Shock-Waves and Irreversibility in Einstein's Theory of Gravity

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1 Introduction

In this article we discuss the issues, and summarize our recent work on the construction of a class of spherically symmetric shock-wave solutions of the Einstein gravitational field equations for a perfect fluid. Our construction entails matching Friedmann-Robertson-Walker (which we refer to as (R-W), c.f. [18], page 412) to Oppenheimer-Tolman[3] (O-T) type metrics at shock-wave interfaces across which the gravitational metric is Lipschitz continuous. In our dynamically matched solution, we imagine the R-W metric as an exploding inner core (of a star or the universe as a whole), and the boundary of this inner core is a shock surface that is driven by the expansion behind the shock into the static O-T solution, which we imagine as the outer layers of a star, or the outer regions of the universe. Our solution addresses the problem first posed by Oppenheimer-Synder (O-S) in 1939 of extending their solution to the case of non-zero pressure. The O-S model is obtained by matching the R-W metric to the *empty* space Schwarzschild metric, and since in this case mass and momentum cannot cross the interface, (which in this case models the surface of a star), and keep the outer solution empty, O-S must make their well known (unphysical) assumption that the pressure be identically zero inside the star. In the classical theory of shock-waves, the interface in the O-S solution is a contact discontinuity, and this means that the solution is time-reversible. In contrast, our shock-wave solution in which $p \neq 0$, is an irreversible solution of the Einstein gravitational field equations in which the irreversibility, loss of information, and increase of entropy in the fluids puts time-irreversibility into the dynamics of the gravitational field:

The shock-waves we identify are described by an autonomous system of ODE's that determine the shock position and the R-W pressure for any given O-T solution. In this article we discuss the derivation of this system of ODE's, as well as an explicit solution of the ODE's that models a shock-wave exploding into a *static, singular, isothermal sphere.* The outer static solution for an isothermal sphere has an inverse-square density profile, and this solution is important in theories of how stars form from gaseous clouds, [1]. The idea is that a star begins as a diffuse gas cloud which slowly contracts under its own gravitational force, until it reaches the point where the mean free path for transmission of light is small enough that the scattering of light within the cloud has the effect of equalizing the temperature

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[3]We choose this as a name for static, spherically symmetric metrics that solve the Einstein equations for a perfect fluid. It appears that O-T solutions have not been given a name in the literature, and we consider this name to be appropriate, c.f., [17]. In the special case when the density is constant, this metric is commonly referred to as the *Interior Schwarzschild* metric.

within the cloud. The gas then drifts toward the static solution for an isothermal equation of state, namely, a static, singular, isothermal sphere. Since this solution is inverse square in the density and pressure, the density tends to infinity at the center of the sphere, and this ignites the thermonuclear reactions. The result is a shock-wave explosion emanating from the center of the sphere. Our explicit solution is an exact, general relativistic model for this shock-wave explosion. In this solution, both the inner R-W sound speed $\sqrt{\sigma}$ and the outer O-T sound speed $\sqrt{\bar{\sigma}}$ are assumed to be constant, and are related by an algebraic relation of the form $\bar{\sigma} = H(\sigma) < \sigma$, which we interpret as describing isothermal fluids at different temperatures. (We assume the speed of light $c = 1$.) In this model, the shock-wave emerges from a singularity that represents the initial *Big Bang* in the R-W metric, and so this model provides a scenario by which the universe begins with a shock-wave explosion. We observe that in this explicit solution, the shock speed remains everywhere less than the speed of light so long as the R-W sound speed $\sqrt{\sigma}$ satisfies $\sigma < \sigma_2 \equiv \sqrt{5}/3 \approx .745$; and the Lax characteristic condition holds so long as $\sigma < \sigma_1 \approx .458$. When $\sigma_1 < \sigma < \sigma_2$, the shock speed becomes supersonic with respect to the fluid on both sides of the shock, and tends to the speed of light as $\sigma \to \sigma_2$.

