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# THE LARGE TIME EXISTENCE OF PERIODIC SOLUTIONS FOR THE COMPRESSIBLE EULER EQUATIONS

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### Abstract

We demonstrate the existence of solutions to the full $3 \times 3$ system of compressible Euler equations in one space dimension, up to an arbitrary time $T > 0$, in the case when the initial data has arbitrarily large total variation, and sufficiently small supnorm. The result applies to periodic solutions of the Euler equations, a nonlinear model for sound wave propagation in gas dynamics. Our result extends Glimm's celebrated 1965 theorem to the case of large total variation, and our analysis establishes a growth rate for the total variation that depends on a new length scale $d$ that we identify in the problem. This length scale plays no role in $2 \times 2$ systems, (or any system possessing a full set of Riemann coordinates), nor in the small total variation problem for $n \times n$ systems, the cases originally addressed by Glimm. Recent work by a number of authors has demonstrated that when the total variation is sufficiently large, solutions of $3 \times 3$ systems of conservation laws can in general blow up in finite time, (independent of the supnorm), due to amplifying instabilities created by the nonvanishing "Lie brackets" of the vector fields that define the elementary waves. For the large total variation problem, there is an interaction between large scale effects that amplify and small scale effects that are stable, and we show that for the class of systems possessing the same (unsigned) non-zero Lie bracket components as the Euler system, the length scale on which this interaction occurs is $d$. In the limit $d \to \infty$, we recover Glimm's theorem, and we show there exist linearly degenerate systems within the class considered for which the growth rate we obtain is sharp.

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## 1. Introduction

We consider the initial value problem for the system of compressible Euler equations in one space dimension,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ E_t + ((E+p)u)_x &= 0, \end{aligned} \tag{1.1}$$

where $\rho$ is the density, $u$ the velocity, $p$ the pressure, and $E$ the energy of the fluid. This is the special case of the general initial value problem for a system of conservation laws,

$$U_t + F(U)_x = 0, \tag{1.2}$$

$$U(x,0) = U_0(x), \tag{1.3}$$

where for (1.1), $U = (\rho, \rho u, E)$. System (1.1) represents the zero dissipation limit of the compressible Navier-Stokes equations. It is well known that shock-waves form in solutions of (1.1) even in the presence of smooth data, and shock-waves introduce time-irreversibility, increase of entropy (in a generalized sense), and loss of information, and this leads to the decay of solutions, c.f. [5,12].

In the case when the equation of state is of the form $p = p(\rho)$, (the case of isothermal and isentropic flow, [12]), the first two equations uncouple from the third, and system (1.1) reduces to the $2 \times 2$ system known as the $p$-system; this includes the isothermal and isentropic equations of state. There is a well developed existence theory for $2 \times 2$ systems of conservation laws, but for three equations, the only general existence theorems that apply to the full nonlinear system (1.1) are based primarily on methods of analysis first introduced by Glimm in 1965. Other methods apply only to systems of conservation laws possessing a full set of Riemann coordinates[3], (e.g. $2 \times 2$ systems), or under the assumption that the initial total variation is sufficiently small. A full set of Riemann coordinates is a coordinate system in which the the coordinate vectorfields are eigenvectors of the flux, and thus have pairwise vanishing Lie

[3]See Serre [11] for an extensive analysis of such systems which he refers to as "rich".

brackets, c.f. [12]. (See e.g., [10], where large total variation is allowed among components that are *almost planar.*) Glimm's theorem can be stated as follows:

**Theorem 1.** *(Glimm 1965) If the total variation of the initial data $U_0(x)$ is smaller than a threshold value $V_{crit}$,*

$$V_0 = TV\{U_0(\cdot)\} < V_{crit}, \tag{1.4}$$

*then a global weak solution with shocks exists for all time and*

$$TV\{U(\cdot,t)\} < CV_0, \tag{1.5}$$

*where $V_{crit}$ and $C$ depend only on the flux function $F$ in the neighborhood of the solution.*

In 1970, Glimm and Lax went on to prove that for $2 \times 2$ systems vlike the $p-$system, periodic solutions decay at a rate $O(1/t)$ in the total variation, so long as the oscillation of the initial data is sufficiently small[4]. (The oscillation is equivalent to the supnorm once an origin $\bar{U}$ is chosen.) This was a triumph for the mathematical theory of shock-waves because it provides a quantitative estimate of the dissipation present in the zero dissipation limit of gas dynamics. However, the methods of Glimm and Lax give only a *short time* existence theorem for periodic solutions of the $3 \times 3$ system (1.1), the case when the true physical entropy effects the time-irreversibility. *The long time existence problem for periodic solutions of (1.1) has remained open since that time.*

The difficulty when the total variation is large and the system does not possess a full set of Riemann coordinates, (e.g., periodic solutions of (1.1)), is that when $TV\{U_0(\cdot)\} > V_{crit}$, there exists a de-stabilizing, amplification effect due to the non-vanishing Lie brackets in the eigenfields $(\lambda_i, R_i)$ of the Jacobian matrix $dF$ :

$$dF \cdot R_i = \lambda_i R_i, \quad i = 1, 2, 3.$$

For $2 \times 2$ systems, the effect vanishes because vector fields in the plane can always be rescaled to have pairwise vanishing Lie brackets. For system (1.1),

[4]The result was extended by Zumbrun to $n \times n$ systems which possess a full set of Riemann coordinates, [18].

the Lie brackets play a dominant role. Indeed, recent well known work based on the geometrical optics approximations of system (1.2) have demonstrated that certain $3\times3$ systems of type (1.2) are *resonant* in the sense that solutions blow up in a finite time, independent of the initial supnorm bound, when $TV\{U_0(\cdot)\} > V_{crit}$, [4, 7, 8].

