}. \tag{28}$$

In other words, at each time t_n, a piecewise constant approximation $U_{\Delta x}(x,t_n)$ is obtained by taking the arithmetic averages of $U_{\Delta x}(x,t_n-)$ at each interval of the mesh, so that the solution in S_n can be constructed by solving the Riemann problems $[U_{j-1}^n, U_j^n]$ posed at each point of discontinuity $(x_j,t_n), j \in Z$. The Courant–Freidrichs–Levy restriction (25) ensures that the Riemann problem solutions in each S_n do not interact before time t_{n+1} [22], [15], [16].

THEOREM 1. *If the neighborhood* $\mathbf{B}$ *containing* U_* *is chosen to be small enough, then the Godunov approximate solutions* $U_{\Delta x}(x,t)$ *are defined for all time. Moreover,*

$$F(J_{n+1}) \le F(J_n), \tag{29}$$

for each $n \ge 0$, *where* J_n *denotes the sequence of elementary waves appearing in the approximate solution* $U_{\Delta x}$ *in the strip* S_n, *and* F *is defined in* (24).

Proof. The supnorm bound on solutions follows from Proposition 1, which asserts the existence of convex invariant regions for Riemann problems in a neighborhood of U_*. For (29), note that the solutions of the Riemann problems used in the construction of the Godunov approximate solutions are admissible solutions of the Riemann problem, and so were selected to minimize the F-value of the elementary waves among all possible solutions of the Riemann problem. Using this, estimate (29) follows by the argument given in Theorem 1 of [15]. □

Theorem 1 leads directly to the following compactness result for approximate solutions generated by the Godunov method.

THEOREM 2. *Assume that the initial data* $U_0(x) \in \mathbf{B}$ *satisfies the condition* $\mathrm{Var}_z\{U_0(\cdot)\} = V_z < \infty$ *and* $\mathrm{Var}\{a(\cdot)\} = V_a < \infty$. *Then* $U_{\Delta x}(x,t) \in \mathbf{B}$ *for all* $x, t \ge 0$, $\mathrm{Var}_z\{U_{\Delta x}(\cdot,t)\} < 4V_z$ *for all* $t \ge 0$, *and a subsequence of* $\{U_{\Delta x}\}$ *converges boundedly, almost everywhere, to a bounded measurable function* $U(x,t)$ *as* Δx *tends to zero.*

Proof. See Theorem 2 of [15].

Thus assume that $U_{\Delta x}(x,t)$ is a sequence of Godunov approximate solutions that converges boundedly, pointwise almost everywhere to a function $U(x,t)$, and satisfies the estimate

$$\mathrm{Var}_z\{U_{\Delta x}(\cdot,t)\} < 4V_z. \tag{30}$$

In the next section we conclude by showing that the limit function $U(x,t)$ is a classical weak solution of (2) when a' has no delta function singularities.

5. Convergence of the residual for the generalized Godunov method. The residual R for system (2) is defined by

$$R(a,u,\varphi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u\varphi_t + f\varphi_x + a'g\varphi\}\ dx\,dt + \int_{-\infty}^{+\infty} u_0(x)\varphi(x,0)\ dx.$$

Then (a,u) is a *weak solution* of (2) if and only if $R(a,u,\varphi) = 0$ for all compactly supported smooth functions $\varphi = \varphi(x,t)$. We prove the following result.

THEOREM 3. *Assume that* $U_{\Delta x}(x,t) = (a_{\Delta x}(x), u_{\Delta x}(x,t)) \to U(x,t) = (a(x), u(x,t))$ *and satisfies* (30) *and the conditions of Theorems* 1 *and* 2. *Assume further that* $a(x)$ *satisfies*

$$|a(x) - a(y)| \leq M|x-y|, \tag{31}$$

$$\int_{-\infty}^{+\infty} |a'(x) - a'_{\Delta x}(x)|\, dx \leq C_1 \Delta x, \tag{32}$$

where $a_{\Delta x}$ *is the piecewise linear interpolant of* a *on the* Δx*-mesh. Then* $R(a,u,\varphi) = \lim_{\Delta x \to 0} R(a_{\Delta x}, u_{\Delta x}, \varphi) = 0$ *for all test functions* φ. *That is,* $U(x,t)$ *is a weak solution of* (1).

