SOLUTIONS TO THE EULER EQUATIONS WITH LARGE DATA

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ABSTRACT. We consider the large time existence problem for 3×3 systems of conservation laws with large total variation and small oscillation. Under these conditions it is known that general solutions can blow up in finite time due to destabilizing geometric wave interaction effects. We discuss a general framework for rigorously addressing this problem, and provide a sufficient condition for existence of solutions. We then restrict to a class of systems which includes the Euler equations, having simplified wave interactions. We obtain an explicit path integral formula for approximate solutions. We then find time–dependent bounds for the total variation. The variation grows exponentially in time, and known examples show that our estimates are sharp.

1. INTRODUCTION

We consider the behavior of solutions to 3×3 systems of strictly hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{1}$$

in one space dimension x, with initial data having total variation V_0 and sup-norm $\epsilon > 0$. We address the problem of long time existence for small ϵ and large V_0. In general the time of existence of solutions is highly dependent on geometric features of the conservation law, determined by the Lie brackets of eigenvectors of the Jacobian matrix Df. For gas dynamics, symmetries arise as a consequence of the Second Law of Thermodynamics, and our primary interest here is to explore the role that such symmetries play in the large-time existence problem for the Euler equations.

Our analysis uses Glimm's method, for which compactness is based on a bound for the total variation of approximate solutions [1]. Glimm obtained this bound as long as V_0 is small, or if ϵV_0 is small when there is a full set of Riemann coordinates

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(so that Lie brackets vanish identically.) In either of these cases, solutions decay due to shock formation and interaction [2, 4], but new methods are required when the total variation is large and the Lie brackets are non-zero.

There are two different types of nonlinearity in systems of three or more conservation laws. The first is the well-known dependence of wave-speed on the solution at each point, known as genuine nonlinearity [3]. This leads to the formation of shock waves and the subsequent decay of solutions in each family. Weak solutions must be introduced to continue the solution beyond the time of shock wave formation. The other nonlinearity is the geometric nonlinear coupling of different wave families [7]. In systems of 3 equations, this coupling is quadratic, and leads to the generation of new waves of quadratic strength when waves of different families interact. We emphasize that genuine nonlinearity is stabilizing, while geometric nonlinearity is destabilizing.

In recent years some progress has been made in understanding the nature of the geometric nonlinearity, both through the method of weakly nonlinear geometric optics, and directly through Glimm's method. The most dramatic results show the resonant blow-up of solutions to general 3×3 systems. This clearly indicates that extra assumptions must be made to ensure nonlinear stability of solutions.

The method of reorderings was developed in [7] to account for quadratic effects of wave interactions, which arise due to geometric coupling between wave fields. Using this framework, a potential for the oscillation was built by including quadratic terms which correspond to future interactions. After interaction, the class of future interactions is smaller, so that the potential decreases, up to errors which are cubic. This led to a separation of the total variation and sup-norms, but those results still required that the total variation lie below some threshold. In order to remove this restriction, we must account for cumulative quadratic effects. Here we describe an extension of the method and derive an algebraic condition which is sufficient to guarantee existence of solutions.

The crucial modification is the definition of a new error potential, analogous to Glimm's, but including all cumulative quadratic effects. The defining property of this functional expresses the fact that the error in a series of interactions should be a sum of individual errors. This is a statement of 'conservation of interaction errors.' We then isolate a continuity property, needed to take higher order errors into account. The important difference between our error potential and Glimm's is that we include the contribution of each interacting pair using wave strengths at the time of interaction rather than initially, thus including all quadratic effects of earlier interactions.

Since our error functional takes quadratic terms into account, continuity implies that the error potential decreases if the sup-norm ϵ is small enough, rather than the total variation. Indeed, we show that continuity of the error potential is a sufficient condition for existence of solutions. We remark that boundedness of the functional is a consequence of continuity. Since solutions to general systems may be unbounded, the error potential cannot be continuous in general.

of systems that includes the 3×3 Euler equations. We assume that the second (entropy) field possesses a Riemann coordinate, and for convenience assume that it has constant wave-speed. This assumption greatly simplifies reordered sequences by simplifying pairwise wave interactions, and allows us to model entropy fluctuations as a background source for scattering sound waves, which does not change across reorderings.