2 General Discussion

The Einstein gravitational field equations can be written in the compact form

$$G = \kappa T, \tag{1}$$

where G denotes the Einstein curvature tensor, T denotes the stress energy tensor, $\kappa = 8\pi/c^2$, and c denotes the speed of light. In Einstein's theory, all properties of the gravitational field are determined by a Lorentzian metric g, (a non-degenerate symmetric 2-tensor of signature $(-1,1,1,1)$), and equation (1) describes the time evolution of g, together with the sources in T, *simultaneously.*

In a given coordinate system $x = (x^0, ..., x^3)$ defined on spacetime, the components of G are given by

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}, \tag{2}$$

where the Ricci tensor R_{ij} and the scalar curvature R are obtained from the Riemann curvature tensor R^i_{jkl}. In the important case of a perfect fluid, the stress energy tensor takes the form

$$T_{ij} = (p + \rho c^2)u_i u_j + p g_{ij}, \tag{3}$$

where ρ is the mass-energy density of the fluid, (as measured in a frame moving with the fluid), p is the pressure and u^i denote the components of the unit 4-velocity vector u of the fluid. When an equation of state for the fluid $p = p(\rho)$ is specified, system (1) imposes ten equations in 14 unknowns: the ten unknown metric components g_{ij}, the density ρ, and three of the four components of the unit vector u. Four remaining equations can then be imposed to specify a coordinate system.

It is well known that G satisfies the fundamental identity

$$div G = 0, \tag{4}$$

where div denotes the covariant divergence for the unique symmetric connection Γ which is compatible with the metric g. The Einstein theory then has the remarkable property that conservation of energy and momentum hold *automatically* on solutions of the field equations (1). For example, in the case of a perfect fluid, if g, ρ, u solve (1), then the conservation law

$$divT = 0 \tag{5}$$

must hold as a consequence of (4). In the case when g is identically equal to the flat Minkowski metric η, div reduces to the ordinary divergence, and (3), (5) reduce to the relativistic Euler equations, [14], which in turn reduce to the classical compressible Euler equations for an inviscid perfect fluid in the limit of small velocities.

One important issue in the subject of general relativity is the problem of explaining how time irreversibility should enter the theory, [12]. Indeed, all of the exact solutions of the Einstein equations (1) for classical sources assume the stress tensor for a perfect fluid, or for electro-magnetism, and it is well known that smooth solutions of the Einstein equations with these sources are time reversible. However, *it is our thesis here that time-irreversibility has been built into the theory of general relativity from the very start by way of the stress tensor for a perfect fluid, and that the irreversibility, loss of information and increase of entropy in solutions of (1) is encoded in the properties of the shock-waves.*

The idea is that, in the case of a perfect fluid (3), the relativistic compressible Euler equations appear as a subsystem of the Einstein gravitational field equations (1) through the identity (5), and the relativistic Euler equations form a system of nonlinear hyperbolic conservation laws, the setting for the mathematical theory of shock-waves. Therefore, like the classical compressible Euler equations, solutions of the Einstein equations should be time reversible only up until the formation of shock-waves. For the classical Euler equations, the dissipation and increase of entropy at shock-waves can be interpreted physically as the non-negligible effects of viscosity that must appear because the inviscid Euler equations represent a *singular limit* of the Navier-Stokes equations as the viscosity is taken to zero: however small the viscosity parameters, the dissipative effects of viscosity are not negligible in the surface layer of a shock, and this dissipation is fully accounted for in the shock-waves that appear in the zero viscosity limit.