We use the following lemma.

LEMMA 2. *Assume that* f, g *are smooth,* u *and* a *are bounded, and* a *satisfies* (31)–(32). *Then for each test function* φ,

$$|R(a,u,\varphi) - R(a_{\Delta x}, u, \varphi)| = O(1)\Delta x,$$

where the constant $O(1)$ *depends only on the bounds for* a, u, *and the test function* φ.

Proof of Lemma 2. First note that

$$|a'(x)| \leq M, \tag{33}$$

$$|a'_{\Delta x}(x)| \leq M, \tag{34}$$

$$|a(x) - a_{\Delta x}(x)| \leq C_0 \Delta x, \tag{35}$$

follow directly from (31) and the definition of $a_{\Delta x}$.

For any test function φ,

$$\begin{aligned}
|R(a,u,\varphi) - R(a_{\Delta x}, u, \varphi)| \leq \int\!\!\int \{ & |f(a,u) - f(a_{\Delta x}, u)||\varphi_x| + |a'(x) g(a,u) \\
& - a'_{\Delta x}(x) g(a_{\Delta x}, u)||\varphi| \}\, dxdt \\
\leq & \int\!\!\int \left|\frac{\partial f}{\partial a}\right| |a - a_{\Delta x}||\varphi_x|\, dxdt \\
& + \int\!\!\int |a' - a'_{\Delta x}||g(a,u)||\varphi|\, dxdt \\
& + \int\!\!\int |a'_{\Delta x}||g(a,u) - g(a_{\Delta x}, u)||\varphi| dxdt \\
= & I + II + III.
\end{aligned}$$

(Here, $\partial f/\partial a$ is evaluated at a point between $(a(x), u(x))$ and $(a_{\Delta x}(x), u(x))$.) The hypotheses imply that

$$|\mathrm{I}| \leq \left\| \frac{\partial f}{\partial a} \right\|_\infty C_0 \Delta x \, \|\varphi_x\|_\infty |\mathrm{Supp}\{\varphi\}| = O(\Delta x),$$

$$|\mathrm{II}| \leq C_1 \Delta x \|g\|_\infty \|\varphi\|_\infty |\mathrm{Supp}_t\{\varphi\}| = O(\Delta x),$$

$$|\mathrm{III}| \leq M \left\| \frac{\partial g}{\partial a} \right\|_\infty C_0 \Delta x \, \|\varphi\|_\infty |\mathrm{Supp}\{\varphi\}| = O(\Delta x).$$

Thus, the lemma is proven. $\square$

We show that $R(a, u_{\Delta x}, \varphi) \to 0$ as $\Delta x \to 0$, where $u_{\Delta x}$ is the approximate solution generated by the 2×2 Godunov method. By Lemma 2, it suffices to show that $R(a_{\Delta x}, u_{\Delta x}, \varphi) \to 0$.

Let $\mathcal{R}_{jn}$ denote the rectangle $x_j \le x \le x_{j+1}$, $t_n \le t \le t_{n+1}$; let

$$R_{jn} = \iint_{\mathcal{R}_{jn}} \{u_{\Delta x}\varphi_t + f(a_{\Delta x}, u_{\Delta x})\varphi_x + a'_{\Delta x}(x) g(a_{\Delta x}, u_{\Delta x})\varphi\}\ dx\, dt;$$

and set

$$\bar{R}_{jn} = \iint_{\mathcal{R}_{jn}} \{u_{\Delta x}\varphi_t + f(a_j, u_{\Delta x})\varphi_x + a'_{\Delta x}(x) g(a_j, u_{\Delta x})\varphi\}\ dx\, dt.$$