In our quadratic model, sound waves add linearly, so that each sound wave can be treated separately. We thus examine the scattering pattern of a single sound wave passing through the fluctuating entropy field. We describe the scattering process in terms of paths of scattered waves, each given an appropriate strength. Combining these linearly, we get a path integral formula for reordered sequences. We thus have an algebraic expression for the piecewise constant approximate solutions in Glimm's scheme which includes all quadratic effects of wave interactions.

Our final task is to use the path integral formula to find bounds for the reordered sequences, and thus show that the functionals are continuous. By bounding the strengths of scattered waves and carefully counting paths, we find time-dependent bounds, with exponential growth for the total variation. We do not expect solutions of gas dynamics to have exponential growth, as the fact that the entropy is convex (which we have not used) introduces extra symmetries, leading to extra cancellation due to different paths, see [6]. On the other hand, there are known examples which satisfy our hypotheses and exhibit exponential growth [8], so that our estimates are sharp.

To find the bounds, we assume a bound for 2-wave strengths, and count the number of paths. In this paper, we present a simplified argument which yields the bound as long as the entropy remains Lipschitz. A more sophisticated path count allows this condition to be dropped, and yields new physics. This paper is a summary of results in [5], and the interested reader is referred to that paper for details.

2. Reorderings

We briefly recall the method of reorderings in Glimm's scheme, and describe the strengths of reordered waves [7]. The Glimm approximations consist of piecewise constant states separated by elementary waves, which are shocks, rarefactions or contact discontinuities. Our notation is as follows. For a fixed mesh size Δx, we consider the sequence of constant states $\{u_0, \ldots, u_n\}$ separated by the wave sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ of index $(c_1, \ldots, c_n)$. This means that the wave separating constant states u_{i-1} and u_i is a wave of strength γ_i from the c_i-th family. Here a positive strength means that the wave is a rarefaction, while a negative strength indicates a shock, and we do not distinguish between a wave and its strength. The total variation and sup-norm of the solution can then be measured by

$$V(\gamma) = \sum |\gamma_i| \quad \text{and} \quad S(\gamma) = \max |\sum_{c_i = r} \gamma_i|, \tag{2}$$

respectively, the last sum being taken over sets of consecutive waves of each family

One of the main ideas in Glimm's scheme is to use induction to reduce complicated large-time wave interaction patterns to the interaction of two Riemann solutions. If α_j and β_j, $j = 1 \ldots 3$, represent the two Riemann solutions, then the resulting Riemann solution is represented by

$$\epsilon_i = \alpha_i + \beta_i + \sum_{j>k} \alpha_j \beta_k \Lambda_i^{jk} + O(1) SD(\alpha, \beta), \tag{3}$$

where the Λ_i^{jk} are (constant) interaction coefficients measuring the strength of geometric nonlinearity, defined by the Lie bracket of eigenvector fields. The local interaction error D is defined by

$$D(\alpha, \beta) = \sum_{App} |\alpha_j \beta_k|, \tag{4}$$

the sum being over all approaching pairs of waves. Our main concern is the appropriate global extension of this error potential.

The method of reorderings is a tool for accounting for the quadratic effects of interactions in Glimm's scheme, through changes in wave sequences. Each pairwise interaction of waves is modeled by transposing the wave pair (in case they are from different families), or combining them into one (if they are of the same family). The only quadratic effect occurs when waves of different families cross, in which case new waves are generated. Instead of introducing a new wave into the model, we adjust the strength of the nearest wave of that family by the appropriate amount; this exactly models Glimm's scheme, in which the number of waves is conserved.