In 1991, Young ([15]) introduced a new method of analysis called the *method of re-orderings*, that essentially accounts for a cancelation that occurs in the waves that are created by multiple interactions of waves with a single wave when the supnorm is small. The leading order effect of interactions is quadratic because the generation of waves at interaction is based on the lack of commutivity of the vector fields $R_i$ that determine the left and right states of the elementary waves. The failure to commute is measured by the Lie brackets of the vector fields, an inherently quadratic quantity. By accounting for the cancelation, the method of re-orderings demonstrates that the Lie bracket errors actually contribute only *third order* to the supnorm estimate of the solution at time $t$, and based on this improved estimate, Young proved that for $n \times n$ systems, solutions generated by the Glimm scheme satisfy the following more refined estimates:

**Theorem 2.** *(Young 1991) Let $U(x,t)$ denote any solutions to which the original assumptions of Glimm apply, so that*

$$TV\{U_0(\cdot)\} < V_{crit}. \tag{1.6}$$

*Then*

$$\sup|U(\cdot,t) - \bar{U}| < C \sup|U_0(\cdot) - \bar{U}|, \tag{1.7}$$

*where $C$ depends only on values of the flux $F$ in a neighborhood of the solution.* While Theorem 2 requires $V < V_{crit}$, it gives the stability of solutions in the supnorm. Thus the method of re-orderings provides an analytical approach to estimating a cancelation (due to oscillation) that occurs in the amplification effects of the Lie brackets, and this stabilizes the supnorm.

## 2. Our New Result

We now discuss recent work of the authors in which we address the large total variation, small oscillation problem for the $3 \times 3$ Euler problem (1.1). For convenience, we study the Lagrangian version of the Euler equation, [12],

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ E_t + (pu)_x &= 0, \end{aligned} \tag{2.1}$$

where $v = 1/\rho$. For smooth solutions, (2.1) implies

$$S_t = 0,$$

where $S$ is the entropy, [12]. In this case the second eigenvalue satisfies $\lambda_2 \equiv 0$, and $S$ is also a Riemann coordinate for the second eigenvector field, $\nabla_U S = L_2$, the corresponding left eigenvector. The 2-waves are thus the *contact discontinuities.* More generally, and with the goal in mind of isolating a particular nonlinear aspect of the Euler equations, we consider any $3 \times 3$ system within the class of conservation laws that have a Riemann coordinate for the second family. It is not difficult to show that any such system will have the same (unsigned) non-zero Lie bracket components at the base state $\bar{U}$ as does the Euler system (2.1), (under a suitable normalization of the eigenvector fields, c.f. [17]). For any system in this class, we prove the following theorem that applies to solutions defined in a small enough neighborhood of the state $U = \bar{U}$ :

**Theorem 3.** *(Temple, Young) For any $V_0 > 0$, $d > 0$ and time $T > 0$, there exists an $\epsilon \equiv \epsilon(V_0, d, T)$, such that, if the initial data $U_0(\cdot)$ of the Cauchy problem (1.2) satisfies*

$$\sup |U_0(\cdot) - \bar{U}| \leq \epsilon, \quad TV(U_0(\cdot)) \leq V_0, \tag{2.2}$$

*and*

$$||U_0(\cdot)||_d < V_{crit}, \tag{2.3}$$

*then the conservation law admits a weak solution up to time $T$ with bounded supnorm and bounded total variation, and*

$$TV\{U(\cdot, t)\} \leq V_0 \exp(KT/d), \tag{2.4}$$

*where $K$ is a constant depending on the equations, and $||U_0(\cdot)||_d$ denotes the maximum total variation of the function $U_0(\cdot)$ over intervals of length $d$.*

Theorem 4 gives the existence of solutions up to an arbitrary time $T$ in the case when the initial data has arbitrarily large total variation and sufficiently small supnorm. As a corollary, we obtain the large time existence of periodic solutions of the Euler equations, a setting which can be described as the nonlinear theory of sound waves. Our proof is based on new functionals and new estimates for the Glimm scheme, the identification of a new length scale in the problem, and the introduction of a new norm, $||\cdot||_d$, which we call the $d$-norm, that is natural for the estimates on the nonlinearities required for our proof.