Unfortunately, shock-waves are exceedingly complicated in the setting of general relativity. For one thing, there is no simple model like the classical Navier-Stokes equation that plays a similar role in relativistic theories, partly because the introduction of second order parabolic terms to model dissipation introduces an infinite speed of propagation similar to that observed in the linear heat equation. Thus the classical interpretation of the Euler equations as the *zero viscosity limit* of the Navier-Stokes equations is not so easily made for the Einstein equations. In the setting of 4-dimensional spacetime, the mathematical theory of the classical Euler equations is woefully incomplete. The theory of conservation laws and shock-waves is well understood mathematically only in the case of one space-one time dimension, but we believe that this theory does set out the *role* we expect shock-waves to play in the theory of the Einstein equations for a perfect fluid. For a classical system of conservation laws in one space-one time dimension, it was shown by Lax [6] that shock-waves always form when the solution is compressive. Lax also made the point that irreversibility, loss of information and increase of entropy could be attributed to a shock-wave in a general system of conservation laws based on the fact that characteristics always impinge on shock- waves. In this setting a celebrated existence theory has been given by Glimm [3], and a rather complete analysis of the irreversibility of shock-waves is described in the work of Glimm, Lax, Diperna and Liu [13] in which the time asymptotic decay of solutions to shock-waves is rigorously demonstrated. In fact, in one space dimension, all solutions with the same values at spatial infinity, (at $t = 0$), will time asymptotically decay to the same solution. The conclusion we draw from the mathematical theory of shock-waves is that the pure conservation law $divT = 0$ contains *all of the information* about the zero viscosity limit, that in this limit irreversibility and dissipation are encoded in the structure of the shock-waves, and that shock-waves will always form in solutions except in very special cases when the fluid is everywhere expansive. We believe that this conclusion carries over to the Einstein equations (1) for a perfect fluid (3). Indeed, in locally Lorentzian coordinates at a point, (coordinates in which $g_{ij} = \eta_{ij} = diag(-1, 1, 1, 1)$, and $g_{ij,k} = 0$ for all $i, j, k = 0, ..., 3$), the covariant divergence equals the classical divergence; and thus, to within higher order errors, the perfect fluid in general relativity satisfies the relativistic Euler equations at each point. Thus if shock-waves were to never form in T, then in essence, the gravitational field would have the effect of removing all of the compressive action of the Euler equations, and thereby would have the effect of eliminating the dissipation in fluids! This we find physically impossible, and thus we maintain that shock-waves are as fundamental to solutions of (1), (3) as they are to the classical Euler equations. This suggests that when viscosity and heat conduction are neglected, (as they are in the case of a perfect fluid), it is the shock-waves in the sources T that fundamentally introduce irreversibility into the dynamics of the gravitational field. This point of view seems to have essentially been overlooked, [12, 5]. In [15], we organize the general theory of matching two metric solutions of (1) across an interface from the point of view of the theory of shock-waves, and we apply this theory to explicitly construct spherically symmetric shock-wave solutions of the Einstein equations. This construction solves the problem first posed by Oppenheimer and Snyder in 1939 of generalizing their model to the case of nonzero pressure.

If the fluid variables shock, the entries in the stress tensor T become discontinuous, and thus by the Einstein equation, the metric loses continuity in the second derivative. In Section 3 of our paper [15], we present a self-contained treatment of the general theory of shock-waves for the Einstein equations by considering the general problem of allowing the metric to be only Lipschitz continuous across a hypersurface Σ in spacetime. This treatment re-organizes and extends previous results of Israel. (See also [9] and references therein where this topic is referred to by the heading *Junction Conditions*). The result is the the following theorem:

Theorem 1 *Let Σ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector* $\mathbf{n}$. *Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are all equivalent:*
(i) $[K] = 0$ at each point of Σ.

(ii) The curvature tensors R^i_{jkl} and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.

(iii) There exist locally Lorentzian coordinate frames at each point $P \in \Sigma$.

(iv) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates.

Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions

$$[G]_i^\sigma n_\sigma = 0, \tag{6}$$

(which express the weak form of conservation of energy and momentum across Σ when $G = \kappa T$), hold at each point on Σ, a covariant statement.

