The Existence of a Global Weak Solution to the Non-Linear Waterhammer Problem

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1. Introduction

Fluid flow in pipelines is usually modeled by the quasilinear hyperbolic system

$$\rho_t + G_x = 0, \qquad (x,t) \in (0,1) \times (0,\infty),$$

(1.1)

$$G_t + (G^2/\rho)_x + p(\rho)_x = -f|G|G/2D\rho,$$

where ρ is mass density, G is momentum density, $p = p(\rho)$ is pressure, $f = f(|G|)$ is the Moody friction factor, and D is pipe diameter, cf. [12]. In this paper, we construct global weak solutions to (1.1) satisfying given initial conditions

(1.2) $$\rho(x,0) = \rho_0(x), \qquad G(x,0) = G_0(x), \qquad x \in [0,1],$$

and given boundary conditions

(1.3a) $$\rho(0,t) = \rho_B(t), \qquad t \in (0,\infty),$$

(1.3b) $$G(1,t) = 0.$$

This poses the classical "waterhammer problem" since the waterhammer phenomena in hydraulics can be created by a sudden valve closure downstream (modeled by the boundary condition $G \equiv 0$) or by a rapid change in the pressure upstream (modeled by a discontinuity in ρ_B). These events create pressure waves which are reflected at the boundaries.

The term $-f|G|G/2D\rho$ accounts for the momentum loss due to viscous friction between the fluid and the pipe wall. Since the flow changes from laminar to turbulent at a flow rate near $G_c = 2000\mu/D$ (where μ is the dynamic viscosity), the properties of f also change at $G = G_c$. In the laminar region

(1.4) $$f(|G|) = 64\mu/|G|D, \qquad |G| < G_c,$$

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but the friction factor is determined experimentally for turbulent flow ($|G| > G_c$) and depends on the pipe roughness (which we assume to be constant in space and time) as well as on the flow rate. In particular, it can be observed from experimental data that there exists a constant $f_1 > 0$ such that

$$\lim_{|G|\to\infty} f(|G|) = f_1. \tag{1.5}$$

Thus, the friction term $f|G|G/2D\rho$ is nearly quadratic in G for turbulent flow. (See [12] for a discussion of the theoretical and experimental basis for the Moody friction factor.) Our analysis will assume only the following properties for $\mathcal{K}(G) = f|G|G/2D$:

$$\mathcal{K}(0) = 0, \tag{1.6}$$

$$\mathcal{K}_G \geqq \frac{\mathcal{K}}{G} \geqq 0, \tag{1.7}$$

$$\mathcal{K} \text{ is locally Lipschitz continuous}. \tag{1.8}$$

Property (1.6) states that there should be no friction when there is no flow. Property (1.7) states that the relative change in the friction (assuming that ρ is fixed) is greater than the relative change in the flow rate. This is obviously valid in the laminar regime ($\mathcal{K}(G) = 64G/\mu D$) and in the completely turbulent regime ($\mathcal{K}(G) = f_1|G|G$). Our study of the Moody diagram (cf. [12], p. 297) has led us to assume its validity in general. We note that (1.7) is equivalent to the condition $(fG)_G \geqq 0$. Property (1.8) is justified for all flow rates, G, except possibly at the transition flow rate $|G| = G_c$. (See [7] where f is allowed to be multi-valued at $|G| = G_c$.)

We also assume that the sound speed, $c > 0$, is constant, i.e.,

$$p'(\rho) = c^2. \tag{1.9}$$

This is valid for an ideal gas which is maintained at a constant temperature by heat exchange between the gas, the pipe wall, and the surrounding environment. For many physical problems property (1.9) is also a good approximation for modelling the flow of liquids. We believe that the perturbation techniques used in [13] can be utilized to obtain results for more general equations of state, but it is our view that the additional technical complications would obscure the ideas presented here.

In [7], Luskin has shown for the initial value problem (1.1)–(1.2) that a unique, global smooth solution exists if the initial data are in an appropriate invariant region and if the first derivatives of the initial data are sufficiently small. However, if the first derivatives of the initial data are too large, then discontinuities can be shown to occur even when the data is smooth. (This can be done using a variant of Lax's ideas for the frictionless case, cf. [4].) To allow for more general data here, we need to consider weak solutions of (1.1). We call $\rho, G \in L^\infty(\Omega)$ a weak solution of (1.1) if

$$\iint_\Omega [\rho\phi_t + G\phi_x]\,dx\,dt = 0,$$

$$\iint_\Omega \left[G\phi_t + \left(\frac{G^2}{\rho} + p(\rho) \right)\phi_x - (f|G|G/2D\rho)\phi \right] dx\,dt = 0, \tag{1.10}$$

for all $\phi \in C_0^\infty(\Omega)$, where $\Omega = (0,1) \times (0,\infty)$.

The principal result of this paper is the following theorem.

THEOREM 1. *Assume that properties* (1.6)–(1.9) *hold and that*

$$\operatorname*{Var}_{t\geqq 0} \ln \rho_B + \operatorname*{Var}_{x\in[0,1]} \ln \rho_0 + \operatorname*{Var}_{x\in[0,1]} \frac{G_0}{c\rho_0} < \ln \frac{3+\sqrt{5}}{2} \simeq .96. \tag{1.11}$$

Then there exists a weak solution $\rho, G \in L^\infty(\Omega)$ *to* (1.1). *The initial values are satisfied in the sense that*

$$\rho(\cdot,t), G(\cdot,t) \in \mathrm{Lip}\big([0,\infty), L^1(0,1)\big) \tag{1.12}$$

and

$$\lim_{t\to 0} \rho(\cdot,t) = \rho_0, \qquad \lim_{t\to 0} G(\cdot,t) = G_0 \quad in \quad L^1(0,1).$$

The boundary values are satisfied in the sense that, for any $T > 0$,

$$\rho(x,\cdot), G(x,\cdot) \in \mathrm{Lip}\big([0,1], L^1(0,T)\big), \tag{1.13}$$

and

$$\lim_{x\to 0} \rho(x,\cdot) = \rho_B, \qquad \lim_{x\to 1} G(x,\cdot) = 0 \quad in \quad L^1(0,T).$$

(Here, e.g., $\rho(\cdot,t) \in \mathrm{Lip}([0,\infty], L^1(0,1))$ means that there exists a constant, C, such that

$$|\rho(\cdot,t_1) - \rho(\cdot,t_2)|_{L^1(0,1)} \leqq C|t_1 - t_2|,$$

for all $t_1, t_2 \in [0,\infty)$.) Without loss of generality we assume that $\rho_0(0) = \rho_B(0)$ and that $G_0(1) = 0$ by redefining $\rho_0(0)$ and $G_0(1)$ if necessary. In this way the incompatibility of initial and boundary data is accounted for in the left-hand side of (1.11) by allowing

$$\lim_{x\to 0} \rho_0(x) \neq \rho_0(0) \quad \text{and} \quad \lim_{x\to 1} G_0(x) \neq 0.$$

The only purpose of (1.11) is to guarantee *a priori* that the flow remains subsonic; i.e.,

$$|v| < c \quad \text{for} \quad (x,t) \in \Omega, \tag{1.14}$$

where $v = G/\rho$ is the velocity of the flow. Note that (1.11) is independent of units, and is a satisfactory bound for subsonic flow since $|v| < c$ is sustained by a bound of as much as $.96c$ on the variation in v and $c \ln \rho$ of the data. The *a priori* bound (1.14) is required to guarantee that the boundary conditions (1.3) can be imposed. In general, boundary value problems for (1.1) in which either the density or the flow rate is assigned at each boundary can be solved uniquely only when the characteristic speeds λ_1, λ_2 satisfy $\lambda_1 < 0$, $\lambda_2 > 0$. Our problem is posed in Eulerian coordinates where the characteristic speeds are $\lambda_1 = v - c$, $\lambda_2 = v + c$; so (1.14) is required for $\lambda_1 < 0$, $\lambda_2 > 0$. Earlier work on the construction of solutions to initial boundary value problems has been done by Nishida and Smoller [11] and Liu [5] for the "piston problem", but *a priori* bounds similar to (1.14) were not required there. This is because the piston problem is posed in Lagrangian coordinates where the boundaries move with the fluid. Thus, for the piston problem, $\lambda_1 = -c$, $\lambda_2 = c$; hence $\lambda_1 < 0$, $\lambda_2 > 0$ is already guaranteed *a priori*.

Note also that boundary condition (1.3b) is a "natural" or "flux" boundary condition and could have been imposed weakly by requiring that

$$\iint_\Omega [\rho\phi_t + G\phi_x]\, dx\, dt = 0$$

for all $\phi \in C_0^\infty((0,1] \times (0,\infty))$. However, the boundary condition (1.3a) is not a natural boundary condition and it is necessary for us to give new results on the regularity of the solution at the boundary in order to make sense of boundary condition (1.3a). This problem, as well, did not arise in [5] or [11] since the boundary conditions for the piston problem are "natural" boundary conditions and could be imposed weakly.

Our method is to obtain a solution to (1.1)–(1.3) as the limit of approximate solutions which are constructed by a fractional step procedure. In the first part of each step we use Glimm's method, cf. [1], to approximate the solution of the system of conservation laws for frictionless flow. The second part of each step accounts for the effect of friction on the flow, and involves solving an ordinary differential equation that is determined by the zero order term. Liu [6] and Ying and Wang [14] have also proven the existence of global solutions for some systems of convervation laws with zero order terms; and the latter used a fractional step method similar to ours. However, their analyses only took account of the magnitude and not the orientation of the vector field given by the zero order terms. These methods are inadequate for our purposes because the physical friction term $f|G|G/2D\rho$ is quadratic in G at infinity; and solutions to systems of conservation laws with quadratic zero order terms will "blow up" in finite time if the associated vector field is allowed to have an arbitrary orientation. Moreover, the methods in [6] and [14] will not imply the *a priori* bound (1.14) unless the orientation of the vector field is considered. Thus, because our friction term is quadratic in G and because our boundary value problem requires an *a priori* bound for the solution, it is crucial that we have found a nonlinear functional which is equivalent to the variation norm and which is nonincreasing on both of the fractional steps. Although the functional introduced by Nishida [9] is nonincreasing for the system of conservation laws, it is inadequate for our purposes since it can increase on the friction step. However, we show that if the zero order term satisfies certain monotonicity conditions, then the functional given by Liu [5] is nonincreasing for both fractional steps. These monotonicity conditions are satisfied by the physical friction term in (1.1) when the flow is subsonic. The fractional step procedure that we have analyzed has recently been developed by Marchesin and Paes-Leme [8] to obtain numerical results for system (1.1).

In Section 2, we discuss the solution of Riemann problems. The fractional step scheme is defined in Section 3 and the basic stability result is given there. In Section 4, the structure of Riemann problem solutions is further analyzed to obtain bounds on the "non-linear interaction"; these bounds are used in Section 5 to prove regularity results for the approximate solutions. Section 6 contains the analysis of the convergence of the approximate solutions.

2. Solution of the Riemann Problem

The Riemann problem is the initial value problem for data which is constant to the left and right of $x = 0$. We study the Riemann problem for the non-linear hyperbolic system

$$u_t + F(u)_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.1}$$

where $u = (\rho, G)^{tr}$, $F(u) = (G, G^2/\rho + p(\rho))^{tr}$, and $p'(\rho) = c^2$. The eigenvalues of dF are

$$\lambda_1(u) = v - c, \qquad \lambda_2(u) = v + c, \tag{2.2}$$

with corresponding right eigenvectors

$$\mathcal{R}_1(u) = (1, v - c)^{tr}, \qquad \mathcal{R}_2(u) = (1, v + c)^{tr}. \tag{2.3}$$

The main existence result is that, for initial data

$$u(x,0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

(we always assume $\rho_L, \rho_R > 0$) there exists a unique solution $u(x,t) = u(x/t)$ such that $u(x/t)$ consists of constant states separated by "shock wave" and "rarefaction wave" solutions, cf. [9].