The length scale $d$ is defined to be the largest interval over which the total variation of the initial data is smaller than the critical total variation $V_{crit}$ required for Glimm's method. (We refine this to require only the total variation of the entropy be less than $V_{crit}$ over intervals of length $d$.) The $d$-norm of a function of bounded variation is the sup of the total variation over intervals of length $d$. It is not difficult to show that for any $\epsilon > 0$, and any function $f$ of bounded variation over $x \in \mathbf{R}$, there is a length $d$ such that

$$||f||_d < \epsilon.$$

The length scale $d$ is a new length scale in the problem that plays no role in $2 \times 2$ systems, or in $n \times n$ systems possessing a full set of Riemann coordinates, or when the initial total variation is less than $V_{crit}$. Indeed, $\exp(KT/d) \to 1$ as $d \to \infty$, and our growth rate estimate reduces to the time independent estimate obtained by Glimm in this limit, (see also [10]). For large initial total variation, our methods show that there is an interaction between large scale effects (that amplify) and small scale effects (that are stable), and the length scale on which this interaction occurs in the Euler problem is $d$. The growth rate $O(1)\exp(KT/d)$ is obtained for all systems within the class of systems that have the same nonzero Lie bracket components as that of the Euler system, and such that one of the fields has a Riemann invariant, just as Euler. In the limit $d \to \infty$, we recover Glimm's theorem, and our results are sharp in the sense that there exist linear degenerate systems within the class considered for which the growth rate $O(1)\exp(KT/d)$ is sharp.

On a mathematical level, the fundamental problem is that, in contrast to the small variation problem, in the large total variation problem the total wave strength generated by a wave at time $t = 0$ *cannot* be estimated by a *small perturbation* of the initial wave strength. Thus our analysis introduces the idea of measuring wave strengths *as they occur in the solution*, instead of by perturbation of their values initially, as was done in Glimm's original paper. To accomplish this, we estimate future wave strengths in terms of Glimm type functionals that account for only the leading order *linear* and *quadratic* effects of interactions. More specifically, the Glimm type functionals are defined at each time step as *sups* over all possible wave configurations that can be generated up to time $T$ assuming that wave strengths are given exactly by the leading order linear and quadratic effects at interaction, these effects being determined by the Lie bracket structure constants at the state $\bar{U}$ alone. Thus, estimating the difference between the full nonlinear problem and the quadratic problem at each interaction diamond of Glimm's method is reduced to the problem of the *continuity* of the functionals defined at the quadratic level; and the finiteness of the functionals at each time step of the full nonlinear problem is reduced to the problem of the *boundedness* of the functionals defined at the quadratic level. To succeed with this strategy, we must obtain supnorm estimates similar to those obtained by Glimm in the case when there exists a full set of Riemann coordinates, (and hence when all Lie bracket terms vanish), [13, 10]. For this we extend the method of re-orderings introduced by Young in [15] by defining functionals for the total variation and the supnorm, as well as a Glimm potential interaction functional $Q$, that account for linear and quadratic effects, so that the change in the functionals at interaction diamonds are at most *third* order in the full nonlinear problem. This is accomplished by incorporating into the functional, all possible "future" (Lie bracket) quadratic errors that could accumulate between the given time $t$ and the final time $T$. In this way, the leading quadratic errors are accounted for by the functionals defined in terms of the "quadratic" system that corresponds to the Euler equations – and the decrease in the potential interaction functional is enough to compensate for the third order errors due to higher order nonlinearity. The $d$-norm works perfectly for this method of analysis because it is stable under perturbation by a set of waves which sum to an arbitrarily small total variation. The third order errors introduce such waves into the scheme at each interaction diamond of Glimm's method. We conclude that the generation of wave strengths in the full nonlinear problem *is* a small perturbation of the strengths computed at the quadratic level alone, thus the *quadratic "system"* plays the same role in our argument that the *linear* system plays in Glimm's original argument.

Although we address the problem of $3 \times 3$ systems that have the same unsigned Lie bracket structure as Euler, our method of reducing the nonlinear problem to the problem of obtaining estimates at the linear and quadratic levels alone, applies to the large total variation small oscillation problem for systems of conservation laws in general. In general, the behavior of our quadratic model determines the qualitative behavior of the solution to the fully nonlinear system.

## 3. Some Details

Let $(\lambda_i, R_i)$, $i = 1, 2, 3$, denote the eigenfamilies for a $3 \times 3$ system of type (1.2) that has the same Lie bracket structure constants as (2.1) at a fixed state $\bar{U}$, and assume that the first and third eigenfamilies are genuinely nonlinear and the second is linearly degenerate, c.f. [12]. Let the components of the Lie brackets of the vector fields $R_i$ be denoted by

$$[R_i, R_j] = \Lambda_{ij}^k R_k, \tag{3.1}$$

where we assume summation on repeated up-down indices. For the Euler equations (2.1), $\lambda_1 = \sigma = -\lambda_3$ and $\lambda_2 \equiv 0$ where $\sigma$ is the sound speed, and one can show that there exists a coordinate system in a neighborhood of $\bar{U}$ in which the components take the canonical form

$$[R_i, R_j] \equiv 0 \quad \text{for} \quad i, j = 1, 3, \tag{3.2}$$

and at $U = \bar{U}$,

$$[R_1, R_2] = \Lambda R_3, \tag{3.3}$$

and

$$[R_2, R_3] = -\Lambda R_1. \tag{3.4}$$

For simplicity we take $\Lambda = 1$, $\sigma = 1$ at $\bar{U} = 0$.