Here $[K]$ denotes the jump in the second fundamental form (extrinsic curvature) K across Σ, (this being determined by the metric separately on each side of Σ because g_{ij} is only Lipschitz continuous across Σ), and by $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous. Theorem 1 should be credited mostly to Israel, who obtained these results in Gaussian normal coordinates. Our contribution was to identify the covariance class of $C^{1,1}$ transformations, and to thereby obtain precise coordinate independent statements for (ii)-(iv). As a consequence of this, we obtain the result that the Ricci scalar curvature R *never* has delta function sources at a Lipschitz continuous matching of the metrics, as well as the following theorem that partially validates the statement that shock-wave singularities in the source free Einstein equations $R_{ij} = 0$ or $G_{ij} = 0$ can only appear as coordinate anomalies, and can be transformed away by coordinate transformation:

Theorem 2 *If a smooth shock surface Σ forms in weak solutions of $R_{\alpha\beta} = 0$ or $G_{\alpha\beta} = 0$ posed in some given coordinate system y, such that the y-components $g_{\alpha\beta}$ of the metric tensor g are Lipschitz continuous across Σ, and C^k functions of y on either side of Σ (continuous up to the boundary on either side separately), then there exists a regular $C^{1,1}$ coordinate transformation taking $y \to x$, such that, the components g_{ij} of g in x-coordinates are actually C^k functions of x in a neighborhood of each point on the surface Σ.*

Note that when there are delta function sources in G on a surface Σ, the surface should be interpreted as a surface layer (because $G = \kappa T$), and not a shock-wave, [4, 9]. In [15] we show that for spherically symmetric solutions, the weak form of conservation of energy and momentum (6) implies the absence of surface layers, so long as the areas of the spheres of symmetry match smoothly at Σ. We use this result in our construction of the shock-waves that extend the Oppenheimer-Snyder model to the case of non-zero pressure.

3 An Exact Shock-Wave Solution

We now discuss how we explicitly construct spherically symmetric shock-wave solutions of the Einstein equations (1) for a perfect fluid by matching the Robertson-Walker (R-W) metric,

$$(R-W) \qquad ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\}, \tag{1}$$

and Oppenheimer-Tolman (O-T) metric,

$$(O-T) \qquad d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + \left(1 - \frac{2M(\bar{r})}{\bar{r}}\right)d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{2}$$

at shock-wave interfaces across which the metrics are Lipschitz continuous, and the conditions (i)-(iv) of Theorem 1 above hold. Here, the quantity $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$ denotes the standard metric on the 2-sphere. The R-W metric is homogeneous at each fixed t, and expands or contracts in time according to the Cosmological Scale Factor $R(t)$. The fluid is assumed to be co-moving with the metric, c.f. [18], and the Einstein equations (1) reduce to a system of two ODE's in two unknowns when an equation of state $p = p(\rho)$ is specified. The R-W metric has been accepted as a model for the universe as a whole. In this interpretation, $R(t)$ determines the red-shift factors for far away objects, and the existence of a singularity in the R-W metric in backwards time has been interpreted as the original *Big Bang*. The O-T metric is a time independent solution of (1) that is used to model the interior of a star. In this case the stress tensor is again taken to be that of a perfect fluid that is co-moving with the metric, and the functions $B(\bar{r})$ and $M(\bar{r})$, (the total mass inside radius $\bar{r}$), are also determined by an autonomous system of two equations in two unknowns when an equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is chosen, and the Einstein equations (1) are imposed.