We first discuss the rarefaction wave solutions. We note that a smooth solution $u(\xi)$, $\xi = x/t$, must satisfy

$$[dF - \xi I]\dot{u}(\xi) = 0.$$

Hence, a smooth solution $u(\xi)$ must satisfy $\dot{u}(\xi) \in \operatorname{span}\{\mathfrak{R}_l(u(\xi))\}$ and $\xi = \lambda_l(u(\xi))$ for $l = 1$ or $l = 2$. An l-rarefaction wave is a continuous solution $u(x,t)$ whose values lie on an integral curve of the eigenvector $\mathfrak{R}_l$. The functions

$$s = \tfrac{1}{2}(v + c\ln\rho), \qquad r = \tfrac{1}{2}(v - c\ln\rho) \tag{2.4}$$

are Riemann invariants; i.e.,

$$\nabla_u s \cdot \mathfrak{R}_1 = 0, \qquad \nabla_u r \cdot \mathfrak{R}_2 = 0,$$

and s (respectively r) is constant on an integral curve of $\mathfrak{R}_1$ (respectively $\mathfrak{R}_2$). Thus, the l-rarefaction curves can be defined by

$$\begin{aligned} R_1(u_L) &= \{u_R \mid r(u_R) \geqq r(u_L),\ s(u_R) = s(u_L)\}, \\ &= \{u_R \mid v_L - v_R = -cz \text{ for } z = \ln\rho_L - \ln\rho_R \geqq 0\}, \end{aligned} \tag{2.5a}$$

$$\begin{aligned} R_2(u_L) &= \{u_R \mid r(u_R) = r(u_L),\ s(u_R) \geqq s(u_L)\} \\ &= \{u_R \mid v_L - v_R = -cz \text{ for } z = \ln\rho_R - \ln\rho_L \geqq 0\}. \end{aligned} \tag{2.5b}$$

It is important to note that

$$\Psi : (\rho, G) \to (r, s)$$

is a $1-1$ regular map of $\mathbb{R}^+ \times \mathbb{R}$ onto $\mathbb{R} \times \mathbb{R}$.

A 1-shock wave (respectively 2-shock wave) of speed σ is a weak solution

$$u(x,t) = \begin{cases} u_L & \text{if } x/t < \sigma, \\ u_R & \text{if } x/t > \sigma, \end{cases} \tag{2.6}$$

which satisfies the Lax entropy condition, cf. [3],

(2.7a) $$\lambda_1(u_L) > \sigma > \lambda_1(u_R)$$

respectively,

(2.7b) $$\lambda_2(u_L) > \sigma > \lambda_2(u_R).$$

Since u is a weak solution, it must satisfy the Rankine–Hugoniot jump condition

(2.8) $$\sigma[u_L - u_R] = F(u_L) - F(u_R).$$

By eliminating σ in (2.8) and applying the Lax entropy condition we obtain the following l-shock wave curves:

(2.9a) $$S_1(u_L) = \{u_R \mid v_L - v_R = c(e^{-z/2} - e^{z/2}) \text{ for } z = \ln\rho_L - \ln\rho_R \leqq 0\},$$

(2.9b) $$S_2(u_L) = \{u_R \mid v_L - v_R = c(e^{-z/2} - e^{z/2}) \text{ for } z = \ln\rho_R - \ln\rho_L \leqq 0\}.$$

Substituting in (2.8) we obtain

(2.10) $$\sigma = v_L - ce^{-z/2} = v_R - ce^{z/2}, \qquad z = \ln\rho_L - \ln\rho_R \leqq 0,$$

for a 1-shock and

$$\sigma = v_L + ce^{-z/2} = v_R + ce^{z/2}, \qquad z = \ln\rho_R - \ln\rho_L \leqq 0,$$

for a 2-shock.

It now easily follows from (2.5) and (2.9) that $z = \ln\rho_L - \ln\rho_R$ (respectively $z = \ln\rho_R - \ln\rho_L$) is a regular parametrization of the C^2 curve $T_1(u_L) = R_1(u_L) \cup S_1(u_L)$ (respectively $T_2(u_L) = R_2(u_L) \cup S_2(u_L)$). We call z the "signed strength" of a given wave (so that the signed strength of a rarefaction wave is positive and the signed strength of a shock wave is negative), and we call $|z|$ the strength of a wave. The existence theorem for Riemann problems follows directly from the fact that, given any two states u_L and u_R, there exists a unique state u_M such that $u_M \in T_1(u_L)$ and $u_R \in T_2(u_M)$, cf. [9]; i.e., the Riemann problem for (2.1) can always be solved uniquely by a 1-wave that connects u_L to u_M and a 2-wave that connects u_M to u_R.

It is useful to view the rarefaction and shock curves in the r,s-plane. Since r (respectively s) is a Riemann invariant, the 1-rarefaction curves (respectively 2-rarefaction curves) are parallel to the r-axis (respectively s-axis). Also, the

1-shock curve (2.9a) is easily shown to be

$$r_L - r_R = \tfrac{1}{2}c(e^{-z/2} - e^{z/2} - z), \qquad z = \ln\rho_L - \ln\rho_R \leqq 0,$$

(2.11a)

$$s_L - s_R = \tfrac{1}{2}c(e^{-z/2} - e^{z/2} + z)$$

and the 2-shock curve (2.9b) is easily shown to be

$$r_L - r_R = \tfrac{1}{2}c(e^{-z/2} - e^{z/2} + z), \qquad z = \ln\rho_R - \ln\rho_L \leqq 0,$$

(2.11b)

$$s_L - s_R = \tfrac{1}{2}c(e^{-z/2} - e^{z/2} - z).$$

From (2.11) we can obtain the following theorem concerning the geometry of shock curves in the r, s-plane.

THEOREM 2.1. (See [9].) *All 1-shock curves of equal strength are translates of one another. The 2-shock curve $S_2(u_L)$ is the reflection of the 1-shock curve $S_1(u_L)$ with respect to the line $\ln\rho = \ln\rho_L$. Moreover, if $u_R = u_R(z)$, $z \leqq 0$, denotes the parameterization of $S_1(u_L)$ given above, then*

$$0 < \frac{d}{dz}s(u_R(z)) < \frac{d}{dz}r(u_R(z))$$

and

$$\frac{d}{dz}s(u_R(z)) \Big/ \frac{d}{dz}r(u_R(z))$$

is monotone decreasing from $1-$ at $-\infty$ to $0+$ at 0.

Finally, we shall need to construct the solutions to certain initial boundary value problems. When the boundary is $x = 0$, we consider the problem

(2.12)

$$u_t + F(u)_x = 0, \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

$$u(x,0) = u_0(x) = u_R, \qquad x \in \mathbb{R}^+,$$

$$\rho(0,t) = \rho_L, \qquad t \in \mathbb{R}^+.$$

It can be checked that there exists G_L such that $u_R \in T_2(u_L)$. But the 2-wave connecting u_L to u_R will take the value u_L at $x = 0$ only if it has positive speed. If $u_R \in R_2(u_L)$, then one must have $\lambda_2(u_L) = v_L + c > 0$ to guarantee that the 2-wave has positive speed. If $u_R \in S_2(u_L)$, then the corresponding 2-wave has positive speed if $v_R > -c$ since

$$\sigma > \lambda_2(u_R) = v_R + c > 0.$$

When the boundary is $x = 1$, we consider the problem

$$\text{(2.13)} \qquad \begin{aligned} &u_t + F(u)_x = 0, && (x,t) \in (-\infty, 1) \times \mathbb{R}^+, \\ &u(x,0) = u_L, && x \in (-\infty, 1), \\ &G(0,t) = 0, && t \in \mathbb{R}^+. \end{aligned}$$

In this case there exists ρ_R such that $u_R = (\rho_R, 0)^{tr} \in T_1(u_L)$, but the 1-wave connecting u_L to u_R will take the value u_R at $x = 1$ only if it has negative speed. This is true if $v_L < c$. Thus, the initial boundary value problems (2.12) and (2.13) can be solved by simple waves if all the velocities occurring in the solution are subsonic.

3. Definition and Stability of the Fractional Step Scheme

In this section, we construct approximate solutions to (1.1)–(1.3). Let $h = 1/N$, N a positive integer, $x_i = ih$, $\mathcal{I}_i = [x_{i-1}, x_i]$, $\mathcal{I} = [0,1]$, and let $k > 0$, $t_j = jk$, $\mathcal{J}_j = [t_{j-1}, t_j]$. Also, let $\alpha = (\alpha_1, \alpha_2, \cdots)$, $\alpha_i \in (0,1)$, be a sequence. We define approximate solutions $\hat{u}_h = (\hat{\rho}_h, \hat{G}_h)^{tr}$ and $u_h = (\rho_h, G_h)^{tr}$ inductively. Assume that $\hat{u}_h$ and u_h are defined for $t \leqq t_j$. Then, on $\mathcal{I} \times \mathcal{J}_{j+1}$, $\hat{u}_h$ is the solution to

$$\text{(3.1)} \qquad \begin{aligned} &\hat{u}_{ht} + F(\hat{u}_h)_x = 0, && (x,t) \in \mathcal{I} \times \mathcal{J}_{j+1}, \\ &\hat{\rho}_h(t,0) = \rho_B(t_{j+1/2}), && t \in \mathcal{J}_{j+1}, \\ &\hat{G}_h(t,1) = 0, && t \in \mathcal{J}_{j+1}, \\ &\hat{u}_h(x, t_j +) = u_h(x_{i-1} + \alpha_j h, t_j -), && x \in \mathcal{I}_i \text{ if } j > 0, \\ &\hat{u}_h(x, 0+) = u_0(x_{i-1/2}), && x \in \mathcal{I}_i \text{ if } j = 0, \end{aligned}$$

where $F(u) = (G, G^2/\rho + p(\rho))^{tr}$ and $u_0(x) = (\rho_0(x), G_0(x))^{tr}$.

Next define the functions $\check{u}(t, \mathring{u}) = (\check{\rho}(t, \mathring{\rho}, \mathring{G}), \check{G}(t, \mathring{\rho}, \mathring{G}))$ by

$$\text{(3.2)} \qquad \begin{aligned} &\check{u}_t = H(\check{u}), && t > 0, \\ &\check{u}(0) = \mathring{u}, \end{aligned}$$

where $H(u) \equiv (0, -f|G|G/2D\rho)^{tr} = (0, -\mathcal{K}(G)/\rho)^{tr}$. Then we set

$$u_h(x,t) = \check{u}(t - t_j, \hat{u}_h(x,t)), \qquad t \in \mathcal{J}_{j+1}. \tag{3.3}$$

Note that (3.1) poses an initial value Riemann problem at each mesh point (x_i, t_j), $0 < i < N$, and a boundary Riemann problem of type (2.12) and (2.13) at $i = 0$ and $i = N$, respectively. Therefore, we can use the Riemann problem solutions of Section 2 to solve (3.1) in $\mathcal{J}_{j+1}$, as long as the waves in these solutions do not intersect in $\mathcal{J}_{j+1}$, and as long as $|v|$ remains less than c in the boundary problems. Given these two conditions, $\hat{u}_h$ and u_h are defined for all time. Choose h/k fixed with

$$\frac{h}{k} \geqq 4c.$$

We guarantee the two conditions above by showing that if (1.11) holds, then $|v| < c$ for the approximate solutions $\hat{u}_h$ and u_h. It then follows from (2.2) that the wave speeds are bounded by $2c$, and so the waves cannot interact in $\mathcal{J}_{j+1}$.