A surjective map $\tau : \{1, \ldots, n\} \to \{1, \ldots, m\}$ defines an action on sequences of n waves through permutation of subscripts. That is, τ acts on the wave sequence and its index by shifting the wave γ_i from position i to position $\tau(i)$. As long as domains and ranges match up correctly, we can define compositions of maps, and so we can factor maps (non-uniquely) into products of elementary maps. These are the transposition or merge of a single adjacent pair of waves.

To extract more information from the map τ, we define the *crossing set* C_τ by

$$C_\tau = \{ (j,k) \mid j < k \text{ and } \tau(j) > \tau(k) \}, \tag{5}$$

which identifies those pairs of waves which cross under the action of τ, and similarly define the *merge set*

$$M_\tau = \{ (j,k) \mid j < k \text{ and } \tau(j) = \tau(k) \}, \tag{6}$$

and the union of these as the interaction set I_τ. Using these sets, we characterize the set of 'physical' admissible reorderings, as follows. In order for a pair of waves to cross, that is to pass through each other, the wave on the left should travel faster than that on the right, so we require that $c_j > c_k$ for all pairs $(j,k) \in C_\tau$. Similarly, only waves in the same family may merge to form one wave, so we require that if $(j,k) \in M_\tau$ then $c_j = c_k$. In particular, this condition implies that the reordered index of a sequence is single-valued. Note that the class of admissible reorderings depends only on the index of the wave sequence, so that perturbing wave strengths does not change the class of reorderings.

In factoring reorderings into interactions of adjacent wave pairs, we allow only admissible factors, and treat different factorizations as different reorderings. Each factorization of $\tau = \lambda_t \dots \lambda_1$ determines a time-like ordering on the set I_τ of interactions which make up the reordering, by ordering pairs corresponding to each factor from right to left. Similarly, to each interacting pair we can associate a 'sub-ordering', namely the largest ordering $\mu = \lambda_m \dots \lambda_1$ under which that pair does not interact.

To accurately model the evolution of the approximate solutions, we must consider changes in the strengths of waves as well as their relative positions. We define the action of a reordering on wave strengths inductively, so we need only describe the action of a single join or transposition of waves. Since our estimates for local interactions are exact to second order, we treat all cubic terms as errors, and ignore them in the model. Interacting waves of the same family add linearly with cubic errors, so if the join ϕ maps p and $p+1$ onto p, then the new wave γ'_p has strength $\gamma'_p = \gamma_p + \gamma_{p+1}$, while no other strengths change (although their positions may shift).

It remains to model the effect of a single transposition, which is to produce a wave of the third family whose strength is the weighted product of incident wave strengths. We do not allow a new wave to be created, but rather adjust the strength of the nearest wave of the appropriate family. This is analogous to sampling in the Glimm scheme, since only 3 waves leave each interaction diamond. The wave whose strength is to be adjusted when a particular pair interacts is determined by an interaction map. These maps are inductively defined on the set C_τ of crossing pairs, and the image $i^\tau(j,k)$ of a crossing pair is that wave of the third family whose strength is to be changed. We thus describe the effect of the transposition κ of γ_k and γ_{k+1} on the wave γ_i, where $i = i^\kappa(k, k+1)$, as

$$\gamma'_i = \gamma_i + \Lambda_{c_i} \gamma_k \gamma_{k+1}, \tag{7}$$

while other strengths are unchanged.

Having described the effect of elementary interactions, we use the factorization to inductively define the reordered wave sequence, so that $\tau\gamma = \lambda_t(\dots \lambda_1(\gamma))$. We have thus modeled the evolution of approximate solutions, including cumulative quadratic effects.

For the class of solutions we consider, quadratic terms dominate higher order effects, so that control of these terms allows us to deduce existence of solutions. Each space-like curve in Glimm's scheme has an associated reordering, so that the waves crossing a curve can be approximated by a reordered sequence. We use these sequences to estimate norms of the solution at large times. In particular, we define the functional F for the (future) total variation to be

$$F(\gamma) = \sup_\tau V(\tau\gamma), \tag{8}$$

where γ is the sequence of waves initially, $\tau\gamma$ is the reordered sequence, and the supremum is over admissible reorderings. This functional is analogous to the functional P for the sup-norm, introduced in [7]. Since our bounds are time–dependent,

we further restrict the class of reorderings to those whose interactions take place within (some large) time T.