Our proof has two main parts:

(I) Use the method of re-orderings to reduce the full nonlinear problem to the problem of obtaining estimates at the quadratic level.

(II) Obtain the estimates at the quadratic level.

We first discuss (II). The problem here is to obtain estimates for wave interactions to quadratic order, assuming constant Lie bracket factors. When two waves interact, the result of the interaction is to transmit waves in the same family and to scatter waves into different families. Because the shock and rarefaction curves have third order tangency in a general $n \times n$ system, (c.f. Figure 1, [12]), it follows that the leading order changes in wave strength at interaction are the quadratic changes due to nonvanishing Lie bracket components (3.0), see Figure 2. In this case, if an i-wave $\gamma^i$ interacts with a wave $\gamma^j$, then the out-going waves will have strength $\tilde{\gamma}^k$ given to quadratic order by

$$\tilde{\gamma}^k = \gamma^k + |\gamma^i||\gamma^j|\Lambda^k_{ij}, \quad \text{(no implied summation)}, \tag{3.5}$$

with a third order error of order $|\gamma^i||\gamma^j|\left(\sup|U_0(\cdot) - \bar{U}|\right)$, and $\Lambda^k_{ij}$ can be assumed to take on its (constant) value at $U = \bar{U}$. A point of clarification is in order here. The Lie bracket coefficients depend on our choice of normalization for the eigenvector fields $R_i$. A natural choice, that is independent of the coordinates of the state space, is to normalize the length of $R_i$ by the condition

$$\nabla\lambda_i \cdot R_i = 1. \tag{3.6}$$

However, this choice need not be optimal for the problem of relating wave strength changes to Lie bracket components. It is shown in [17] that by choosing

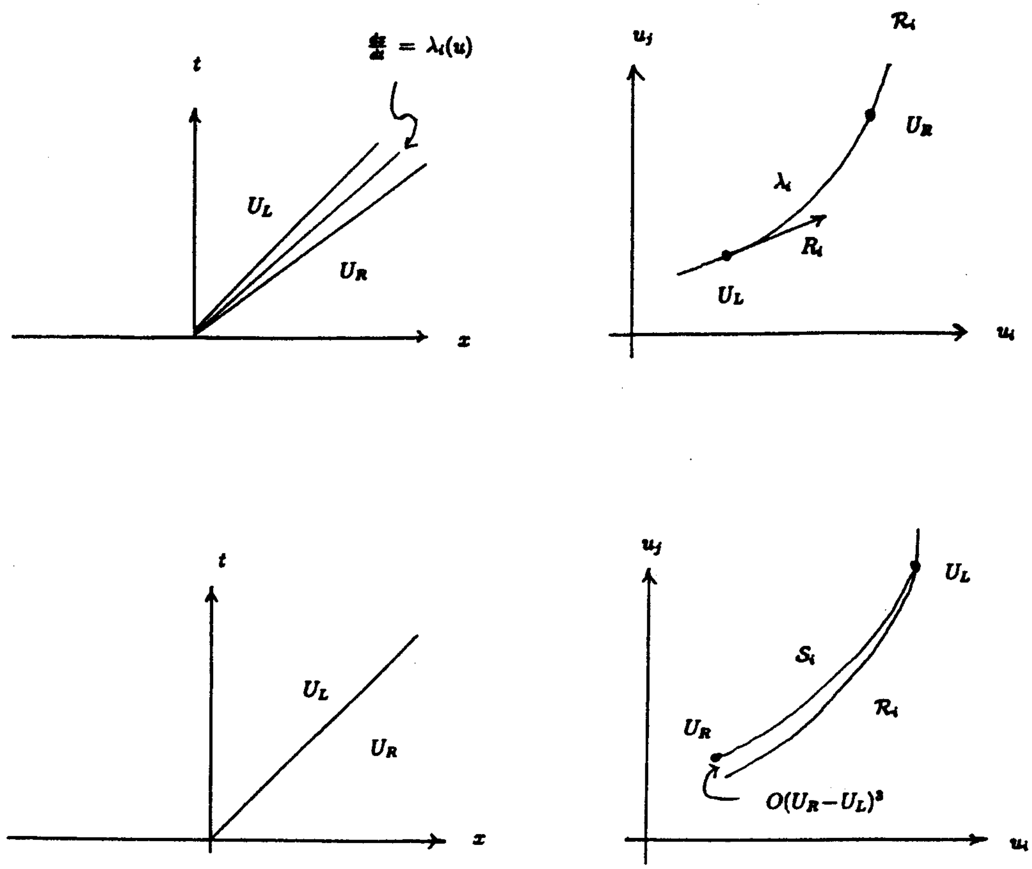


Figure 1: The third order contact between rarefaction and shock curves means that the leading order contribution to reflected waves is due to the nonvanishing Lie brackets.

the normalization of the eigenvector fields to be of unit length in a coordinate system that is as close as possible to a full set of Riemann coordinates at the point $\bar{U}$, we can arrange that, for the Euler system (2.1), the structure constants are given at $\bar{U}$ by

$$\begin{aligned} \Lambda^1_{32} &= -\Lambda^1_{23} = 1, \\ \Lambda^3_{21} &= -\Lambda^3_{12} = -1, \\ \Lambda^k_{ij} &= 0, \quad \text{otherwise.} \end{aligned} \tag{3.7}$$