We shall use "I-curves" in our proof of the stability of the scheme, cf. [1]. We call the points

$$(x_i + \alpha_j h, t_j), \qquad 0 \leqq i \leqq N-1, j \geqq 0,$$

$$(x_i, t_{j+1/2}), \qquad 0 \leqq i \leqq N, j \geqq 0,$$

vertex points. An I-curve, J, is a curve in x,t-space which successively connects vertex points on t_j to adjacent vertex points on $t_{j\pm 1/2}$ such that the index i is a non-decreasing function of x on J, and such that J is linear between adjacent vertex points (see Figure 3.1). It is important to note that the unique I-curve, $J(j)$, which connects the vertex points on t_j to the vertex points on $t_{j+1/2}$ crosses all the waves in the Riemann problem solutions of $\hat{u}_h$ in $\mathcal{J}_{j+1}$. We partially order

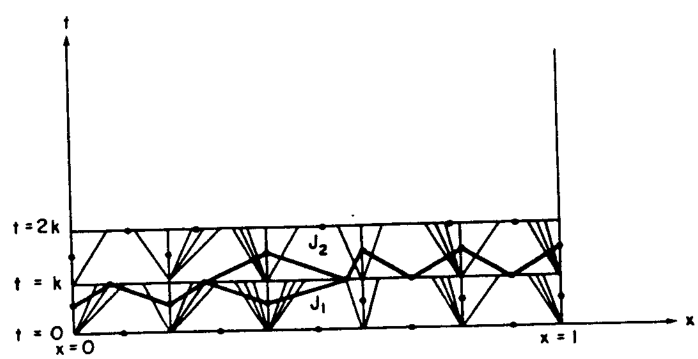

Figure 3.1

the I-curves by saying that larger curves lie toward larger time, and we call J_2 an immediate successor of J_1 if J_l connect the same vertex points except one and if $J_2 > J_1$. We also let $t(J)$ denote the value of t where J intersects the line $x = 0$. For any I-curve, J, define

$$L(J) = \sum_J |\gamma_A|,$$

$$L_B(J) = \sum_{t_j > t(J)} |\ln \rho_B(t_{j+1/2}) - \ln \rho_B(t_{j-1/2})|,$$

where $\sum_J$ is the sum over all waves which cross J and where γ_A denotes the strength of a wave in the solution $\hat{u}_h$.

LEMMA 3.1. *Suppose that*

$$L(J(j)) + L_B(J(j)) \equiv V_j < \ln \tfrac{1}{2}(3 + \sqrt{5}). \tag{3.4}$$

Then

$$\sup_{(x,t) \in \mathcal{I} \times \mathcal{I}_{j+1}} \left| \ln \frac{\hat{\rho}_h(x,t)}{\rho_\infty} \right| \leqq V_j, \tag{3.5}$$

and

$$\sup_{(x,t) \in \mathcal{I} \times \mathcal{I}_{j+1}} \left| \frac{\hat{v}_h(x,t)}{c} \right| \leqq \frac{e^{V_j} - 1}{e^{V_j/2}} < 1, \tag{3.6}$$

where

$$\rho_\infty = \lim_{t \to \infty} \rho_B(t). \tag{3.7}$$

Proof: The proof of (3.5) is an immediate consequence of our definition of wave strength. To prove (3.6) note that $\hat{v}_h(1,t) = 0$ and that $\hat{u}_h(1,t)$ is connected to $\hat{u}_h(x,t)$, $t \in \mathcal{I}_{j+1}$, by a curve which is composed of a sequence of shock and rarefaction curves the sum of whose strengths is bounded by V_j. By viewing l-waves in the r,s-plane, it is evident from Theorem 2.1 that $|\hat{v}_h(x,t)|$ must be smaller than the change in v across an l-shock which starts at $u_L = (\rho_L, 0)^{tr}$ and has strength V_j. For such a shock wave, we can use (2.11a) to calculate that

$$\left| \frac{v_R}{c} \right| = \frac{e^{V_j} - 1}{e^{V_j/2}} < 1.$$

We next prove

THEOREM 3.1. *Suppose that* (1.11) *holds. Then*

$$V_0 \leqq V \equiv \operatorname*{Var}_{t \geqq 0} \ln \rho_B + \operatorname*{Var}_{x \in [0,1]} \ln \rho_0 + \operatorname*{Var}_{x \in [0,1]} \frac{v_0}{c} < \ln \tfrac{1}{2}(3+\sqrt{5}) \tag{3.8}$$

and

$$L_B(J(j+1)) + L(J(j+1)) \leqq L_B(J(j)) + L(J(j)) \quad \text{for} \quad j = 0, 1, \cdots. \tag{3.9}$$

The inequality (3.8) follows easily from Theorem 2.1, so that it remains to verify (3.9). We shall show that

$$L_B(J_2) + L(J_2) \leqq L_B(J_1) + L(J_1) \tag{3.10}$$

whenever J_2 is an immediate successor of J_1. To verify (3.10), it suffices to study the "interaction" that occurs in the region Δ that lies between J_1 and J_2; i.e., it suffices to compare the strength of the waves that cross J_2/J_1 to the strength of the waves that cross J_1/J_2. If Δ does not contain a mesh point (x_i, t_j), then

$$L_B(J_2) + L(J_2) = L_B(J_1) + L(J_1).$$

Thus, we need only consider the case when Δ is a "diamond" centered at (x_i, t_j) for $1 \leqq i \leqq N-1$ or a "half-diamond" centered at (x_0, t_j) or (x_N, t_j). We denote these diamonds by Δ_{ij}. We let γ_{ij}^l denote the signed strength of the l-wave in the solution of the Riemann problem which is posed at (x_i, t_j). Here, γ_{ij}^l denotes both the name as well as the signed strength of a wave. It is convenient to introduce the I-curve, $\tilde{J}_2$, which is identical to J_2 as a curve in the x,t-plane, but where the waves which cross $\tilde{J}_2 \backslash J_1$ are taken to be the waves which solve the Riemann problem with constant states u_L^- and u_R^-, instead of u_L^+ and u_R^+ (defined in Figure 3.2). Thus, for example, $L(\tilde{J}_2) = \sum_{\tilde{J}_2} |\gamma_A|$ is the sum over all waves which cross $J_2 \cap J_1$ plus the sum of the waves which solve the Riemann problem with constant states u_L^- and u_R^-. We also define $L_B(\tilde{J}_2) = L_B(J_2)$.

LEMMA 3.2 (See Liu [5].) *If J_2 is an immediate successor of J_1, then*

$$L_B(\tilde{J}_2) + L(\tilde{J}_2) \leqq L_B(J_1) + L(J_1).$$

The next lemma says that the functional $L_B + L$ is also nonincreasing for the friction step.

LEMMA 3.3. *If J_2 is an immediate successor of J_1, then*

$$L_B(J_2) + L(J_2) \leqq L_B(\tilde{J}_2) + L(\tilde{J}_2). \tag{3.11}$$

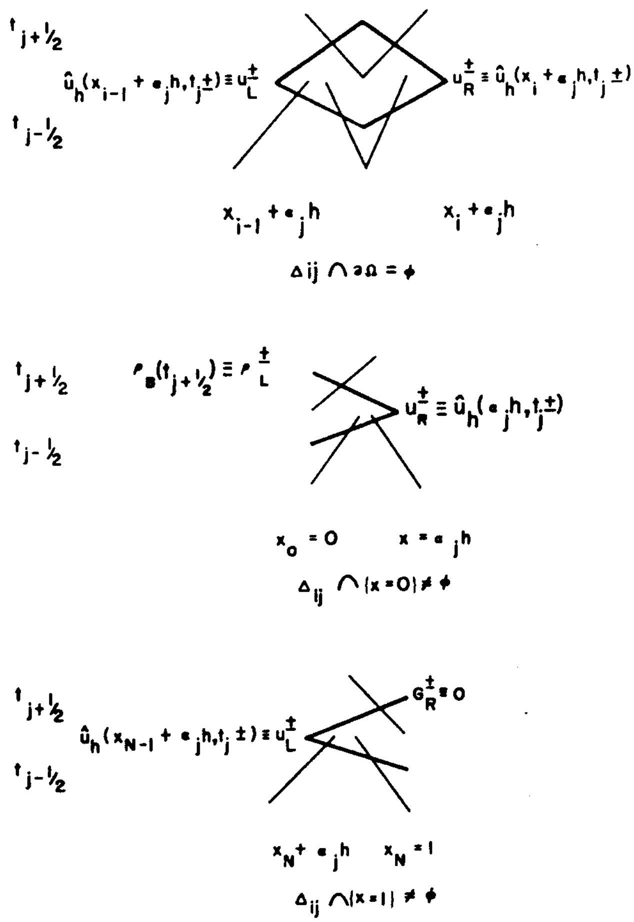

Figure 3.2

The proof of Lemma 3.3 is given in Appendix A. Now (3.10) follows from Lemma 3.2 and Lemma 3.3. Relation (3.10) implies that if $J_2 > J_1$, then

$$L_B(J_2) + L(J_2) \leqq L_B(J_1) + L(J_1). \tag{3.12}$$

The stability result (3.9) is a special case of (3.12) since $J(j+1) > J(j)$. Finally, we note that the stability results in this section did not require any continuity properties for $\mathcal{K}(G)$. In Appendix A, it is shown that the properties (1.6) and (1.7) are sufficient when $\mathcal{K}$ is extended so as to be a maximal monotone function. However, the proof of convergence for our scheme will require the further assumption that $\mathcal{K}$ be locally Lipschitz continuous.

4. Bounds for the Total Non-Linearity

In this section, we shall give bounds for the total non-linearity. These bounds are required for the regularity results in the next section. We assume that $V < \ln\frac{1}{2}(3+\sqrt{5})$, so that by Lemma 3.1 and Theorem 3.1 we have

$$\left|\frac{\hat{v}_h}{c}\right| \leqq \frac{e^V - 1}{e^{V/2}} < 1 \tag{4.1}$$

and

$$\left|\ln\frac{\hat{\rho}_h}{\rho_\infty}\right| \leqq V. \tag{4.2}$$

Hence, there exists a compact set $K \subset (0,\infty)\times\mathbb{R}$, independent of h, k, and α, such that

$$\hat{u}_h(x,t) \in K \quad \text{for} \quad (x,t) \in (0,1)\times(0,\infty). \tag{4.3}$$

In the following, let C denote a generic constant (dependent on c, K, and V) which is independent of h, k, and α.

Recall that γ_{ij}^l is the signed strength of the l-wave in the solution of the Riemann problem for $\hat{u}_h$ at (x_i, t_j) and let $\bar{\gamma}_{ij}^l$ denote the sum of the signed strengths of l-waves which enter Δ_{ij}. We define the non-linearity of the interaction at (x_i, t_j), N_{ij}^l, to be

$$N_{ij}^l = \gamma_{ij}^l - \bar{\gamma}_{ij}^l, \qquad 0 < i < N,\ j > 0,$$

$$N_{0j}^l = \begin{cases} 0 & \text{for } l = 1, \\ \gamma_{0j}^2 - \bar{\gamma}_{0j}^2 + \bar{\gamma}_{0j}^1 & \text{for } l = 2, \end{cases}$$

$$N_{Nj}^l = \begin{cases} \gamma_{Nj}^1 - \bar{\gamma}_{Nj}^1 - \bar{\gamma}_{Nj}^2 & \text{for } l = 1, \\ 0 & \text{for } l = 2. \end{cases}$$

Similarly, if $\tilde{\gamma}_{ij}^l$ is the signed strength of the l-wave that crosses $\tilde{J}_2/J_1$, we denote the non-linearity $\tilde{N}_{ij}^l$ that occurs in the conservation law step by

$$\tilde{N}_{ij}^l = \tilde{\gamma}_{ij}^l - \bar{\gamma}_{ij}^l, \qquad 0 < i < N,$$

$$\tilde{N}_{0j}^l = \begin{cases} 0 & \text{for } l = 1, \\ \tilde{\gamma}_{0j}^2 - \bar{\gamma}_{0j}^2 + \bar{\gamma}_{0j}^1 & \text{for } l = 2, \end{cases}$$

$$\tilde{N}_{Nj}^l = \begin{cases} \tilde{\gamma}_{Nj}^1 - \bar{\gamma}_{Nj}^1 - \bar{\gamma}_{Nj}^2 & \text{for } l = 1, \\ 0 & \text{for } l = 2. \end{cases}$$

Finally, we set

$$N_{ij} = |N_{ij}^1| + |N_{ij}^2|,$$

$$\tilde{N}_{ij} = |\tilde{N}_{ij}^1| + |\tilde{N}_{ij}^2|.$$

We shall prove the following theorem.

THEOREM 4.1. *For any $T > 0$, there exists $C = C(T)$ such that*

$$N(T) \equiv \sum_{t_j \leqq T} N_{ij} \leqq C(T), \tag{4.4}$$

where the sum is over (i, j) such that $t_j \leqq T$.