3. Error Potential

Glimm's scheme is based on the construction of a decreasing global error potential. This is a measure of interaction errors for future wave interactions, and is defined by

$$G = \sum_{App} |\alpha||\beta|, \tag{9}$$

where α and β are wave strengths, and the sum is over all approaching wave pairs. The key idea is that after an interaction, the local interaction error D is balanced by the removal of terms from G, as fewer waves approach. Thus although the total variation (and other norms) may grow, some combination $V + cG$ is decreasing, so that the total variation is bounded. The change in G across a diamond is

$$\Delta G \leq -D + O(1)VD, \tag{10}$$

where D is the local error (removed from G), and the second term is due to higher order errors. Thus in order for G to decrease, we require the variation V to be small.

Our goal is to write down a similar error potential which takes into account quadratic effects, and thus allow larger total variation. The functional G includes all future interactions, but with first order accuracy in wave strengths. We define a functional Q which includes all quadratic terms by means of reorderings. That is, we consider approaching wave pairs, but instead of using initial wave strengths, we use the wave strengths at the time of interaction. Suppose we are given a wave sequence γ and reordering τ, together with an interacting pair (j,k). If μ is the subordering associated to this pair, then we let γ_j^μ be the wave in $\mu\gamma$ corresponding to γ_j; this is the strength of γ_j when that wave interacts with γ_k. We define the error potential by

$$Q(\gamma,\tau) = \sum_{I_\tau} |\gamma_j^\mu||\gamma_k^\mu|, \quad \text{and} \quad Q = \sup_\tau Q(\gamma,\tau), \tag{11}$$

where μ is different for each interacting pair $(j,k) \in I_\tau$. We have the functional relation

$$Q(\gamma,\sigma\tau) = Q(\tau\gamma,\sigma) + Q(\gamma,\tau) \tag{12}$$

for the composition of reorderings, which is the statement that the error potential decreases by the local error after interaction. This functional relation thus expresses the 'conservation of interaction error.' Again, since our bounds are time-dependent, we restrict ourselves to those reorderings spanning time T.

To deal with higher order effects, we consider continuity of the functional. Suppose we are given sequences γ and γ' with the same index (and reorderings), but which differ in a single wave strength, say $\gamma_q \neq \gamma'_q$. We say that Q is continuous, if

$$Q(\gamma,\tau) - Q(\gamma',\tau) \leq K_Q|\gamma_q - \gamma'_q|, \tag{13}$$

where K_Q does not depend on the reordering τ or mesh-dependent parameters, although it may depend on time and the initial data.

Across a diamond, the local interaction error is $O(1)SD$, so that continuity gives

$$\Delta Q \leq -D + O(1)SK_QD, \tag{14}$$

Thus for a decreasing error potential, we require SK_Q to be small. The following theorem holds for general systems, and says that the large–time behavior of solutions can be determined by the quadratic model.

Theorem 1. *Suppose that the functional Q is continuous, with K_Q depending on time T and norms of the initial data. Then given V_0, there is an $\epsilon > 0$ such that if the initial data satisfies*

$$\|u_0\|_\infty \leq \epsilon \quad \textit{and} \quad TV(u_0) \leq V_0, \tag{15}$$

then a weak solution with bounded oscillation and total variation exists up to time T. Moreover, the initial bound for the functional F yields a bound for the total variation of the solution.

Here ϵ is defined in terms of K_Q and V_0, so any restrictions made in finding K_Q apply to the solution. Continuity and boundedness of the functionals are essentially equivalent, so only one of these needs to be checked. Since there are known counter-examples, the functionals F and Q will not have time–independent bounds. Since F and Q are defined in similar ways, $Q \approx F^2$, we make the following conjecture.