In our argument we will *not* be accounting for (a further) cancelation that can occur in solutions due to the relative *signs* of the coefficients $\Lambda^1_{32}$ and $\Lambda^3_{21}$. For

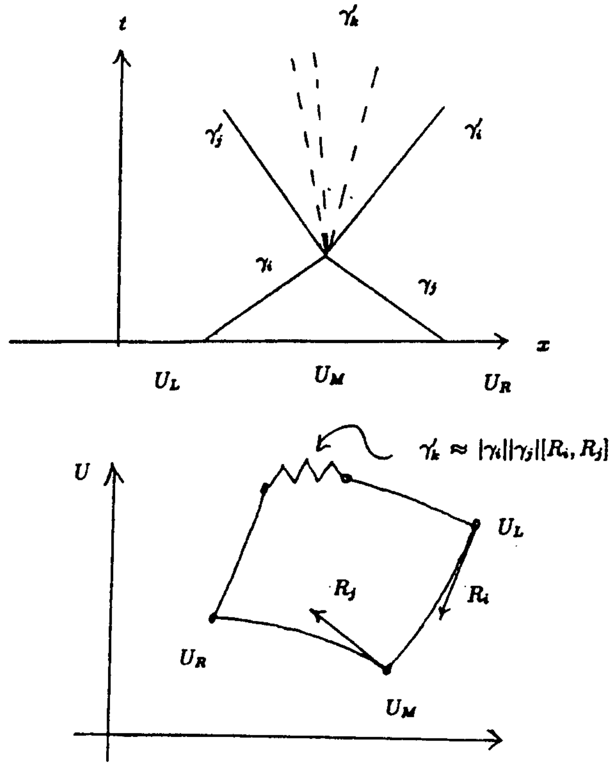


Figure 2: The interaction of the waves $\gamma_i$ and $\gamma_j$ scatter waves $\gamma_k'$ in other families.

this reason, we consider the broader class of systems for which, at the state $\bar{U}$,

$$\begin{aligned} |\Lambda_{32}^1| &= |\Lambda_{12}^3| = 1, \\ \Lambda_{ij}^k &= 0, \quad \text{otherwise,} \end{aligned} \tag{3.8}$$

which clearly includes the Euler system (3.7). Our restriction essentially includes all $3 \times 3$ systems for which any one of the coefficients $\Lambda_{32}^1$, $\Lambda_{21}^3$ or $\Lambda_{31}^2$ vanishes at the point $\bar{U}$.

Our restricting to this class of systems means effectively that we will estimate the total wave strength generated by a single wave by summing the absolute value of all the Lie bracket terms that appear at each interaction with a contact discontinuity. Thus, the cancelation due to oscillating signs of Lie bracket terms due to interaction of a single wave with consecutive waves of different families will be accounted for in our estimate for the supnorm, (i.e., we use the antisymmetry of the bracket), but cancelation will not be accounted for in our estimate for the the total variation of the solution. (This would require using more exact information about the positions of the reflected waves.)

In [16], Young constructed examples of linearly degenerate systems in conservation form in which the structure constants $\Lambda_{ij}^k$ agree with the structure constants for Euler problem (3.1) at a base state $\bar{U}$. We now discuss an example from [16] demonstrating that there exists periodic data for solutions of linearly degenerate systems satisfying (3.7a) for which the growth rate $O(1)\exp(Kt/d)$ is sharp. To this end, fix $d > 0$, and define a grid in the plane $-\infty < x < +\infty$, $t \geq 0$, by $x_i = id$, $t_j = jd$. We construct an exact, weak, periodic solution which takes constant values $U_{ij}$ on $x_i < x < x_{i+1}$, $t = t_j$, in the (worst) case of a system in which

$$\begin{aligned} \Lambda_{32}^1 &= \Lambda_{21}^3 = 1, \\ \Lambda_{ij}^k &= 0, \quad \text{otherwise,} \end{aligned} \tag{3.9}$$

$$\lambda_i = -1, 0, 1, \quad i = 1, 2, 3. \tag{3.10}$$

The following system of conservation laws satisfying (3.7), (3.8) was constructed in [16]:

$$\begin{aligned} u_t + (w + 2uv)_x &= 0, \\ v_t &= 0, \\ w_t + (u(1 - 4v^2) - 2vw)_x &= 0. \end{aligned} \tag{3.11}$$

To construct the periodic solution, let the initial data $U_{i0} = U_0$ consist of the four states $\{U_0, U_1, U_2, U_3\}$ repeated every four mesh spaces in the pattern $U_0, U_1, U_2, U_3$; i.e., $U_i = U_k$ if $i = k \text{ mod } 4, k = 0, \ldots, 3$. To define $U_0, \ldots, U_3$, choose any 2-wave with left state $U_0$ and call it $\beta$. Let

$$|\beta| \equiv \delta > 0.$$

Let $U_I$ denote the right state of $\beta$, and let $\gamma$ denote any 3-wave with left state $U_I$. Let $U_1$ be the right state of $\gamma$. Then let $U_2$ denote the state reached after starting at $U_1$ and following a 2-wave of strength $-\beta$, and to the right of that, a 3-wave of strength $-\gamma$; and let $U_3$ denote the state reached after starting at $U_2$ and following a 2-wave of strength $\beta$, and to the right of that, a 3-wave of strength $\gamma$. It is not difficult to see, then, that to within errors that are order $\delta|\gamma|$, the state $U_3 = U_1$; and to within errors that are order $\delta|\gamma|$, the state $U_1$ connects to $U_0$ on the right by a 2-wave of strength $-\beta$ followed by a 3-wave of strength