Two waves γ_A and γ_B that cross J are said to approach if they are both of the same family and if γ_A or γ_B is a shock wave. Define

$$Q(J) = \sum_{\mathrm{App}(J)} |\gamma_A||\gamma_B|, \tag{4.5}$$

where the sum is over all pairs of approaching waves that cross J. Also, let

$$D_{ij} = \sum_{\mathrm{App}(J_1/J_2)} |\gamma_A||\gamma_B| \tag{4.6}$$

be the sum of the products of approaching waves that enter Δ_{ij}. For a simple wave γ, we define the shock wave strength and rarefaction wave strength of γ by

$$S(\gamma) = \begin{cases} 0 & \text{if } \gamma \geqq 0, \\ |\gamma| & \text{if } \gamma \leqq 0, \end{cases}$$

$$R(\gamma) = \begin{cases} |\gamma| & \text{if } \gamma \geqq 0, \\ 0 & \text{if } \gamma \leqq 0. \end{cases}$$

Then we decompose D_{ij} into

$$D_{ij} = D_{ij}^1 + D_{ij}^2,$$

where

$$D_{ij}^1 = \sum_{\mathrm{App}(J_1 \setminus J_2)} S(\gamma_A) R(\gamma_B)$$

and

$$D_{ij}^2 = \sum_{\mathrm{App}(J_1 \backslash J_2)} S(\gamma_A)S(\gamma_B).$$

We shall need the following lemmas in the proof of Theorem 4.1.

LEMMA 4.1. *There exists a constant C, such that*

$$\tilde{N}_{ij} \leqq CD_{ij}, \qquad 0 < i < N, \tag{4.7}$$

$$\tilde{N}_{0j} \leqq |\Delta L_B|, \tag{4.8}$$

$$\tilde{N}_{Nj} \leqq |\Delta L|, \tag{4.9}$$

where $\Delta L = L(\tilde{J}_2) - L(J_1)$ and $\Delta L_B = L_B(\tilde{J}_2) - L_B(J_1)$.

Proof: Note that if a single 1-wave and a single 2-wave enter Δ_{ij}, then $\tilde{N}_{ij} = 0$. Thus, (4.7) follows from the results in [10], [13]. The estimates (4.8)–(4.9) are easily verified by checking all possible boundary interactions for the conservation law step, cf. [5].

Now let $\gamma_{ij} = (\gamma_{ij}^1, \gamma_{ij}^2)$ and $\tilde{\gamma}_{ij} = (\tilde{\gamma}_{ij}^1, \tilde{\gamma}_{ij}^2)$ (where $\gamma_{0j} = (0, \gamma_{0j}^2)$ and $\gamma_{Nj} = (\gamma_{Nj}^1, 0)$).

LEMMA 4.2. *There exists a constant, C, such that*

$$|\gamma_{ij} - \tilde{\gamma}_{ij}| \leqq Ck|\tilde{\gamma}_{ij}|.$$

Proof: First, let $0 < i < N$. It follows from the results in Section 2 that there exists a regular C^2 diffeomorphism $\theta: \mathbb{R}^2 \to \mathbb{R}^2$, such that for constant states u_L and u_R we have

$$\gamma = \theta(\Psi(u_L) - \Psi(u_R)), \tag{4.10}$$

where $\gamma = (\gamma^1, \gamma^2)$ is the vector of signed strengths of waves in the Riemann problem $[u_L, u_R]$ and where $\Psi((\rho, G)) = (r, s)$. Now, since $\Psi(\check{u})_t = d\Psi(\check{u})H(\check{u})$, we obtain

$$\begin{aligned} &\left|\left[\Psi(\check{u}(t,u_L^-)) - \Psi(\check{u}(t,u_R^-))\right] - \left[\Psi(u_L^-) - \Psi(u_R^-)\right]\right| \\ &\qquad\qquad \leqq Ct|\Psi(u_L^-) - \Psi(u_R^-)|. \end{aligned} \tag{4.11}$$

Thus, since $u_L^+ = \check{u}(k, u_L^-)$ and $u_R^+ = \check{u}(k, u_R^-)$ we have

$$\left|\left[\Psi(u_L^+) - \Psi(u_R^+)\right] - \left[\Psi(u_L^-) - \Psi(u_R^-)\right]\right| \leqq Ck|\Psi(u_L^-) - \Psi(u_R^-)|. \tag{4.12}$$

Hence, (4.10) and (4.12) yield the estimate

$$
\begin{aligned}
|\gamma_{ij} - \tilde{\gamma}_{ij}| &\leqq C\left|\left[\Psi(u_L^+) - \Psi(u_R^+)\right] - \left[\Psi(u_L^-) - \Psi(u_R^-)\right]\right| \\
&\leqq Ck|\Psi(u_L^-) - \Psi(u_R^-)| \qquad (4.13)\\
&\leqq Ck|\tilde{\gamma}_{ij}|.
\end{aligned}
$$

This is the result for $0 < i < N$. An easy check establishes the result for $i = 0, N$.

We now define two non-linear functionals $F_0(J)$ and $F_1(J)$ such that $F_m(J)$ decreases by $C_1\tilde{N}_{ij}$ between successors J_1 and $\tilde{J}_2$ whenever Δ_{ij} does not intersect the boundary which lies opposite $x = m$. First, define

$$
(4.14) \qquad
\begin{aligned}
L_1(J) &= \sum_J |\gamma_A^1|, \\
L_2(J) &= \sum_J |\gamma_A^2|;
\end{aligned}
$$

i.e., $L_l(J)$ is the sum of the strengths of all l-waves that cross J, cf. [5]. Next, define

$$
(4.15) \qquad
\begin{aligned}
F_0(J) &= L(J) + 2L_B(J) + M_1L_1(J) + M_2Q(J), \\
F_1(J) &= L(J) + M_1L_2(J) + M_2Q(J),
\end{aligned}
$$

where M_1 and M_2 are positive constants which will be chosen later. We note that by Theorem 3.1

$$
(4.16) \qquad F_m(J) \leqq C.
$$

We adopt the notation

$$
\begin{aligned}
\Delta L_l &= L_l(\tilde{J}_2) - L_l(J_1), && l = B, 1, 2, \text{ or absent,} \\
\Delta Q &= Q(\tilde{J}_2) - Q(J_1), \\
\Delta S^l &= S(\tilde{\gamma}_{ij}^l) - S(\bar{\gamma}_{ij}^l), && l = 1, 2, \\
\Delta R^l &= R(\tilde{\gamma}_{ij}^l) - R(\bar{\gamma}_{ij}^l), && l = 1, 2,
\end{aligned}
$$

where $S(\bar{\gamma}_{ij}^l)$ (respectively $R(\bar{\gamma}_{ij}^l)$) denotes the sum of l-shock wave strengths (respectively l-rarefaction wave strengths) which enter Δ_{ij}. The following three lemmas give bounds for the above quantities.

LEMMA 4.3. *There is a positive constant, $\delta = \delta(K) > 0$, such that, for $0 < i < N$,*

$$\Delta L \leqq -\delta \Delta S^l, \qquad l = 1,2. \tag{4.17}$$

Proof: See [13].

LEMMA 4.4. *If $0 < i < N$, then*

$$|\Delta R^l| \leqq \tfrac{1}{2} D_{ij}^2, \qquad l = 1,2. \tag{4.18}$$

Proof: See Appendix B.

LEMMA 4.5. *The following estimates hold:*

$$\begin{aligned} |\tilde{\gamma}_{0j}^2 - \bar{\gamma}_{0j}^2| &\equiv |\Delta\gamma^2| \leqq |\Delta L_1| + |\Delta L_B|, \\ |\tilde{\gamma}_{Nj}^1 - \bar{\gamma}_{Nj}^1| &\equiv |\Delta\gamma^1| \leqq |\Delta L_2|. \end{aligned} \tag{4.19}$$

Proof: These estimates are easily checked by using Theorem 2.1 of [5]. The following is the crucial lemma in our proof of Theorem 4.1.

LEMMA 4.6. *Set*

$$M_1/M_2 = \xi, \qquad M_2 \leqq 1, \tag{4.20}$$

and

$$M_2 \leqq \delta/2(V+1),$$

for ξ satisfying $V < \xi < 1$. Then there exists a positive constant C_1 such that

$$F_m(\tilde{J}_2) - F_m(J_1) \leqq -C_1 \tilde{N}_{ij}, \tag{4.21}$$

whenever Δ_{ij} does not intersect the boundary that lies opposite $x = m$.

Proof: We treat case $m = 0$ (the argument for $m = 1$ is similar). First, assume that $i > 0$ ($\Delta_{ij} \cap \partial\Omega = \varnothing$). In this case, we have

$$\begin{aligned} \Delta L_B &= 0, \\ \Delta L_1 &= \Delta R^1 + \Delta S^1, \\ \Delta Q &= \sum_{J_1 \cap J_2} |\gamma_A^1| \Delta S^1 + S(\gamma_A^1) \Delta R^1 \\ &\quad + \sum_{J_1 \cap J_2} |\gamma_A^2| \Delta S^2 + S(\gamma_A^2) \Delta R^2 - D_{ij}^1 - D_{ij}^2. \end{aligned} \tag{4.22}$$

Since $|\Delta R^l| \leqq \frac{1}{2} D_{ij}^2$ and $\sum_{J_1 \cap J_2} |\gamma_A| \leqq V < 1$,

$$\Delta Q \leqq \sum_{J_1 \cap J_2} |\gamma_A^1| \Delta S^1 + \sum_{J_1 \cap J_2} |\gamma_A^2| \Delta S^2 - D_{ij}^1 - \tfrac{1}{2} D_{ij}^2. \tag{4.23}$$

Thus,

$$\Delta F_0 \leqq \Delta L + \left\{ M_1 + M_2 \sum_{J_1 \cap J_2} |\gamma_A^1| \right\} \Delta S^1$$
$$+ \left\{ M_2 \sum_{J_1 \cap J_2} |\gamma_A^1| \right\} \Delta S^2 - M_2 D_{ij}^1 + M_1 \Delta R^1 - M_2 \tfrac{1}{2} D_{ij}^2.$$

Now since $M_l \leqq \delta / 2(V+1)$, Lemma 4.3 implies that

$$\Delta F_0 \leqq -M_2 D_{ij}^1 - \frac{1}{2} M_2 \left[D_{ij}^2 - \frac{2M_1}{M_2} \Delta R^1 \right].$$

But $M_1 / M_2 = \xi < 1$; hence, we see from Lemma 4.4 that

$$\Delta F_0 \leqq -M_2 D_{ij}^1 - \left[M_2 \tfrac{1}{2}(1-\xi) \right] D_{ij}^2 \leqq -\left[M_2 \tfrac{1}{2}(1-\xi) \right] D_{ij}. \tag{4.24}$$

The result (4.21) now follows immediately from Lemma 4.1.

Next, we consider the case $i = 0$ (when $\Delta_{ij} \cap \partial\Omega \neq \emptyset$). Then, by Lemma 4.5,

$$\begin{gathered} \Delta L + 2\Delta L_B \leqq \Delta L_B \leqq 0, \qquad \Delta L_1 \leqq 0, \\ \Delta Q \leqq \sum_{J_1 \cap J_2} |\gamma_A^2| \, |\Delta \gamma^2| \leqq \sum_{J_1 \cap J_2} |\gamma_A^2| (|\Delta L_B| + |\Delta L_1|) \leqq V |\Delta L_B| + V |\Delta L_1|. \end{gathered} \tag{4.25}$$

Hence,

$$\Delta F_0 \leqq (1 - M_2 V) \Delta L_B + (M_2 V - M_1) |\Delta L_1|.$$

Now $M_2 V - M_1 = M_2(V - \xi) < 0$ and $1 - M_2 V > 0$; thus

$$\Delta F_0 \leqq (1 - M_2 V) \Delta L_B.$$

Therefore, by Lemma 4.1,

$$\Delta F_0 \leqq -(1 - M_2 V) \tilde{N}_{0j}. \tag{4.26}$$

The arguments for $F_1(J)$ are similar.

LEMMA 4.7. *Under the conditions of Lemma* 4.6,

$$F_m(J_2) - F_m(J_1) \leqq -C_1 N_{ij} + Ck|\tilde{\gamma}_{ij}|, \tag{4.27}$$

whenever Δ_{ij} *does not intersect the boundary that lies opposite* $x = m$.