The idea in the matching is to define a coordinate transformation mapping $(t, r) \to (\bar{t}, \bar{r})$ such that, under this identification of coordinates, the metrics R-W and O-T agree Lipschitz continuously on a 3-dimensional shock surface. We set $\bar{r} = Rr$ in order that the spheres of symmetry match smoothly at the surface, and with this it remains to define $\bar{t} = \bar{t}(t, r)$. The following general theorem states that a Lipschitz continuous matching of metrics can be achieved for arbitrary R-W and O-T metrics,[15]

Theorem 3 *Let $B(\bar{r}), M(\bar{r}), \bar{\rho}(\bar{r}), \bar{p}(\bar{r})$ denote any O-T solution of the Einstein equations (1), and let $R(t), \rho(t), p(t)$ denote any R-W solution of (1). Then, (under non-degeneracy assumptions, [15]), there exists a coordinate mapping $(t, r) \to (\bar{t}, \bar{r})$ of the form $\bar{r} = Rr$ and $\bar{t} = \bar{t}(t, r)$ such that, under this identification, the metrics agree and are Lipschitz continuous at the shock surface*

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3. \tag{3}$$

The equation (3) defines the shock surface $\bar{r} = \bar{r}(t)$ implicitly so long as $\bar{\rho}'(\bar{r}) < 0$.

Note that for this theorem, arbitrary equations of state $p = p(\rho)$ and $\bar{p} = \bar{p}(\bar{\rho})$ can be assigned. The identity (3) can be interpreted as a global conservation of mass principle for such matchings: in words, (3) says that the total mass that the O-T solution would see inside the shock interface $\bar{r} = \bar{r}(t)$ if it were continued on into the origin $\bar{r} = 0$, equals the total mass inside a sphere of radius $\bar{r} = \bar{r}(t)$ and constant density $\rho(t)$.

A further constraint on the metrics must be imposed to ensure that the Rankine-Hugoniot jump conditions (6) for conservation hold across a shock. By the general theory, it suffices to match the second fundamental forms $[K] = 0$ across the surface, and this alone will ensure that the surface is not a surface layer, and the conditions of Theorem 1 will then all apply at the shock surface. In [15] we show that in the presence of spherical symmetry, $[K] = 0$ is equivalent to the single condition

$$[T^i_j]n_i n^j = 0. \tag{4}$$

Using this, we derive a system of two autonomous ordinary differential equations in the shock position $r(t)$ and the cosmological scale factor $R(t)$ that determine the R-W metrics that will match any given O-T metric at the shock surface (3) such that energy and momentum are conserved across the shock. In order to impose the extra constraint (4), we must allow the R-W pressure $p(t)$ to be determined by the equations, rather than by an equation of state. This is summarized in the following theorem,[15]:

Theorem 4 *Let $B(\bar{r}), M(\bar{r}), \bar{\rho}(\bar{r}), \bar{p}(\bar{r})$ denote any fixed O-T solution of the Einstein equations. Then, (under non-degeneracy assumptions), the R-W metric $R(t), \rho(t), p(t)$ will satisfy conservation across the shock surface (3), (under the identification $\bar{r} = Rr, \bar{t} = \bar{t}(t,r)$), if and only if $(r(t), R(t))$ solve the system of ODE's*

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - k, \tag{5}$$

$$\alpha \dot{r}^2 + \beta \dot{r} + \gamma = 0, \tag{6}$$

where all of the functions $\rho, \alpha, \beta, \gamma$ appearing in (5) and (6) can be expressed in terms of the unknowns $(r(t), R(t))$ through the shock surface equation (3) and the identity $\bar{r} = Rr$. The R-W pressure is then given by

$$p = -\frac{\frac{d}{dt}(\rho R^3)}{3R^2\dot{R}}. \tag{7}$$

Thus, for any given O-T solution, system (5), (6) gives the conservation constraint in terms of a first order *autonomous* ODE in the unknowns $(r(t), R(t)$, and the solutions of this ODE determine the R-W metrics that will match a given O-T metric across a shock surface, such that the additional condition of conservation is maintained across the shock. The solutions that solve this ODE reduce to the Oppenheimer-Snyder solutions when M is constant and the O-T metric reduces to the empty space Schwarzschild metric. In the O-S case, the solution of (5), (6) reproduces $p \equiv 0$. Note that for a given O-T solution, we can, in principle, construct a shock-wave at radius $\bar{r} = Rr$ with arbitrary R-W pressure p assigned at that radius by specifying the appropriate initial conditions for r and R.