Conjecture. *For general systems, boundedness of the functional F independent of mesh parameters is necessary and sufficient for existence of solutions. Any assumptions on the initial data and time of existence used in bounding F translate into conditions on solutions to the Cauchy problem.*

4. Path Integral Formula

Although we have found an algebraic criterion for existence of solutions, this is difficult to analyze without extra assumptions. In general, reordered sequences become unmanageable, and we know that the functionals are unbounded. Motivated by gas dynamics, we make the assumption that our system possesses a Riemann coordinate w. This means that ∇w is a left eigenvector,

$$\ell \cdot Df = \lambda \ell, \quad \text{where} \quad \ell = \nabla w, \tag{16}$$

so that w does not change across rarefactions of other wave families. We also assume that the family corresponding to w has constant wave-speed λ. This is true

for gas dynamics if the Lagrange formalism is used, and allows us to derive a scalar transport equation for smooth solutions, namely

$$w_t + \lambda(u) w_x = 0. \tag{17}$$

Note that this is not the same as assuming the existence of an entropy–flux pair.

As a consequence of our assumptions, the second (entropy) field is degenerate in two ways : first, since we have a Riemann coordinate, the coefficient Λ_2 vanishes, so that (at the quadratic level) no new wave is generated when two sound waves interact. Also, since the wave-speed is constant, 2-waves do not interact, as only one 2-wave enters or leaves each diamond. These observations allow us to treat the entropy as a constant background source at the quadratic level.

With these assumptions, we can now describe the effects of a reordering on a wave sequence precisely. We know that 2-waves do not change, so we need only consider changes in sound waves due to interactions. Moreover, sound waves of different families produce no quadratic effects, and waves of the same family add linearly, so that we can consider each sound wave separately. We thus consider the effect of passing a single sound wave through a static pattern of entropy fluctuations. If a sound wave ϵ crosses an entropy jump γ_{k_1}, a sound wave of strength $\epsilon^{(1)} = \Lambda \epsilon \gamma_{k_1}$ of the opposite family is reflected. This new wave then interacts with other entropy waves, say γ_{k_2}, to produce a wave $\epsilon^{(2)} = \Lambda' \epsilon^{(1)} \gamma_{k_2}$ in the same family as ϵ. This process continues inductively, and after r interactions, say r even, the wave generated by this series of interactions has strength

$$\epsilon^{(r)} = \epsilon^{(r-1)} \gamma_{k_r} \Lambda' = \epsilon \gamma_{k_1} \dots \gamma_{k_r} (\Lambda \Lambda')^{r/2}. \tag{18}$$

This scattering process occurs for each distinct pattern of reflections of the wave as it passes through the field of entropy fluctuations. We shall refer to one of these interaction patterns as a path.

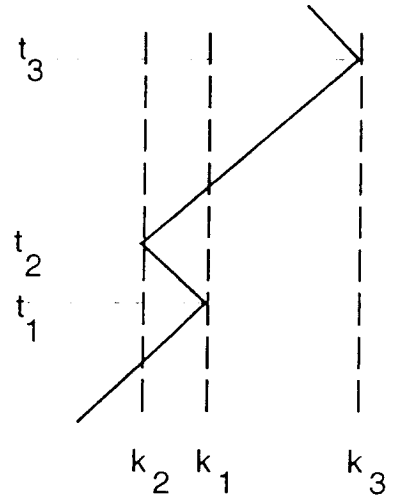

FIGURE 1. A single path

Formally, a path is a set of crossing wave pairs $\{(j_p, k_p)\} \subset C_\tau$, where each k_p represents a 2-wave, and satisfying $j_{p+1} = \imath^\tau(j_p, k_p)$. We say that it begins at j_1 and

ends at $\tau\imath^\tau(j_d, k_d)$, where (j_d, k_d) is the last interacting pair in the path. The position j_{p+1} represents the position of the scattered wave after the first p interactions, and that scattered wave has strength given as above. Since all sound waves combine linearly, we can add up each of these contributions to get a discretized path integral formula.