Proof: It follows from Lemma 4.2 that

$$|N_{ij} - \tilde{N}_{ij}| \leqq Ck|\tilde{\gamma}_{ij}|. \tag{4.28}$$

Thus, Lemma 4.6 implies that

$$F_m(\tilde{J}_2) - F_m(J_1) \leqq -C_1 N_{ij} + Ck|\tilde{\gamma}_{ij}|. \tag{4.29}$$

Note also that, by Appendix A,

$$\gamma_{0j}^2 \leqq \tilde{\gamma}_{0j}^2;$$

and

$$|\gamma_{Nj}^1| \leqq |\tilde{\gamma}_{Nj}^1|.$$

Thus, if $\Delta_{ij} \cap \{x = 0\} \neq \emptyset$, then

$$F_0(J_2) - F_0(J_1) = F_0(\tilde{J}_2) - F_0(J_1) \leqq -C_1 N_{0j} + Ck|\tilde{\gamma}_{0j}|,$$

and if $\Delta_{ij} \cap \{x = 1\} \neq \emptyset$, then

$$F_1(J_2) - F_1(J_1) \leqq F_1(\tilde{J}_2) - F_1(J_1) \leqq -C_1 N_{Nj} + Ck|\tilde{\gamma}_{Nj}|.$$

Next, let $m = 0$ and $0 < i < N$. (The argument for $m = 1$ is similar.) Since $L(J) \leqq C$ for all I-curves J, it follows from Lemma 4.2 that

$$|F_0(J_2) - F_0(\tilde{J}_2)| \leqq Ck|\tilde{\gamma}_{ij}|. \tag{4.30}$$

The result (4.6) follows from (4.29) and (4.30).

Proof of Theorem 4.1: Let $J(0) = J_0 \leqq J_1 \leqq \cdots \leqq J_\mu$ be a maximal sequence of immediate successors such that $\Delta_{ij} \cap \{x = 1\} = \emptyset$ (respectively $\Delta_{ij} \cap \{x = 0\} = \emptyset$), and let Y_0 (respectively Y_1) be the set of diamonds Δ_{ij}, crossed by the above sequence. Note that $t_j \leqq (k/h)(1 - x_i)$ if and only if $\Delta_{ij} \in Y_0$. It follows from Lemma 4.7 that

$$\begin{aligned} C_1 \sum_{\Delta_{ij} \in Y_0} N_{ij} &\leqq \sum_{j=1}^{\mu} \left[F_0(J_{j-1}) - F_0(J_j) \right] + Ck \sum_{\Delta_{ij} \in Y_0} |\tilde{\gamma}_{ij}| \\ &= \left[F_0(J_0) - F_0(J_\mu) \right] + Ck \sum_{\Delta_{ij} \in Y_0} |\tilde{\gamma}_{ij}|. \end{aligned} \tag{4.31}$$

By the results of Section 3, $\sum_{i=0}^{N} |\tilde{\gamma}_{ij}| \leqq V$ and $F_0(J) \leqq C$ for all I-curves J. Thus,

$$k \sum_{\Delta_{ij} \in Y_0} |\tilde{\gamma}_{ij}| \leqq \frac{k}{h} V_0,$$

and hence, in view of (4.31),

$$\sum_{\Delta_{ij} \in Y_0} N_{ij} \leqq C. \tag{4.32}$$

By a similar estimate using $F_1(J)$ we obtain

$$\sum_{\Delta_{ij} \in Y_1} N_{ij} \leqq C. \tag{4.33}$$

Thus, we have

$$\sum_{t_j \leqq k/2h} N_{ij} \leqq \sum_{\Delta_{ij} \in Y_0} N_{ij} + \sum_{\Delta_{ij} \in Y_1} N_{ij} \leqq C. \tag{4.34}$$

Similarly, since $V_j \leqq V_0 \leqq V$ for $j = 1, 2, \cdots$, there exists, for every $q = 0, 1, 2, \cdots$, a constant C, independent of h, k, α, and q, such that

$$\sum_{qk/2h < t_j < (q+1)k/2h} N_{ij} \leqq C. \tag{4.35}$$

Thus, the estimate of Theorem 4.1 follows since

$$\sum_{t_j \leqq T} N_{ij} \leqq qC = C(T),$$

for any $q \in \mathbb{N}$ satisfying $qk/2h \geqq T$. This completes the proof of Theorem 4.1.

5. Regularity of the Approximate Solutions

In this section, we show that the approximate solutions u_h are L^1 continuous in time and space to within an error dominated by the mesh length. These results, stated in Lemmas 5.1 and 5.2, are needed for the proof of the fact that the constructed solution to (1.1)–(1.3) actually takes on its boundary values in the L^1 sense.

LEMMA 5.1. *There exists a constant C, such that*

$$\int_0^1 |u_h(x, \tau_2) - u_h(x, \tau_1)| \, dx \leqq C[|\tau_2 - \tau_1| + k]. \tag{5.1}$$

Proof: Suppose that

$$0 \leqq t_{m-1} \leqq \tau_1 < t_m < \cdots < t_M < \tau_2 \leqq t_{M+1}. \tag{5.2}$$

Then

$$\begin{aligned}\int_0^1 |u_h(x,\tau_2) - u_h(x,\tau_1)|\,dx \leqq \int_0^1 |u_h(x,t_m-) - u_h(x,\tau_1)|\,dx \\ + \sum_{j=m}^{M-1} \int_0^1 |u_h(x,t_{j+1}-) - u_h(x,t_j+)|\,dx \\ + \int_0^1 |u_h(x,\tau_2) - u_h(x,t_M+)|\,dx \\ + \sum_{j=m}^{M} \int_0^1 |u_h(x,t_j+) - u_h(x,t_j-)|\,dx.\end{aligned} \tag{5.3}$$

For $s,t \in [t_j, t_{j+1}]$ we have

$$\begin{aligned}&\int_0^1 |u_h(x,s) - u_h(x,t)|\,dx \\ &\leqq \int_0^1 |u_h(x,s) - \hat{u}_h(x,s)|\,ds + \int_0^1 |\hat{u}_h(x,s) - \hat{u}_h(x,t)|\,dx \\ &\quad + \int_0^1 |\hat{u}_h(x,t) - u_h(x,t)|\,dx.\end{aligned} \tag{5.4}$$

But

$$\int_{x_{i-1}}^{x_i} |\hat{u}_h(x,s) - \hat{u}_h(x,t)|\,dx \leqq Ch\left(|\gamma_{i-1,j}^2| + |\gamma_{i,j}^1|\right).$$

Thus

$$\int_0^1 |\hat{u}(x,s) - \hat{u}_h(x,t)|\,dx \leqq ChL(J(j)). \tag{5.5}$$

Moreover, in view of (3.3) it follows that

$$\max_x |u_h(x,s) - \hat{u}_h(x,s)| \leqq C|s - t_j| \quad \text{for} \quad s \in \mathcal{I}_{j+1}, \tag{5.6}$$

so that, by (5.4),

$$\int_0^1 |u_h(x,s) - u_h(x,t)|\,dx \leqq ChL(J(j)) + Ck. \tag{5.7}$$

Furthermore, from (5.6),

$$\int_{x_{i-1}}^{x_i} |u_h(x,t_j+) - u_h(x,t_j-)|\,dx = \int_{x_{i-1}}^{x_i} |u_h(x_{i-1}+\alpha_j h, t_j-) - u_h(x,t_j-)|\,dx$$

$$\leqq Ch\big(|\gamma^2_{i-1,j-1}| + |\gamma^1_{i,j-1}|\big) + Chk,$$

so that

$$\int_0^1 |u_h(x,t_j+) - u_h(x,t_j-)|\,dx \leqq ChL(J(j-1)) + Ck. \tag{5.8}$$

Now putting (5.7) and (5.8) into (5.3) and summing, we obtain

$$\int_0^1 |u_h(x,\tau_2) - u_h(x,\tau_1)|\,dx \leqq Ch(M-m+1)V_0 + Ck(M-m+1)$$

$$\leqq C\big[|\tau_2-\tau_1| + k\big],$$

where we have applied Theorem 3.1 to estimate $L(J(j))$. This completes the proof of Lemma 5.1.

Next, we prove a similar but more difficult estimate involving L^1 continuity in space.

LEMMA 5.2. *Assume that* $\alpha = (\alpha_1,\alpha_2,\alpha_3,\cdots,)$ *is equidistributed,* $T<\infty$, *and* $y_1, y_2 \in [0,1]$. *Then there exists a constant* $C = C(T)$ *such that*

$$\int_0^T |u_h(y_2,t) - u_h(y_1,t)|\,dt \leqq C|y_2-y_1|, \tag{5.9}$$

for K sufficiently small.

We shall need the concept of approximate characteristics to prove Lemma 5.2. It is convenient for us to modify the definitions given by Glimm and Lax [2] and by Liu [6]. We shall define the set of approximate characteristics $\{\Gamma^J_\phi\}_{\phi\in\mathfrak{M}_J}$ by induction on J, $J \in \mathbb{N}$. An approximate characteristic Γ^J_ϕ is a map

$$\Gamma^J_\phi : [t_{j_\phi}, t_J] \to [0,1]\times\{1,2\}\times\mathbb{R},$$

where $\Gamma^J_\phi(t) = (x_\phi(t), l_\phi(t), \gamma_\phi(t))$ indicates that the characteristic at time t is at position $x_\phi(t)$, is travelling on the $l_\phi(t)$ characteristic, and has signed strength $\gamma_\phi(t)$. Here $x_\phi(t)$ is a continuous, piecewise linear function, and $l_\phi(t)$ and $\gamma_\phi(t)$ are piecewise constant functions with respect to the partition $\{t_{j_\phi}, t_{j_\phi+1},\cdots,t_J\}$.

At fixed J, $\phi \in \mathfrak{M}_J$ indexes the characteristics in $[0, t_J]$, and $\mathfrak{M}_J$ depends on J due to both the creation and the splitting of characteristics at $t > t_J$. To begin the definition, let at first $J = 0$. In this case set

$$\mathfrak{M}_0 = \{0, \cdots, N\} \times \{1,2\},$$

and, for $\phi = (i, l) \in \mathfrak{M}_0$, set $j_\phi = 0$ and define

$$\Gamma_\phi^0(0) = (x_i, l, \gamma_{i,0}^l).$$

Next, suppose $\mathfrak{M}_{J-1}$ and $\{\Gamma_\phi^{J-1}\}_{\phi \in \mathfrak{M}_{J-1}}$ are defined, and define

$$\mathfrak{M}_J = \{\phi_\theta \mid \phi \in \mathfrak{M}_{J-1}, \theta \in \{L, R\}\} \cup \{\{0, \cdots, N\} \times \{1,2\}\},$$

with

$$j_{\phi_\theta} = j_\phi \quad \text{for} \quad \phi \in \mathfrak{M}_{J-1},$$

$$j_\phi = J \quad \text{for} \quad \phi = \{i, l\}.$$

We first define $\Gamma_{\phi_L}^J(t)$ and $\Gamma_{\phi_R}^J(t)$ for $t \in [t_{j_\phi}, t_J]$ and $\phi \in \mathfrak{M}_{J-1}$. Thus, assume

$$\Gamma_\phi^{J-1}(t_{J-1}) = (x_i, l, \gamma).$$

We define $\Gamma_{\phi_\theta}^J(t)$. Let $\mu_1 \gamma_{i,J-1}^1$ (respectively $\mu_2 \gamma_{i,J-1}^2$) denote the signed strength of the 1-wave (respectively 2-wave) that lies to the left of the point $(x_{i-1} + \alpha_J h, t_J)$ (respectively $(x_i + \alpha_J h, t_J)$) in the solution of the Riemann problem which is posed at (x_i, t_{J-1}) in the approximate solution $\hat{u}_h$, so that $(1 - \mu_1)\gamma_{i,J-1}^1$ (respectively $(1 - \mu_2)\gamma_{i,J-1}^2$) is the signed strength of the 1-wave (respectively 2-wave) that lies to the right of $(x_{i-1} + \alpha_J h, t_J)$ (respectively $(x_i + \alpha_J h, t_J)$) in this Riemann problem solution, $0 \leqq \mu_1, \mu_2 \leqq 1$. Now if $l = 1$, define