$-\gamma$. Thus, to within errors that are order $\delta|\gamma|$, the periodic initial data in the first time step consists of 2-waves of strength $\pm\beta$, speed zero, alternating in sign at every mesh point $d$ units apart in $x$, superimposed with 3-waves of strength $\pm\gamma$, alternating in sign every two mesh points. (See Figure 3.) Now, in the limit of small oscillation, $\delta \to 0$, $|\gamma| \to 0$, the leading order effects dominate, and since we are addressing the small supnorm problem, we are justified in neglecting these quadratic errors in calculating the total variation at time $T$. Thus, wlog, assume that the waves at time zero consist of 2-waves and 3-waves of strength $\delta$ and $\gamma$, respectively. Since 2-waves move at speed 0 and 3-waves at speed $+1$, the waves propagate as discontinuities that interact only at the mesh points $(x_i, t_0)$. Moreover, when the 3-waves interact with 2-waves at the first time step, to leading order, they transmit 3-waves of equal strength and reflect 1-waves of strength $\pm\delta|\gamma|$, where the sign is determined by the sign of the 2-wave times the sign of the 3-wave, according to (3.4) and (3.7). These reflected 1-waves then interact with 2-waves at mesh points located at future times, and the total variation of the solution at time $T = Nd > 0$ is the sum of the strengths of the waves generated at $t_N$, based on this procedure. (This procedure, when we include the higher order waves required to complete the Riemann problems in the data, generates an exact, periodic solution of the system (3.8a) of period $4d$.) To determine this sum, (neglecting higher order terms), let $\mathcal{P}$ denote a continuous time-like path in the $xt$-plane taking slopes $\pm 1$ as it connects successive mesh points, c.f. Figure 3. Then it is not difficult to see that each path starting at $(x_{i_0}, 0)$ and ending at $(x_{i_1}, T)$ will contribute, (at leading order), to the strength of the waves at $(x_{i_1}, T)$ by an amount $\pm\delta^k|\gamma|$, where $k$ denotes the number of reflections (changes of sign) of P between $t = 0$ and $t = T$. Although the sign is determined by a complicated formula involving the positions of the reflection points, one can verify that *this periodic problem has been set up so that there is no cancelation*, and (to leading order) it suffices to take absolute values in summing over all paths that end at the same wave at time $t = T$. (See Figure 3.) Thus, to leading order, the total strength $S_\gamma(T)$ at time $t = T$ generated by a wave $\gamma$ at time $t = 0$ can be expressed in a "path integral" form as follows:

$$S_\gamma(T) = \left|\sum_k \sum_{\mathcal{P}_k} \delta^k \gamma\right|, \tag{3.12}$$

where P denotes any path starting at the mesh position of $\gamma$ at time $t = 0$, and having $k$ reflections up to time $t = T$. The total variation of the solution up to time $T$ is therefore given, to leading order, by

$$S(T) = \sum_\gamma S_\gamma(T). \tag{3.13}$$

To estimate (3.9a), we first count the number of paths $P_k$. First note that $T = Nd$ implies that there are at most $N = T/d$ possible reflection points. Moreover, each choice of $k$ times between $t = 0$ and $t = T$ determines a unique $k$-path starting at some fixed initial wave $\gamma$. Thus, $\binom{N}{k}$ gives the number of $k$-paths up to time $T$, starting at $\gamma$. Thus we obtain

$$\begin{aligned} \left|\sum_k \sum_{\mathcal{P}} \delta^k \gamma\right| &= \sum_k \binom{N}{k} \delta^k \gamma \\ &= (1+\delta)^N \gamma \\ &= (1+\delta)^{(1/\delta)\delta N} \gamma \approx \exp\left(\tfrac{\delta T}{d}\right)\gamma. \end{aligned} \tag{3.14}$$

We are now justified in neglecting the quadratically weak waves when the supnorm is small, because, by (3.9b), they can only contribute a negligible amount to the wave strength at $\gamma$, and thus we conclude that in this example,

$$TV\{U(\cdot,T)\} \approx \sum_i \exp\left(\frac{\delta T}{d}\right)|\gamma_{i0}| = V_0 \exp\left(\frac{\delta T}{d}\right), \tag{3.15}$$

gives a sharp estimate of the total variation of the solution over intervals whose domain of dependence at $t = 0$ has total variation $V_0$.