Then, integrating by parts, we obtain

$$\begin{aligned}
\bar{R}_{jn} = &\left\{\int_{x_j}^{x_{j+1}} u_{\Delta x}(x, t_{n+1}-)\varphi(x, t_{n+1})\ dx \right.\\
&\left. - \int_{x_j}^{x_{j+1}} u_{\Delta x}(x, t_n+)\varphi(x, t_n)\ dx\right\}\\
&+ \left\{\int_{t_n}^{t_{n+1}} f(a_j, u_{\Delta x}(x_{j+1}-, t))\varphi(x_{j+1}, t) dt\right.\\
&\left. - \int_{t_n}^{t_{n+1}} f(a_j, u_{\Delta x}(x_j+, t))\varphi(x_j, t)\ dt\right\}\\
&+ \left\{\int_{t_n}^{t_{n+1}} \int_{x_j}^{x_{j+1}} a'_{\Delta x}(x) g(a_j, u_{\Delta x}(x, t))\varphi(x, t) dx dt\right\}\\
= &\ \mathrm{I}_{j,n} + \mathrm{II}_{j,n} + \mathrm{III}_{j,n}.
\end{aligned}$$

Then

$$\sum_{j,n\ge 0} \mathrm{I}_{j,n} + \int_{-\infty}^{+\infty} u_0(x)\varphi(x,0) dx = - \sum_{j,n\ge 1} \int_{x_j}^{x_{j+1}} [u]_n(x)\varphi(x, t_n) dx,$$

where

$$[u]_n(x) = u_{\Delta x}(x, t_n+) - u_{\Delta x}(x, t_n-),$$

and we take $u_{\Delta x}(x, 0-) \equiv u_0(x)$. Similarly,

$$\sum_{j,n} \mathrm{II}_{j,n} = - \sum_{j,n} \int_{t_n}^{t_{n+1}} [f]_j(t)\varphi(x_j, t)\ dt,$$

where

$$[f]_j(t) = f(a_j, u_{\Delta x}(x_j+, t)) - f(a_{j-1}, u_{\Delta x}(x_j-, t)).$$

Then

$$\begin{aligned}
&\int_{x_j}^{x_{j+1}} [u]_n(x)\varphi(x, t_n)\ dx \\
&\quad = \int_{x_j}^{x_{j+1}} [u]_n(x)\varphi(x_j, t_n) dx + O(1)(\Delta x)^2 \mathrm{Var}_{x_j < x < x_{j+1}} u_{\Delta x}(\cdot, t_n+),
\end{aligned} \tag{36}$$

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} [f]_j(t)\varphi(x_j,t)\;dt \\
&\qquad = \int_{t_n}^{t_{n+1}} [f]_j(t)\varphi(x_j,t_n)\;dt + O(1)(\Delta x)^2 \mathrm{Var}_{x_{j-1}<x<x_{j+1}} u_{\Delta x}(\cdot, t_n+),
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} a'_{\Delta x}(x) g(a_j, u_{\Delta x}(x,t))\varphi(x,t)\;dx\,dt \\
&\qquad = \int_{t_n}^{t_{n+1}}\int_{x_j}^{x_{j+1}} a'_{\Delta x}(x) g(a_j, u_{\Delta x}(x,t))\varphi(x_j,t_n)\;dx\,dt + \|\varphi\|_{C^1} M \|g\|_\infty (\Delta x)^3.
\end{aligned}
$$

(We used that $\sup_{x_j<x<x_{j+1}} |[u]_n| \le \mathrm{Var}_{x_j<x<x_{j+1}} u(\cdot, t_n+)$; cf. [22].)

Suppose now that there is a single standing wave at (x_j, t_n). (The argument when there are two standing waves separated by a 0-speed shock is similar.) Denote the standing wave that connects $(a_{j-1}, u_{\Delta x}(x_j-, t_n+))$ on the left to $(a_j, u_{\Delta x}(x_j+, t_n+))$ on the right by $(a_j(x), u_{jn}(x))$. Since a standing wave can be given any parameterization and the standing wave at (x_j, t_n) takes a_{j-1} to a_j, and since $a_{\Delta x}$ is monotone from x_{j-1} to x_j, we can take $a_j(x) = a_{\Delta x}(x)$. Then

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} [f]_j(t)\;dt &= \int_{t_n}^{t_{n+1}}\int_{x_{j-1}}^{x_j} \frac{d}{dx} f(a_{\Delta x}(x), u_{jn}(x))\;dx\,dt \\
&= \int_{t_n}^{t_{n+1}}\int_{x_{j-1}}^{x_j} a'_{\Delta x}(x) g(a_{\Delta x}(x), u_{jn}(x))\;dx\,dt\;.
\end{aligned} \tag{38}
$$