(5.10a)

$$\Gamma_{\phi_L}^J(t) = \begin{cases} (x_\phi(t), l_\phi(t), \mu_1 \gamma_\phi(t)), & t < t_{J-1}, 0 \leqq i \leqq N, \\ \left(x_i - \dfrac{h}{k}(t - t_{J-1}), 1, \mu_1 \gamma_\phi(t_{J-1})\right), & t \in \mathfrak{g}_J, 0 < i \leqq N, \\ (0, 2, -\mu_2 \gamma_\phi(t_{J-1})), & t \in \mathfrak{g}_J, i = 0, \end{cases}$$

$$\Gamma_{\phi_R}^J(t) = \begin{cases} (x_\phi(t), l_\phi(t), (1 - \mu_1)\gamma_\phi(t)), & t < t_{J-1}, 0 \leqq i \leqq N, \\ (x_i, 1, (1 - \mu_1)\gamma_\phi(t_{J-1})), & t \in \mathfrak{g}_J, 0 < i \leqq N, \\ \left(\dfrac{h}{k}(t - t_{J-1}), 2, -(1 - \mu_2)\gamma_\phi(t_{J-1})\right), & t \in \mathfrak{g}_J, i = 0, \end{cases}$$

and if $l = 2$, define

$$\Gamma^{J}_{\phi_L}(t) = \begin{cases} (x_\phi(t), l_\phi(t), \mu_2\gamma_\phi(t)), & t < t_{J-1}, 0 \leqq i \leqq N, \\ (x_i, 2, \mu_2\gamma_\phi(t_{J-1})), & t \in \mathfrak{I}_J, 0 \leqq i < N, \\ \left(1 - \frac{h}{k}(t - t_{J-1}), 1, \mu_1\gamma_\phi(t_{J-1})\right), & t \in \mathfrak{I}_J, i = N, \end{cases}$$

(5.10b)

$$\Gamma^{J}_{\phi_R}(t) = \begin{cases} (x_\phi(t), l_\phi(t), (1-\mu_2)\gamma_\phi(t)), & t < t_{J-1}, 0 \leqq i \leqq N, \\ \left(x_i + \frac{h}{k}(t - t_{J-1}), 2, (1-\mu_2)\gamma_\phi(t_{J-1})\right), & t \in \mathfrak{I}_J, 0 \leqq i < N, \\ (1, 1, (1-\mu_1)\gamma_\phi(t_{J-1})), & t \in \mathfrak{I}_J, i = N. \end{cases}$$

The fact that $l_{\phi_0}(t)$ changes value at $t = t_{J-1}$ when $i = 0$ or $i = N$ accounts for the reflection of characteristics at the boundaries $x = 0$ or 1. Moreover, when 1-characteristics are reflected at the boundary $x = 0$, the sign of $\gamma_\phi(t)$ changes due to the way in which waves reflect but, for $\phi \in \mathfrak{M}_J$, $\gamma_\phi(t)$ has a constant magnitude which we denote by $|\gamma_\phi|$.

Finally, when $\phi = (i, l)$, we have $t_\phi = t_J$, and we define

$$\Gamma^{J}_{\phi}(t_J) = (x_i, l, N^l_{ij}). \tag{5.11}$$

This completes the definition of the approximate characteristics. For convenience, we let Γ^J_ϕ refer to both the function defined above as well as the piecewise linear curve in x, t-space given by the graph of x_ϕ.

The following three lemmas give the important properties of the approximate characteristics.

LEMMA 5.3. *Let $T = t_J$. Then*

$$\sum_{\phi \in \mathfrak{M}_J} |\gamma_\phi| = \sum_{i,l} |\gamma^l_{i0}| + \sum_{\substack{j<J \\ i,l}} |N^l_{ij}| \leqq C(T), \tag{5.12}$$

and, for $j < J$,

$$\gamma^l_{ij} = \sum \gamma_\phi(t_j), \tag{5.13}$$

the latter sum being over all ϕ such that $t_\phi \leqq t_j$, $x_\phi(t_j) = x_i$, and $l_\phi(t_j) = l$.

Proof: Statement (5.13) follows from the method of partitioning wave strengths in (5.10) and (5.11), and can be proven by a simple induction argument. For example, assume by induction that the sum of the strengths of the characteristics that enter a diamond Δ_{ij} and have $l_\phi(t_j) = l$ is equal to the sum of the strengths of the l-waves that enter Δ_{ij}. Then the sum of the strength of the characteristics that leave Δ_{ij} and have $l_\phi(t_j) = l$ must be equal to the sum of the strengths of the associated characteristics that enter Δ_{ij}, plus N_{ij}^l, the strength of the characteristic that is created at (x_i, t_j). By the definition of N_{ij}^l, this latter sum equals γ_{ij}^l, verifying (5.13). Similarly, (5.12) follows by an easy induction argument.

The next lemma follows immediately by our choice of μ_l in the definition of Γ_ϕ^J, once we note that, by (4.1),

$$\lambda = \min_{i,h} |\lambda_i(\hat{u}_h)| > 0.$$

LEMMA 5.4. *Let* $\phi \in \mathfrak{M}_J, j_\phi \leqq j \leqq J-1$, *and assume* $|\gamma_\phi| \neq 0$. *If* $l_\phi(t_j) = 1$ *and* $\alpha_{j+1} > 1 - \lambda k/h$, *then* $x'_\phi(t) = -h/k$ *for* $t \in \mathcal{I}_{j+1}$. *If* $l_\phi(t_j) = 2$ *and* $\alpha_{j+1} < \lambda k/h$, *then* $x'_\phi(t) = h/k$ *for* $t \in \mathcal{I}_{j+1}$.

The following theorem is an easy consequence of the definition of equidistributed sequence. A sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, $\alpha_i \in (0,1)$, is equidistributed if, for any $0 \leqq a < b \leqq 1$,

$$\lim_{J \to \infty} \frac{N(J, [a,b])}{J} = b - a,$$

where $N(J, [a,b]) = \text{Card}\{ j \leqq J \mid \alpha_j \in [a,b]\}$.

THEOREM 5.1. *Let* α *be any equidistributed sequence,* $\alpha_i \in (0,1)$. *For natural numbers* $m < n$, *and for any* $0 \leqq a < b \leqq 1$, *let* $A(m,n) = \{m, m+1, \cdots, m+n\}$, *and define*

$$\bar{N}(n, [a,b], m) = \text{Card}\{ j \in A(m,n) \mid \alpha_j \in [a,b]\},$$

$$N(n, [a,b], J) = \underset{m+n \leqq J}{\text{Min}} \{ \bar{N}(n, [a,b], m)\}.$$

Then, for any $w \in (0,1)$,

$$\lim_{J \to 0} \frac{N([wJ], [a,b], J)}{wJ} = b - a,$$

where $[x]$ *denotes the integer part of* x.

Now let $T > 0$ be arbitrary, and let $J = J(k)$ satisfy $t_{J-1} \leqq T < t_J$.

LEMMA 5.5. *Let α be a given equidistributed sequence. Let $0 \leqq y_1 < y_2 \leqq 1$. Then there exists a k_0 such that, if $k \leqq k_0$, then*

$$T_\phi \leqq \left(2 + t_J \frac{h}{k}\right)\left(\frac{t_J}{T}\right) 4 \frac{y_2 - y_1}{\lambda} \leqq C(T)(y_2 - y_1),$$

where T_ϕ denotes the total amount of time that an approximate characteristic Γ_ϕ^J with non-zero strength lies in $[y_1 - h, y_2 + h] \times [0, t_J]$.

Proof: Let

$$w = \frac{4(y_2 - y_1)}{\lambda T}$$

and without loss of generality assume T large so that $w < 1$. Since $t_{J-1} \leqq T < t_J$, we have $J = [T/k + 1]$, so that we can make J arbitrarily large by choosing k sufficiently small. Thus, by Theorem 5.1, we can choose k_0 so that if $k \leqq k_0$, then $h < \frac{1}{2}(y_2 - y_1)$ (recall that h/k is fixed),

$$\frac{N([wJ], [0, \lambda k/h], J)}{wJ} \geqq \frac{1}{2} \frac{\lambda k}{h}$$

and

$$\frac{N([wJ], [1 - \lambda k/h, 1], J)}{wJ} \geqq \frac{1}{2} \frac{\lambda k}{h}.$$

This implies that, if $k \leqq k_0$, then in any $[wJ]$ consecutive time steps in $[0, t_J]$, we must have $\alpha_j \leqq \lambda k/h$ (respectively $\alpha_j \geqq 1 - \lambda k/h$) in at least $[(\lambda k/2h)wJ]$ of those time steps. Thus by Lemma 5.4, during any $[wJ]$ consecutive time steps in $[0, t_J]$, an approximate characteristic with non-zero strength has a speed of magnitude h/k in at least $[(\lambda k/2h)wJ] = [2(y_2 - y_1)/h]$ of those time steps. Therefore, if $k \leqq k_0$, a characteristic with non-zero strength must pass between $y_1 - h$ and $y_2 + h$ through $[y_1 - h, y_2 + h] \times [0, t_J]$ in a maximum of $[4(y_1 - y_1)/\lambda k] = [wJ]$ time steps. But since an approximate characteristic has a maximum speed of h/k, it can reflect off a boundary at most $1 + t_J h/k$ times, so it can pass through $[y_1 - h, y_2 + h] \times [0, t_J]$ at most $2 + t_J h/k$ times. Thus the total time T_ϕ that an approximate characteristic with non-zero strength spends in

$[y_1 - h, y_2 + h] \times [0, t_J]$ must satisfy

$$T_\phi \leqq \left(2 + t_J \frac{h}{k}\right)[wJ]k \leqq \left(2 + t_J \frac{h}{k}\right)4\left(\frac{t_J}{T}\right)\frac{y_2 - y_1}{\lambda} \leqq C(T)(y_2 - y_1).$$

Note that k_0 depends only on α and $w = 4(y_2 - y_1)/\lambda T$.

We now turn to the estimate of Lemma 5.2. We first estimate

$$\int_0^T |u_h(y_2,t) - u_h(y_1,t)|\,dt \leqq \int_0^{t_J} |u_h(y_2,t) - u_h(y_1,t)|\,dt$$

$$= \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} |u_h(y_2,t) - u_h(y_1,t)|\,dt$$

and

$$\int_{t_{j-1}}^{t_j} |u_h(y_2,t) - u_h(y_1,t)|\,dt$$

$$\leqq \int_{t_{j-1}}^{t_j} |\hat{u}_h(y_2,t) - \hat{u}_h(y_1,t)|\,dt$$

$$+ \int_{t_{j-1}}^{t_j} |\{u_h(y_2,t) - u_h(y_1,t)\} - \{\hat{u}_h(y_2,t) - \hat{u}_h(y_1,t)\}|\,dt.$$

However, for $t \in \mathcal{J}_j$,

$$|\{u_h(y_2,t) - u_h(y_1,t)\} - \{\hat{u}_h(y_2,t) - \hat{u}_h(y_1,t)\}|$$

$$\leqq C|t - t_{j-1}|\,|\hat{u}_h(y_2,t) - \hat{u}_h(y_1,t)|,$$

and

$$|\hat{u}_h(y_2,t) - \hat{u}_h(y_1,t)| \leqq C\left(\sum_{i=m+1}^{M} |\gamma_{ij}^1| + \sum_{i=m}^{M+1} |\gamma_{ij}^2|\right),$$

where $x_m \leqq y_1 < x_{m+1} < \cdots < y_2 \leqq x_M$. Hence we see that

$$\int_0^{t_J} |u_h(y_2,t) - u_h(y_1,t)|\,dt \leqq Ck \sum_{j=0}^{J-1}\left(\sum_{i=m+1}^{M} |\gamma_{ij}^1| + \sum_{i=m}^{M-1} |\gamma_{ij}^2|\right). \tag{5.14}$$

Now by substituting (5.13) into the right-hand side of (5.14), we obtain

$$k\sum_{j=0}^{J-1}\left(\sum_{i=m+1}^{M}|\gamma_{ij}^1|+\sum_{i=m}^{M-1}|\gamma_{ij}^2|\right)\leqq\sum_{\phi\in\mathfrak{M}_J}T_\phi|\gamma_\phi|, \tag{5.15}$$

Where T_ϕ is the amount of time that a characteristic Γ_ϕ^J with non-zero strength is in $[x_m, x_M]\times[0,t_J]$, and $|\gamma_\phi|$ is the strength of that characteristic. But, by Lemma 5.5, we can estimate $T_\phi\leqq C(T)(y_2-y_1)$ if $k\leqq k_0$, and by Lemma 5.3, $\sum_{\phi\in\mathfrak{M}_J}|\gamma_\phi|\leqq C(T)$; thus we can estimate (5.15) by

$$k\sum_{j=0}^{J-1}\left(\sum_{i=m+1}^{M}|\gamma_{ij}^1|+\sum_{i=m}^{M-1}|\gamma_{ij}^2|\right)\leqq C(T)|y_2-y_1|. \tag{5.16}$$

Therefore, substituting (5.16) into (5.14) verifies Lemma 5.2.