We obtain an explicit solution of the ODE's (5), (6) when we take $\bar{p} = \bar{\sigma}\bar{\rho}$, for $\bar{\sigma} \equiv const. < c$. Substituting this into the O-T metric (2) yields the following exact solution which represents a relativistic version of a *static, singular isothermal sphere.* Let γ be defined by

$$\gamma = \frac{1}{2\pi\mathcal{G}}\left(\frac{\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}\right). \tag{8}$$

Then the following define an exact solution of O-T type:

$$\bar{p} = \bar{\sigma}\bar{\rho}, \tag{9}$$

$$\bar{\rho}(\bar{r}) = \frac{\gamma}{\bar{r}^2}, \tag{10}$$

$$M(\bar{r}) = 4\pi\gamma\bar{r} \tag{11}$$

$$A(\bar{r}) = 1 - 8\pi\mathcal{G}\gamma \equiv const, \tag{12}$$

$$B(\bar{r}) = \bar{r}^{\frac{4\sigma}{1+\sigma}}. \tag{13}$$

Note that when $\bar{\sigma} \to 0$, we must have $A \to 1$ and $B \to 1$, which is the Newtonian limit. Moreover, $\bar{\rho} \to \infty$ and $\bar{p} \to \infty$ as $\bar{r} \to 0$, and thus this "isothermal sphere" is "singular" in the sense that it takes an infinite pressure at $\bar{r} = 0$ to hold the configuration up. When we match this solution to the R-W metric with $k = 0$, and impose conservation, we obtain that $p = \sigma\rho$ gives the equation of state in the R-W metric where

$$\bar{\sigma} = H(\sigma) \equiv \frac{1}{2}\sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2}\ sigma - \frac{7}{2}, \tag{14}$$

and the shock positions $\bar{r}(t)$ and R-W density $\rho(t)$ and scale factor $R(t)$ are given by the formulas:

$$\bar{r}(t) = \pm\sqrt{18\pi\mathcal{G}\gamma}(1+\sigma)(t-t_0)+\bar{r}_0, \tag{15}$$

$$\rho(t) = \frac{3\gamma}{\bar{r}^2}, \tag{16}$$

$$R(t) = R_0\left(\frac{\bar{r}(t)}{\bar{r}_0}\right)^{\frac{2}{3+3\sigma}}. \tag{17}$$

Here the + in (15) corresponds to the out-going shock-wave, which is physical because $\rho(t) = 3\bar{\rho}(\bar{r}) > \bar{\rho}(\bar{r})$ on the shock surface $\bar{r} = \bar{r}(t)$. Note also that all functions in (15)-(17) are functions of $\bar{r}(t)$, and thus we see immediately that there is a singularity in this solution in backward time $t = t^*$,

$$t^* = t_0 - \frac{\bar{r}_0}{\sqrt{18\pi\mathcal{G}\gamma(1+\sigma)}}, \tag{18}$$

at which point $\bar{r}(t^*) = 0$, $R(t^*) = 0$, $\rho(t^*) = \infty$ and $p(t^*) = \infty$.

Thus this explicit solution provides a scenario by which the *Big Bang* begins with a shock-wave explosion. We observe that the shock-wave solution (15)-(17) removes the singularity in the O-T solution for $\bar{p} = \bar{\sigma}\bar{\rho}$ after time $t = t^*$, and we interpret this as saying that *the shock-wave provides the pressure required to hold the O-T configuration up.* Also note that the Newtonian limit is acheived as $\sigma, \bar{\sigma} \to 0$.

It is not difficult to show that $H'(\sigma) > 0$ for $0 < \sigma < 1$, and that

$$\bar{\sigma} = \frac{3}{7}\sigma + O(\sigma^2)$$

as $\sigma \to 0$. Moreover, $\bar{\sigma} = \sqrt{17} - 4 \approx .1231$ when $\sigma = 1/3$; and $\bar{\sigma} = \frac{\sqrt{112}}{2} - 5 \approx .2915$ when $\sigma = 1$.