We now discuss the problem of general initial data for the linearly degenerate systems having structure (3.7)-(3.8). For the limit of small supnorm, it suffices to assume that $\Lambda^k_{ij}$ are equal to their values at $\bar{U}$. In this case we prove that if

$$TV\{U_0(\cdot)\} = V_0 \le V, \tag{3.16}$$

we can obtain

$$TV\{U(\cdot,t)\} \le V_0 \exp\left(\frac{KT}{d}\right), \tag{3.17}$$

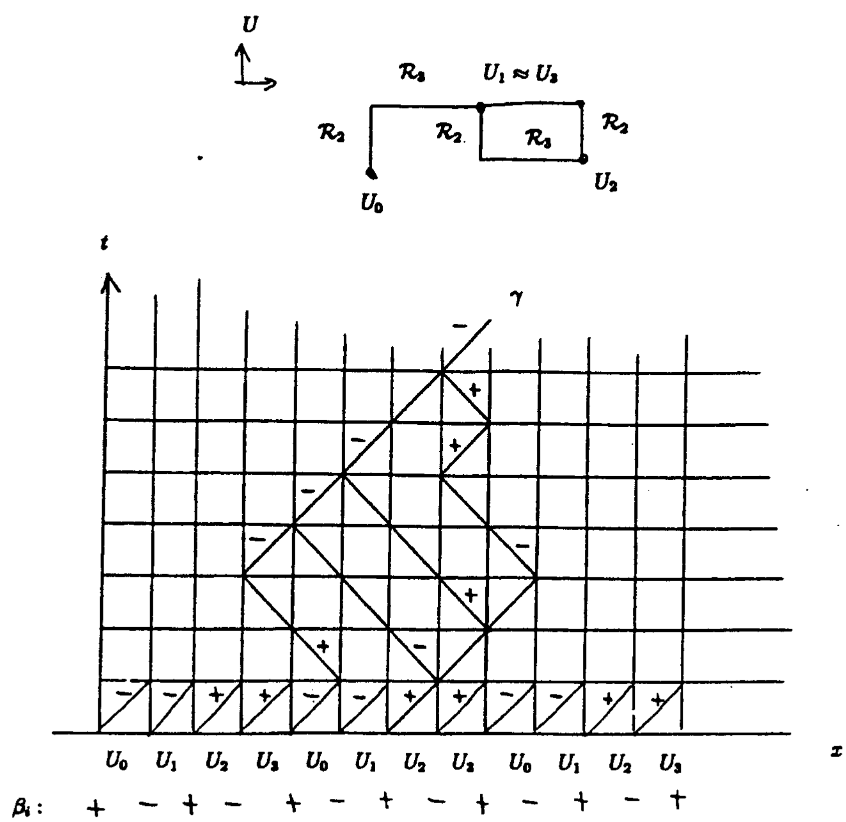


Figure 3: A periodic solution with paths contributing constructively (neglecting higher order terms)

where $d$ is the largest interval such that the total variation of the initial data $U_0(\cdot)$ over intervals of length $d$ is less than $V_{crit}$. Here we can allow the total variation bound $V$ to be arbitrarily large. Let $I_l = (ld, (l+1)d)$, $l \in Z$, denote the intervals in the $d$-mesh on $xt$-space, so that each interval has length $d$. Let $\beta_i$ denote the contact discontinuity at position $x_i$ that remains stationary for all time, and maintains a constant strength in the linearly degenerate field when we assume (3.5) is exact. The idea is to estimate clusters of consecutive waves in the intervals $I_l$ as above, bounding the generation of waves due to interactions within each cluster by the small total variation methods of Glimm. Thus, consider a wave $\gamma$ reflecting any number of times among waves $\beta_i \in 3I_l \equiv I_{l-1} \cup I_l \cup I_{l+1}$, chosen such that $\sum \beta_i < V_{crit} < 1$. Then the total strength $S(T)$

of waves generated up to time $T$ is given by

$$\begin{aligned} S(T) &= \sum_k \sum_{\mathbf{i}\in\mathcal{P}_k} \beta_{i_1}\cdots\beta_{i_l}\gamma \\ &\le \sum_k \left(\sum_{3I_l}\beta_j\right)^k \gamma \le \frac{1}{1-V_{crit}}\gamma. \end{aligned}$$

Here $\mathbf{i} = (i_1, \ldots, i_k)$ denotes the reflection positions for a $k$-path $\mathcal{P}$, and note that our normalizations have made $V_{crit} = 1$. Consider further all paths in the $d$-mesh. Any one is an equivalence class of paths on the $\Delta x < d$ mesh. If $t_1, ..., t_k$ denote the times at which the $d$-path passes through one complete interval $I_p$, then the total variation increase due to interactions in $[t_1, t_2]$ amplifies by no more than a factor of $\frac{1}{1-V_{crit}}$. Thus we can argue as follows:

$$S_\gamma(T) = \sum_k \sum_{\mathbf{i}\in\mathcal{P}_k} A_{I_1}, \cdots, A_{i_k}\gamma,$$

where the second sum is over all $d$-grid characteristics having $k$ interaction points, and $A_{i_p}$ denotes the amplification factor for all $\Delta x$-characteristics associated with a given $d$-characteristics due to interactions in $3I_p$. The $A_{i_p}$ then can be estimated by

$$A_{i_p} \le \sum_l \sum_{\mathbf{j}\in\mathcal{P}_l^p} \beta_{j_1}\cdots\beta_{j_l} \le \frac{1}{1-V_{crit}},$$

where $\mathcal{P}_l^p$ denotes the $l$-paths inside $3I_{i_p}$. Thus

$$S_\gamma(T) \le \sum_k \sum_{\mathbf{i}\in\mathcal{P}_k} 2^k \left(\frac{1}{1-V_{crit}}\right)^k \gamma. \tag{3.18}$$