6. Convergence of Approximate Solutions

For $\phi\in C_0^\infty((0,1)\times(0,\infty))$, we define

$$D(\alpha,h,\phi)=-\int\int\left[u_h\phi_t+F(u_h)\phi_x+H(u_h)\phi\right]dx\,dt. \tag{6.1}$$

From (3.3) it follows that

$$u_h(x,t)=\hat{u}_h(x,t)+\int_{t_j}^{t}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\,ds, \qquad t\in\mathfrak{J}_{j+1}. \tag{6.2}$$

Thus,

$$\begin{aligned}-\int_{t_j}^{t_{j+1}}\int_0^1u_h\phi_t\,dx\,dt=&-\int_{t_j}^{t_{j+1}}\int_0^1\hat{u}_h\phi_t\,dx\,dt\\&-\int_{t_j}^{t_{j+1}}\int_0^1\left[\int_{t_j}^{t}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\,ds\right]\phi_t\,dx\,dt.\end{aligned} \tag{6.3}$$

Now

$$\begin{aligned}-\int_{t_j}^{t_{j+1}}\int_0^1\hat{u}_h\phi_t\,dx\,dt=&\int_{t_j}^{t_{j+1}}\int_0^1F(\hat{u}_h)\phi_x\,dx\,dt\\&+\int_0^1\left[\hat{u}_h(x,t_j+)\phi(x,t_j)-\hat{u}_h(x,t_{j+1}-)\phi(x,t_{j+1})\right]dx\end{aligned} \tag{6.4}$$

and from integration by parts and (6.2) we get

$$-\int_{t_j}^{t_{j+1}}\int_0^1\left[\int_{t_j}^{t}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\,ds\right]\phi_t\,dx\,dt$$

(6.5)
$$\begin{aligned}
&= -\int_0^1\left[\int_{t_j}^{t_{j+1}}H(\check{u}(s-t_j,\hat{u}_h(x,t_{j+1})))\,ds\right]\phi(x,t_{j+1})\,dx\\
&\quad+\int_0^1\int_{t_j}^{t_{j+1}}\frac{d}{dt}\left[\int_{t_j}^{t}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\,ds\right]\phi\,dt\,dx\\
&= -\int_0^1[u_h(x,t_{j+1}-)-\hat{u}(x,t_{j+1}-)]\phi(x,t_{j+1})\,dx\\
&\quad+\int_0^1\int_{t_j}^{t_{j+1}}H(u_h(x,t))\phi\,dt\,dx\\
&\quad+\int_0^1\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}\frac{d}{dt}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\phi(x,t)\,ds\,dt\,dx.
\end{aligned}$$

Hence, substituting (6.4) and (6.5) into (6.3), we obtain

(6.6)
$$\begin{aligned}
-\int_{t_j}^{t_{j+1}}\int_0^1 u_h\phi_t\,dx\,dt &= \int_{t_j}^{t_{j+1}}\int_0^1[F(\hat{u}_h)\phi_x+H(u_h)\phi]\,dx\,dt\\
&\quad+\int_0^1[u_h(x,t_j+)\phi(x,t_j)-u_h(x,t_{j+1}-)\phi(x,t_{j+1})]\,dx\\
&\quad+\int_{t_j}^{t_{j+1}}\int_0^1\int_{t_j}^{t}\frac{d}{dt}H(\check{u}(s-t_j,\hat{u}_h(x,t)))\phi(x,t)\,ds\,dx\,dt.
\end{aligned}$$

Thus, after summing over j, we have

$$D(\alpha,h,\phi)=\sum_{j=1}^{\infty}D_j(\alpha,h,\phi)+\sum_{j=1}^{\infty}E_j(\alpha,h,\phi), \tag{6.7}$$

where

$$D_{j+1}(\alpha,h,\phi)=\int_0^1[u_h(x,t_{j+1}+)-u_h(x,t_{j+1}-)]\phi(x,t_{j+1})\,dx \tag{6.8}$$

and

$$E_{j+1}(\alpha, h, \phi) = \int_{t_j}^{t_{j+1}} \int_0^1 [F(\hat{u}_h) - F(u_h)]\phi\, dx\, dt$$

(6.9)

$$+ \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^{t} \frac{d}{dt} H(\check{u}(s - t_j, \hat{u}_h(x,t)))\phi\, ds\, dx\, dt.$$

Now

$$D_{j+1}(\alpha, h, \phi) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} [u_h(x_{i-1} + \alpha_{j+1}h, t_{j+1}-) - u_h(x, t_{j+1}-)]\phi(x, t_{j+1})\, dx$$

so that, by (4.16),

$$|D_{j+1}(\alpha, h, \phi)| \leqq C\Big(\sum_{i,l} |\gamma_{i,j}^l|\Big) h|\phi|_\infty \leqq CVh|\phi|_\infty. \tag{6.10}$$

Hence,

$$\left|\sum_{j=1}^{\infty} D_j(\alpha, h, \phi)\right| \leqq C\frac{VTh}{k}|\phi|_\infty = C|\phi|_\infty, \tag{6.11}$$

where the support of ϕ is in $(0,1)\times(0,T)$. We also have

$$\left|\int_0^1 D_{j+1}(\alpha, h, \phi)\, d\alpha_{j+1}\right| \leqq C\Big(\sum_{i,l}|\gamma_{i,j}^l|\Big) h^2|\phi_x|_\infty \leqq CVh^2|\phi_x|_\infty = Ch^2|\phi_x|_\infty. \tag{6.12}$$

Consequently, for $0 < j_1 < j_2$,

(6.13)

$$\left|\int D_{j_1}(\alpha, h, \phi) D_{j_2}(\alpha, h, \phi)\, d\alpha\right| = \left|\int\left[\int D_{j_2}(\alpha, h, \phi)\, d\alpha_{j_2}\right] D_{j_1}(\alpha, h, \phi)\, d\bar{\alpha}_{j_2}\right|$$

$$\leqq Ch^3|\phi|_\infty|\phi_x|_\infty,$$

where $\bar{\alpha}_{j_2}$ denotes the sequence obtained from α by deleting the j_2-th element. By (6.10),

$$\int D_j(\alpha, h, \phi)^2\, d\alpha \leqq Ch^2|\phi|_\infty^2. \tag{6.14}$$

Thus,

(6.15)

$$\int\Big(\sum D_j(\alpha, h, \phi)\Big)^2 d\alpha = 2\sum_{j_1<j_2}\int D_{j_1}D_{j_2}\, d\alpha + \sum_j \int D_j^2\, d\alpha$$

$$\leqq \left(\frac{T}{k}\right)^2 CV_0^2h^3|\phi|_\infty|\phi_x|_\infty + \left(\frac{T}{k}\right)CV_0^2h^2|\phi|_\infty^2 = Ch(|\phi|_\infty^2 + |\phi_x|_\infty^2).$$

Therefore, for each $\phi \in C_0^\infty((0,1)\times(0,\infty))$ there exists a sequence $h_m \to 0$ and a set $A \subset \prod_{i=1}^\infty [0,1]$ with measure one such that, if $\alpha \in A$, then

$$\sum_j D_j(\alpha, h_m, \phi) \to 0 \quad \text{as} \quad h_m \to 0. \tag{6.16}$$

Next, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a countable dense set in $C_0^\infty((0,1)\times(0,\infty))$ with respect to the norm $|\ |_\infty$. Then, after refining h_m above by a diagonal argument and after taking the appropriate countable intersection of sets A_n of measure one, we can conclude that there exists a sequence $h_m \to 0$ and a set $A \subset \prod_{i=1}^\infty [0,1]$ with measure one such that, if $\alpha \in A$, $n \in \mathbb{N}$, then

$$\sum_j D_j(\alpha, h_m, \phi_n) \to 0 \quad \text{as} \quad m \to \infty. \tag{6.17}$$

It follows by (6.11) that, if $\alpha \in A$, $\phi \in C_0^\infty((0,1)\times(0,\infty))$, then

$$\sum D_j(\alpha, h_m, \phi) \to 0. \tag{6.18}$$

Now by (4.10)

$$\int_{t_j}^{t_{j+1}}\int_0^1 [F(\hat{u}_h) - F(u_h)]\phi\, dx\, dt \leqq Ck^2|\phi|_\infty. \tag{6.19}$$

Also, for $x \in \mathscr{I}_i$,

$$\begin{aligned} \left|\int_{t_j}^{t_{j+1}}\int_{t_j}^{t}\frac{d}{dt}H(\check{u}(s-t_j, \hat{u}_h(x,t)))\phi\, ds\, dt\right| &= \left|\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}} H_u \check{u}_u \hat{u}_{ht}\phi\, dt\, ds\right| \\ &\leqq Ck|\phi|_\infty \int_{t_j}^{t_{j+1}}|\hat{u}_{ht}|\, dt \\ &\leqq Ck|\phi|_\infty(|\gamma_{i-1,j}^2| + |\gamma_{i,j}^1|). \end{aligned} \tag{6.20}$$

Hence,

$$\begin{aligned} &\int_{t_j}^{t_{j+1}}\int_0^1\int_{t_j}^{t}\frac{d}{dt}H(\check{u}(s-t_j, \hat{u}_h(x,t)))\phi\, ds\, dx\, dt \\ &\leqq C\Big(\sum_{i,l}|\gamma_{i,j}^l|\Big)kh|\phi|_\infty \leqq CVkh|\phi|_\infty \leqq Ckh|\phi|_\infty. \end{aligned} \tag{6.21}$$

Thus, it follows from (6.19) and (6.21) that

$$\left|\sum_{j=1}^\infty E_j(\alpha, h, \phi)\right| \leqq CTk|\phi|_\infty + CTh|\phi|_\infty. \tag{6.22}$$

Consequently, from (6.17) and (6.22) we can conclude that, if $\alpha \in A$ and $\phi \in C_0^\infty((0,1)\times(0,\infty))$, then

$$D(\alpha, h_m, \phi) \to 0 \quad \text{as} \quad m \to \infty. \tag{6.23}$$

Next, choose $T > 0$. Let $\{t_\beta\}_{\beta\in B} \subset (0,T)$ be countable and dense and let $\alpha \in A$ be fixed. Since

$$\operatorname*{Var}_{x\in[0,1]} u_h(\cdot,t) \leqq C$$

for $t \in (0,\infty)$, Helly's theorem implies that there exists a function $u(\cdot,t):\{t_\beta\} \to L^1(0,1)$ such that (after further refining $\{h_m\}$), for $\beta \in B$, $u_{h_m}(\cdot,t_\beta) \to u(\cdot,t_\beta)$ in $L^1(0,1)$ as $m\to\infty$. It follows from Lemma 5.1 that $u_{h_m}(\cdot,t)$ converges to $u(\cdot,t)$ in $L^1(0,1)$ for each $t\in(0,T)$ and

$$|u(\cdot,\tau_2) - u(\cdot,\tau_1)|_{L^1(0,1)} \leqq C|\tau_2-\tau_1| \tag{6.24}$$

for $\tau_1,\tau_2 \in (0,T)$. Hence,

$$\begin{aligned} &-\int_0^\infty\int_0^1 \left[u_{h_m}\phi_t + F(u_{h_m})\phi_x + H(u_{h_m})\phi\right] dx\,dt \\ &\quad \to -\int_0^\infty\int_0^1 \left[u\phi_t + F(u)\phi_x + H(u)\phi\right] dx\,dt \end{aligned} \tag{6.25}$$

as $m \to 0$.