(Note that the presence of a 0-speed shock wave at (x_j, t_n) does not affect this formula since $[f] = 0$ across such a wave. In this case, the "spread out" standing wave $(a_{\Delta x}, u_{\Delta x})$ would consist of two standing waves, separated by a 0-speed shock wave, and the concatenation again connects $(a_{j-1}, u_{\Delta x}(x_j-, t_n+))$ on the left to $(a_j, u_{\Delta x}(x_j+, t_n+))$ on the right.) Thus,

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}} [f]_j(t)\varphi(x_{j-1}, t_n)\;dt \\
&\quad + \int_{t_n}^{t_{n+1}}\int_{x_{j-1}}^{x_j} a'_{\Delta x}(x) g(a_{j-1}, u_{\Delta x}(x,t))\varphi(x_{j-1}, t_n)\,dx\,dt \\
&\qquad = -\varphi(x_{j-1}, t_n)\int_{t_n}^{t_{n+1}}\int_{x_{j-1}}^{x_j} \{a'_{\Delta x}(x) g(a_{\Delta x}(x), u_{jn}(x)) \\
&\qquad\qquad\qquad - a'_{\Delta x}(x) g(a_{j-1}, u_{\Delta x}(x,t))\}\;dx\,dt \\
&\qquad \le O(1)\Delta x^2 \mathrm{Var}_{x_{j-1}\le x\le x_{j+1}} u_{\Delta x}(\cdot, t_n+) + O(1)\Delta x^3.
\end{aligned} \tag{39}
$$

Therefore,

$$
\begin{aligned}
R(a_{\Delta x}, u_{\Delta x}, \varphi) &= \sum_{j,n} R_{j,n}(a_{\Delta x}, u_{\Delta x}, \varphi) + \int_{-\infty}^{+\infty} u_0(x)\varphi(x,0)\, dx \\
&= \left\{ \sum_{j,n} \int_{x_j}^{x_{j+1}} [u]_n(x)\varphi\, dx \right\} \\
&\quad - \left\{ \sum_{j,n} \int_{t_n}^{t_{n+1}} [f]_j(t)\varphi\, dt \right\} \\
&\quad + \left\{ \sum_{j,n} \int_{t_n}^{t_{n+1}} \int_{x_j}^{x_{j+1}} a'_{\Delta x} g(a_j, u_{\Delta x})\varphi\, dx\, dt \right\} + O(1)\Delta x \\
&= \mathrm{I} - \mathrm{II} + \mathrm{III} + O(1)\Delta x.
\end{aligned} \tag{40}
$$

But by the definition of the Godunov method,

$$\int_{x_j}^{x_{j+1}} [u]_n(x)\, dx = 0,$$

and thus by (36),

$$\mathrm{I} = \sum_{j,n} O(1)(\Delta x)^2 \mathrm{Var}_{x_j \le x \le x_{j+1}} u_{\Delta x}(\cdot, t_n+).$$

Also, we have already shown that

$$\mathrm{II} + \mathrm{III} = \sum_{j,n} O(1)(\Delta x)^2 \mathrm{Var}_{x_j \le x \le x_{j+1}} u_{\Delta x}(\cdot, t_n+),$$

and thus we have

$$R(a_{\Delta x}, u_{\Delta x}, \varphi) = O(1)\Delta x^2 \sum_{j,n} \mathrm{Var}_{x_j \le x \le x_{j+1}} u_{\Delta x}(\cdot, t_n+) + O(1)\Delta x\ , \tag{41}$$

where the sum is over those (j, n) with (x_j, t_n) in the support of φ.

The estimate (41), together with $\mathrm{Var}_z u_{\Delta x}(\cdot, t) \le K$, is all that is required to apply the argument in Theorem 3 of [15] (in the case $p = 2$) and conclude that

$$\Delta x^2 \sum_{j,n} \mathrm{Var}_z u_{\Delta x}(\cdot, t_n+) = O((\Delta x)^{1/3}).$$

This gives a rate of convergence on the residual and completes the proof of the theorem. □

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