Theorem 2. *Given the wave sequence γ and reordering τ, let $\delta = \tau(\gamma)$ be the reordered wave sequence. Then the wave δ_i is given by*

$$\delta_i = \sum_j \gamma_j \sum_d \Lambda(i,d) \sum_{\Pi_d(j,i,\tau)} \gamma_{k_1} \cdots \gamma_{k_d}. \tag{19}$$

where $\Pi_d(j,i,\tau)$ is the collection of all d-paths in τ beginning at j and ending at $i = \tau\imath^\tau(j_d, k_d)$, the γ_k's are 2-waves, and Λ is the appropriate weight.

5. Functional Bounds

Having found a formula describing reordered sequences, we look for bounds which will allow us to conclude existence of solutions. It is evident from the path integral formula that the large-time behavior of the solution is highly dependent on the entropy of the solution. Our approach is to bound the 2-wave strengths by some number β, and then count the number of possible paths. This leads to time-dependent bounds for the reordered sequence, and we have the following theorem.

Theorem 3. *If the system of conservation laws admits a 2-Riemann coordinate, and τ is a reordering corresponding to a set of diamonds spanning time T, then the error potential is continuous,*

$$Q(\gamma,\tau) - Q(\gamma',\tau) \le |\gamma_q - \gamma'_q| V(\gamma)^2 O(T), \tag{20}$$

when the sequences differ in the single wave strength γ_q. Moreover, the total variation of the reordered sequence $\tau\gamma$ grows at most exponentially in time,

$$V(\tau\gamma) \le V(\gamma) e^{kT}, \tag{21}$$

where k depends on the flux and the entropy of the initial data.

Since our bounds are independent of mesh parameters, we can now invoke Theorem 1 to obtain existence of solutions, with total variation growing exponentially in time.

We briefly describe the idea of proof of the total variation bound, which in turn yields a bound for Q. Using the path integral formula, we have

$$V(\tau\gamma) = \sum |\delta_i| \le V(\gamma) \max_j |\sum \Lambda(i,d) \gamma_{k_1} \cdots \gamma_{k_d}|, \tag{22}$$

where the sum is over d, i and $\Pi_d(j,i,\tau)$. We now suppose that each $|\gamma_{k_p}| \le \beta$, and use the bound $|\Lambda(i,d)| \le \Lambda^d$, to get

$$V(\tau\gamma) \le V(\gamma) \sum_d (\beta\Lambda)^d \#\Pi_d(J,\cdot,\tau), \tag{23}$$

where $\#\Pi_d(J,\cdot,\tau) = \max_j \sum_i \#\Pi_d(j,i,\tau)$ is the maximum number of d-paths starting at a fixed point.

We now count the number of paths as follows : a path is determined by its starting position, together with the 2-waves $k_1,\ldots,k_d$ with which the scattered wave interacts. Now only one 2-wave passes through each diamond, and each sound wave passes through only one diamond per time step. Therefore the path can be determined by a knowledge of its starting position and a list of (discrete) interaction times. That is, k_p is determined by position j_p and the time t_p at which the p-th interaction occurs, see Figure 1. In particular, the number of paths starting at a fixed point is bounded by the number of lists of interaction times $t_1 < \cdots < t_d$. Since the path spans a time T, we must have $t_d < T/\Delta t$, so that the number of paths is

$$\#\Pi_d(J,\cdot,\tau) \le \binom{T/\Delta t}{d}. \tag{24}$$

Substituting this path count in and using the binomial theorem, gives

$$V(\tau\gamma) \le V(\gamma)(1+\beta\Lambda)^{T/\Delta t} \le V(\gamma)e^{\beta\Lambda T/\Delta t}. \tag{25}$$

As stated here, this bound explodes as the mesh size Δt decreases to 0, unless the entropy remains Lipschitz continuous, which is a major assumption. However, a more sophisticated estimate based on these ideas yields a time–dependent bound with no extra assumptions on the solution. For details of the estimate together with a physical interpretation, the reader is referred to [5].

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