Since $D(\alpha,h_m,\phi)\to 0$, u is a weak solution to (1.1). Furthermore, since $u_h(0)\to u_0$ in $L^1(0,1)$, (6.24) implies that

$$\lim_{t\to 0} u(\cdot,t) = u_0. \tag{6.26}$$

To obtain (1.13) note that the equidistributed sequences have measure one in $\prod_{i=1}^\infty [0,1]$, so that we can further assume that α is equidistributed. Since

$$u_{h_m} \to u \quad \text{in} \quad L^1((0,1)\times(0,T)),$$

we also have

$$u_{h_m}(x,\cdot) \to u(x,\cdot) \quad \text{in} \quad L^1(0,T) \tag{6.27}$$

for almost all x. An application of Lemma 5.2 now implies that

$$u_{h_m}(x,\cdot) \to u(x,\cdot) \quad \text{in} \quad L^1(0,T)$$

for all x and

$$|u(y_1,\cdot) - u(y_2,\cdot)|_{L^1(0,T)} \leqq C|y_1-y_2|. \tag{6.28}$$

But

$$\rho_{h_m}(0,\cdot)\to\rho_B \quad \text{in} \quad L^1(0,T)$$

so that we can finally conclude from (6.28) that

$$\lim_{x\to 0}\rho(x,\cdot)=\rho_B,\quad \lim_{x\to 1}G(x,\cdot)=0 \quad \text{in} \quad L^1(0,T). \tag{6.29}$$

This completes the proof of Theorem 1.

Appendix A

In this appendix, we give the proof of Lemma 3.3. We relax the assumption that $\mathcal{K}$ is locally Lipschitz at $|G|=G_c$ so that we can better model the transition from laminar flow to turbulent flow. We replace property (1.8) by:

$$\begin{aligned}&\mathcal{K}\text{ is Lipschitz continuous on } [-G_c,G_c] \text{ and } \mathcal{K} \text{ is locally}\\ &\text{Lipschitz continuous on } (-\infty,-G_c]\cup[G_c,\infty).\end{aligned} \tag{A.1}$$

Since (1.7) is to remain valid at $|G|=G_c$ in the sense of distributions, we have

$$\begin{aligned}\mathcal{K}(-G_c-)&\equiv\lim_{G\uparrow -G_c}\mathcal{K}(G)\leqq\lim_{G\downarrow -G_c}\mathcal{K}(G)\equiv\mathcal{K}(-G_c+),\\ \mathcal{K}(G_c-)&\equiv\lim_{G\uparrow G_c}\mathcal{K}(G)\leqq\lim_{G\downarrow G_c}\mathcal{K}(G)\equiv\mathcal{K}(G_c+).\end{aligned} \tag{A.2}$$

For the remainder of this appendix, we shall consider $\mathcal{K}$ to be multi-valued at $\pm G_c$ so that

$$\mathcal{K}(-G_c)=[\mathcal{K}(-G_c-),\mathcal{K}(-G_c+)]$$

and

$$\mathcal{K}(G_c)=[\mathcal{K}(G_c-),\mathcal{K}(G_c+)].$$

For $G\neq\pm G_c$, $\mathcal{K}$ remains single-valued. Note that since $\mathcal{K}$ is a maximal monotone function, the differential equation

$$\begin{aligned}&\check{u}_t\in H(\check{u}), \qquad t>0,\\ &\check{u}(0)=\mathring{u},\end{aligned} \tag{A.3}$$

(where $H(u)=(0,-\mathcal{K}(G)/\rho)^{tr}$ is now a multi-valued map) always has a unique solution, cf. [7]. Therefore, the friction step (3.2), (3.3) is still well defined.

Now let $\mathcal{F}(r,s) \equiv -\mathcal{K}(G)/\rho$. Only the following monotonicity property of $\mathcal{F}$ shall be used in the proof of Lemma 3.3:

(A.4) $$\mathcal{F}(r_1,s) \leqq \mathcal{F}(r_2,s) \quad \text{when} \quad r_1 > r_2, |v| < c,$$

(A.5) $$\mathcal{F}(r,s_1) \leqq \mathcal{F}(r,s_2) \quad \text{when} \quad s_1 > s_2, |v| < c.$$

When $\mathcal{F}$ is multi-valued at (r,s), we take this to mean that (A.4) and (A.5) are valid for every element of $\mathcal{F}(r,s)$. The monotonicity assumptions (A.4) and (A.5) are consequences of (1.7) since

$$\frac{\partial \mathcal{F}}{\partial r} = -\frac{1}{\rho}\left[\mathcal{K}_G\left(1-\frac{v}{c}\right) + \frac{\mathcal{K}}{G}\cdot\frac{v}{c}\right] \leqq 0,$$

$$\frac{\partial \mathcal{F}}{\partial s} = -\frac{1}{\rho}\left[\mathcal{K}_G\left(1+\frac{v}{c}\right) - \frac{\mathcal{K}}{G}\cdot\frac{v}{c}\right] \leqq 0,$$

for $|v| < c$. Given these assumptions, we now state and prove:

LEMMA 3.3. *If J_2 is an immediate successor of J_1, then*

(A.6) $$L_B(J_2) + L(J_2) \leqq L_B(\tilde{J}_2) + L(\tilde{J}_2).$$

Proof: First consider the boundary cases $i = 0$ and $i = N$. It is easy to see that

$$\gamma_{0j}^2 = \tilde{\gamma}_{0j}^2$$

and

$$|\gamma_{Nj}^1| \leqq |\tilde{\gamma}_{Nj}^1|;$$

hence (A.6) follows for these cases. Now let $L[u_L, u_R]$ denote the sum of the strengths of the waves in the Riemann problem solution $[u_L, u_R]$. Also, let $u_L(t) = \check{u}(t, u_L)$ and $u_R(t) = \check{u}(t, u_R)$. To verify (A.6) it suffices to show that

(A.7) $$L[u_L(t), u_R(t)] \leqq L[u_L, u_R], \qquad t \geqq 0.$$

Thus, consider the case $\operatorname{sign} v_L \neq \operatorname{sign} v_R$. Note that since $\mathcal{K}(0) = 0$, we know that $\operatorname{sign} v_R(\zeta) \neq \operatorname{sign} v_L(\zeta)$ from some $\zeta \in [0,\infty)$ if and only if $\operatorname{sign} v_R(t) \neq \operatorname{sign} v_L(t)$ for all $t \in [0,\infty)$. In this case, (A.7) follows since $\ln \rho_L(t) - \ln \rho_R(t)$ is constant and since $|v_R(t) - v_L(t)|$ is decreasing for $t \in [0,\infty)$.

For the other case, assume $\operatorname{sign} v_L(t) = \operatorname{sign} v_R(t) > 0$ for all $t \in [0,\infty)$ (the case $v_L(t), v_R(t) < 0$ is similar). Refer to Figure A.1 to see that if $u_R(t)$ lies in

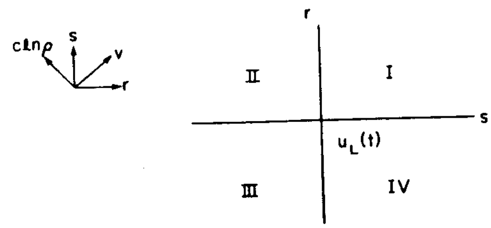

Figure A.1

quadrant II or quadrant IV with respect to $u_L(t)$, then

$$L[u_L(t), u_R(t)] \equiv |\ln \rho_L(t) - \ln \rho_R(t)|$$
$$= |\ln \rho_L - \ln \rho_R| \leqq L[u_L, u_R].$$

Finally, suppose that $u_R(t)$ lies in quadrant I (respectively quadrant III) with respect to $u_L(t)$. By (A.4) and (A.5) it follows that if u_{I} (respectively u_{III}) is in quadrant I (respectively III) with respect to u_L, then

$$|H(u_{\mathrm{III}})| \geqq |H(u_L)| \geqq |H(u_{\mathrm{I}})|. \tag{A.8}$$

Thus, we conclude that $u_R(\zeta)$ is in quadrant I (respectively III) for all earlier times $\zeta \in [0,t]$. But, by (A.8), $v_R(\zeta) - v_L(\zeta) \geqq 0$ (respectively $v_L(\zeta) - v_R(\zeta) \geqq 0$) for all times $\zeta \in [0,t]$. This implies that $L[u_L(\zeta), u_R(\zeta)]$ is decreasing throughout $[0,t]$, so that (A.7) follows for this final case.

Appendix B

In this appendix, we give the proof of Lemma 4.4.

LEMMA 4.4. *If* $0 < i < N$, *then*

$$|\Delta R^l| \leqq \tfrac{1}{2} D_{ij}^2, \qquad l = 1,2. \tag{B.1}$$

Proof: We treat the case $l = 2$. It suffices to consider the case when only two 1-shocks, S_1^1 and S_2^1, enter Δ_{ij}, cf. [13]. In this case, a 1-shock, S_{out}^1, and 2-rarefaction, R_{out}^2, leave Δ_{ij}. Without loss of generality, assume $|S_1^1| \geqq |S_2^1|$. In order to obtain (B.1), we must first estimate derivatives on 1-shock curves. So, if $v = v(z)$ is the velocity along the 1-shock curve that starts at u_L, then, by (2.9),

$$v - v_L = c(e^{-z/2} - e^{z/2}),$$
$$z = \ln \rho_L - \ln \rho_R \leqq 0.$$

Thus,

$$\frac{dv}{dz} = -c \cosh \frac{1}{2} z. \tag{B.2}$$

We estimate dv/dz using the following technical lemma:

LEMMA B.1. *If $w \geqq 0$, then*

$$\tfrac{1}{2} w \geqq 1 - \operatorname{sech} w. \tag{B.3}$$

Proof: Let $P(w) = \frac{1}{2} w - \{1 - \operatorname{sech} w\}$. If $P'(w)$ is non-negative, then P is increasing; and since $P(0) = 0$, this would imply that $P(w) \geqq 0$ for $w \geqq 0$. Now

$$P'(w) = \tfrac{1}{2} - \operatorname{sech} w \tanh w.$$

But it is easily verified that the maximum of $\operatorname{sech} w \tanh w$ is $\frac{1}{2}$, so that $P'(w) \geqq 0$.

Now consider the interaction of two 1-shocks diagrammed in Figure B.1. Here

$$\begin{aligned} A &= cR^2_{\text{out}}, \\ C = cS^1_2 \leqq D &= |v(S^1_{\text{out}}) - v(S^1_1)|, \\ B &= c|S^1_{\text{out}} - S^1_1| = C - A, \end{aligned} \tag{B.4}$$

(recall that we use the same notation to denote both a wave and its strength). By (B.2),

$$\frac{B}{D} = \eta \geqq \left| c^{-1} \frac{dz}{dv}(2S^1_1) \right| = \operatorname{sech} S^1_1, \tag{B.5}$$

where $|(dz/dv)(2S^1_1)|$ denotes the derivative dz/dv of the 1-shock curve starting

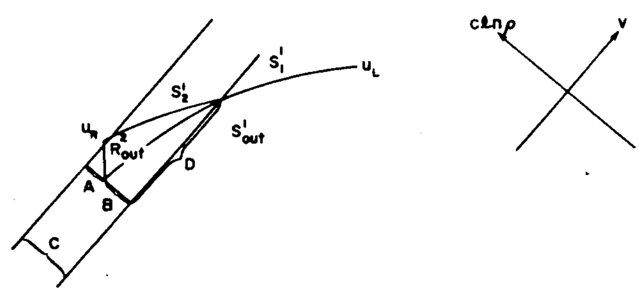

Figure B.1

at u_L, evaluated at $z = 2S_1^1$. But

$$\frac{A}{C} = 1 - \frac{B}{C} \leqq 1 - \frac{B}{D} = 1 - \eta.$$

Thus, by (B.4) and (B.5),

$$R_{\text{out}}^2 \leqq (1-\eta)S_2^1 \leqq \{1\text{-sech}\, S_1^1\}S_2^1. \tag{B.6}$$

In view of Lemma B.1,

$$\{1 - \text{sech}\, S_1^1\} \leqq \tfrac{1}{2}S_1^1, \tag{B.7}$$

so that substituting (B.7) into (B.5) one obtains

$$R_{\text{out}}^2 \leqq \tfrac{1}{2}S_1^1 S_1^2 = \tfrac{1}{2}D_{ij}^2.$$

This completes the proof of Lemma 4.4.

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