The factor $2^k$ is due to the fact that on the $d$-grid, paths need not reflect, (i.e., change slope) at interaction points, thus there exist $2^k$ *more* $k$-paths than the estimate for the case when there is reflection at the interaction points. Therefore,

$$\begin{aligned} S_\gamma(T) &\le \sum_k \sum_{\mathbf{i}\in\mathcal{P}_k} \left(2\frac{1}{1-V_{crit}}\right)^k \gamma \\ &\le \sum_k \binom{N}{k} \left(2\frac{1}{1-V_{crit}}\right)^k \gamma \\ &= \left(1+\frac{2}{1-V_{crit}}\right)^{\frac{T}{d}} \gamma. \\ &\le O(1)\exp\left(KT/d\right). \end{aligned} \tag{3.19}$$

Thus we conclude that

$$TV\{U(\cdot,T)\} \le O(1)S(T) \le O(1)\exp(KT/d)V. \tag{3.20}$$

This outlines the idea in obtaining (3.11) for the linearly degenerate model (3.7), (3.8) when arbitrary initial data of total variation $V$ is given.

For the general nonlinear problem, we can reduce the problem to the argument above because the linearly degenerate problem, in which we estimate the total variation by taking absolute values of wave strengths at interaction, represents the *worst case.* The nonlinear functionals maximize over all possible future positions of waves based on an absolute bound on wave speeds, taking into account only quadratic effects. Thus, we get the worst case when we allow the largest set of re-orderings over which to maximize, this largest set representing possible wave positions up to time $T$ : thus the worst case for the functionals is when we assume the largest possible wave speed. Estimating with the largest possible wave speed reduces the analysis, (at the quadratic level), to essentially the linearly degenerate case considered above.

We now indicate the idea in reducing the full nonlinear problem to estimates for the quadratic wave interaction problem. To this end, let $\alpha_i, \beta_i, \gamma_i$ denote the $1,2,3$-waves located at $x_i$ at some time $t = t_j < T$ in the full nonlinear problem. Let $\alpha \equiv \{\alpha_i\}$, etc. We define three functionals, $L, P$ and $Q$ which evaluate a given sequence of waves, so, e.g. we write $L[\alpha,\beta,\gamma]$, etc. Specifically,

$L[\alpha,\beta,\gamma]$ = the total possible wave strength generated at time $T$ starting with $\alpha,\beta,\gamma$ at time $t$, obtained by maximizing over all possible re-orderings of waves up to time $T$, and quadratic effects are included.

$P[\alpha,\beta,\gamma] = \sup\left|\sum_{i_2}^{i_1}\alpha'_i\right| + \sup\left|\sum_{i_2}^{i_1}\gamma'_i\right|$, where the sup is over all possible re-orderings of waves up to time $T$, and only quadratic effects are included. The sup of the sum of signed strengths of consecutive waves in the same family is equivalent to the supnorm, and increases only 3rd order at interactions.

$Q[\alpha,\beta,\gamma] = \sup \sum_{App} \epsilon_i \epsilon_j$, where the sup is over all possible approaching waves that can be created by re-orderings of waves up to time $T$, when quadratic effects alone are included.

Here $L$ measures the total variation, $P$ the supnorm, and $Q$ decreases at interaction by third order terms that bound the increase in $L$ and $P$ in the full nonlinear system. It is important that the functionals are defined entirely in terms of the quadratic model; and since we are maximizing over all possible re-orderings of waves consistent with a bounded wave speed, the nonlinearity of the wave speeds does not affect this analysis – from the point of view of the analysis of the functionals, we might as well assume all wave speeds are constant at the maximum possible actual value. If the functionals are finite on a finite sequence of waves then we say that the functionals are *bounded*; and if the functionals evaluated on nearby sequences of waves are bounded by the total variation of the difference between the wave sequences, then we say that the functionals are *continuous*. We prove that all of these functionals are bounded and continuous for the class of Euler type systems studied here. We then prove that, across an interaction diamond in the full nonlinear problem, the following estimates hold, c.f. [2]:

$$L(J_+) - L(J_-) \leq K_L S D, \tag{3.21}$$

$$P(J_+) - P(J_-) \leq K_P S D, \tag{3.22}$$

$$Q(J_+) - Q(J_-) \leq -(1 - K_Q S) D, \tag{3.23}$$

and

$$F(J_+) - F(J_-) \leq K_F S D, \tag{3.24}$$

where $F(J) = L(J) + KQ(J)$ for $K$ sufficiently large. These estimates enable us to follow the proof strategy outlined in [13] to obtain the exponential growth in $L$ when the supnorm is sufficiently small.

As a final comment, see [1] for an interesting new regularity result for solutions generated by Glimm's method that is related to the problem of the stability of solutions in the $d$-norm.

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