A further calculation shows that the shock speed relative to the R-W fluid particles is a constant, depending only on σ and independent of $\bar{r}(t)$. The shock speed tends to the speed of light $c = 1$ as $\sigma \to \sigma_2 = \sqrt{5}/3 \approx .745$, and the Lax shock conditions hold for $0 < \sigma < \sigma_1 \approx .458$. When $\sigma_1 < \sigma < \sigma_2$, the shock-wave is supersonic relative to the fluid on both sides of the shock. The solution (15)-(17) represents one very special solution of (5)-(6) in which $\bar{r}$ is always larger than the Schwarzschild raduis for the mass $M(\bar{r})$.

4 Conclusion

Our shock-wave solution opens up intriguing possibilities in cosmology, as well as in the classical theory of shock-waves. For cosmology, our solution opens up the possibility (with an *explicit* model) that there is a shock-wave at the edge of the universe. Consequently, the 2.7 degree Kelvin background radiation could well be due to the blackbody radiation from the surface of this shock-wave. This suggests the possibility of using our explicit formulas, (together with radiation conditions), to obtain interesting new relations between the speed of the shock, its position, and the average energy density in the universe. Moreover, our dynamical matched solution need not have the same type of singularity in the past as does the R-W metric because, in backward time, the shock-wave can reach $\bar{r} = 0$ at the same time that the singularity in the R-W metric forms. Thus, in our model, the nature of the *Big Bang* appears to depend on the equation of state and initial conditions taken in the outer O-T solution. It also suggests interesting implications for the Oppenheimer-Volkoff and Chandrasekhar limits for stable stars. Indeed, a well known theorem states that whenever the total mass inside the radius $\bar{r}$ in an O-T solution falls within 9/8 of the Schwarzschild radius for that mass, the solution is unstable because it takes an infinite pressure at $\bar{r} = 0$ to "hold the star up", [16]. This implies a maximum red-shift factor of two for light emitted from the surface of stable stars. Thus it has been argued that exotic stars like quasars, point sources of light which emit frequencies with red-shift factors much greater that two, must be very far away to account for the large red-shift, which leads to the conclusion that they are extremely large in order to explain their brightness. Our general shock-wave model applies to highly compressed O-T solutions allowing for arbitrary red-shift factors because the shock-wave in the core can generate the enormous pressures required to hold the star up. (This possibilty was realized in the explicit solution (15)-(17) which, however, contains no Schwarzschild singularity.) All of these scenarios can be studied within the context of our models which allow $p \neq 0$.

Our construction also has interesting implications for the classical theory of shock-waves. Our analysis demonstrates that there are remarkable simplifying features of the Einstein equations over the classical Euler equations that allow us to construct multi-dimensional shock-waves in general relativity. Indeed, the Einstein equations are obtained from the Euler equations by introducing the metric potentials g_{ij}, and it is easier to match the metric across a shock-wave than it is to solve the Rankine-Hugoniot jump relations for conservation directly. Once we match the metrics, the number of conservation constraints is reduced by one, and this enables us to assign a non-zero pressure behind the shock for *any* choice of O-T solution on the outside. The covariance of the Einstein equations enables us to obtain an explicit formula for the shock surface, as well as a global conservation of mass principle, and we do this without ever having to tackle the more difficult problem of expressing the metric components in a single coordinate system at the shock. It is also surprising that, by Theorem 1, for every shock-wave solution of $G = \kappa T$, there exists special coordinate systems in which the metric is $C^{1,1}$ across the shock. Thus in computing the shock-wave solution in these coordinates, one never has to difference a discontinuous function when one computes with the Einstein equations (1) instead of the Euler equations $divT = 0$. This suggests new possibilities for the numerical simulation of shock dynamics.

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